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FROBENIUS SPLITTING AND GEOMETRY OF G-SCHUBERT VARIETIES

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Abstract. Let X be an equivariant embedding of a connected reductive group G over an algebraically closed field k of positive characteristic. Let B denote a Borel subgroup of G. A G-Schubert variety in X is a subvariety of the form $\operatorname{diag}(G) \cdot V$, where V is a $B \times B$-orbit closure in X. In the case where X is the wonderful compactification of a group of adjoint type, the G-Schubert varieties are the closures of Lusztig's G-stable pieces. We prove that X admits a Frobenius splitting that compatibly splits all the G-Schubert varieties. Moreover, any G-Schubert variety admits stable Frobenius splittings along ample divisors in case X is projective. Although this indicates that G-Schubert varieties have nice singularities we give an example, in the wonderful compactification of a group of adjoint type, which is not normal. Finally we also extend the Frobenius splitting results to the more general class of R-Schubert varieties.

1. Introduction

Let G be a connected reductive group over an algebraically closed field k of positive characteristic and let B denote a Borel subgroup of G. An equivariant embedding X of G is a $G \times G$-variety which contains $G = (G \times G)/\operatorname{diag}(G)$ as an open $G \times G$-invariant subset, where $\operatorname{diag}(G)$ is the diagonal image of G in $G \times G$. When G is a semisimple group of adjoint type there exists a canonical equivariant embedding $\mathbf{X}$ which is called the *wonderful compactification* and which has been the subject of much attention in recent years. Actually the wonderful compactifications are the primary examples which we have in mind, but as the more general setup follows almost identical, we have decided to state the obtained results in full generality.

Any equivariant embedding X of G contains finitely many $B \times B$-orbits. In recent years the geometry of closures of $B \times B$-orbits has been studied by several authors. The most general result was obtained in [H-T2] where it was proved that $B \times B$-orbit closures are normal, Cohen-Macaulay and have (F-)rational singularities (actually, even stronger results were obtained). In the present paper we will need a Frobenius splitting result of [H-T2] which is : the equivariant embedding X admits a $B \times B$-canonical Frobenius splitting which compatibly splits the closure of every $B \times B$-orbit. From this result we will derive that X admits a $\operatorname{diag}(B)$-canonical Frobenius splitting which compatibly splits every subset of the form $\operatorname{diag}(G) \cdot V$, where V denotes the closure of a $B \times B$-orbit. In this paper, we will consider subsets of the form $\operatorname{diag}(G) \cdot V$ which we call the *G-Schubert varieties* of X. If X is the wonderful compactification, then $\operatorname{diag}(G) \cdot V$ is the closure of some G-stable piece in X and any closure of G-stable piece is of the form $\operatorname{diag}(G) \cdot V$ for some $B \times B$-orbit closure V.

Before discussing the Frobenius splittings on the G-Schubert varieties, let us make a short digression and discuss some motivations to study the G-stable pieces and G-Schubert varieties (in the wonderful compactification).

The decomposition (of the wonderful compactification) into G-stable pieces was introduced by Lusztig in [L] to construct and study a class of perverse sheaves, which generalizes his theory of character sheaves on reductive groups. More precisely, these perverse sheaves are the intermediate extensions of the so-called "character sheaves" on a G-stable piece. This is one of the motivations to study the geometry of the closures of G-stable pieces.

When G is a simple group, the boundary of the closure of the unipotent subvariety of G in the wonderful compactification is a union of certain G-Schubert varieties (see [He] and [H-T]). Thus knowing the geometry of these G-Schubert varieties will help us to understand the geometry of the closure of the unipotent variety.

There is another motivation to study the G-stable pieces and G-Schubert varieties which comes from Poisson geometry. Let $\mathrm{Lie}(G)$ denote the Lie algebra of G and $\ll, \gg$ denote a fixed symmetric non-degenerate ad-invariant bilinear form. Let $<,>$ be the bilinear form on $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$ defined by $< (x, y), (x', y') > = \ll x, x' \gg - \ll y, y' \gg$. In [E-L], Evens and Lu showed that each splitting $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G) = l \oplus l'$, where l and l' are Lagrangian subalgebras of $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$, gives rise to a Poisson structure $\Pi_{l,l'}$ on $\mathbf{X}$. If moreover, one starts with the Belavin-Drinfeld splitting, then all the G-stable pieces/G-Schubert varieties and $B \times B^-$-orbits of $\mathbf{X}$ are Poisson subvarieties, where B^- is a Borel subgroup opposite to B. Thus to understand the Poisson structure on $\mathbf{X}$ corresponding to the Belavin-Drinfeld splitting, one needs to understand the geometry of the G-stable pieces/G-Schubert varieties. However, if we start with another splitting, then we obtain a different Poisson structure on $\mathbf{X}$ and in order to understand these Poisson structures, one needs to study the R-stable pieces [L-Y] instead, which generalize both the G-stable pieces and the $B \times B^-$-orbits.

Now let us turn our attention back to the Frobenius splitting properties. To obtain the described Frobenius splitting properties of the G-Schubert varieties we first prove that $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} X$ admits a $\mathrm{diag}(B)$-canonical Frobenius splitting which compatibly splits all closed subvarieties of the form $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} \overline{V}$, where $\overline{V}$ denotes the closure of a $B \times B$-orbit. By general theory on canonical Frobenius splitting this would follow if X admits a $\mathrm{diag}(B)$-canonical Frobenius splitting compatibly splitting all $B \times B$-orbit closures [B-K, Prop.4.1.17]. However, we only know and expect, that X admits a $B \times B$-canonical Frobenius splitting, which is less restrictive. In particular, we cannot apply the result [loc.cite] directly. Still the proof of [loc.cite] can be modified to the present situation. Actually we prove a result which both contain the statement in [loc.cite] and also the statement which we need. Having obtained the described Frobenius splitting properties of $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} X$ we may apply a push forward argument along the natural morphism : $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} X \to X$ to conclude that X admits a $\mathrm{diag}(B)$-canonical Frobenius splitting which compatibly splits all the G-Schubert varieties.

When X is a projective variety, a closer study of the obtained Frobenius splitting reveals that when restricted to a G-Schubert variety $\mathcal{X}$ then it is actually a Frobenius splitting along the support of an ample divisor . This has strong implication on the cohomology of line bundles. E.g. the higher cohomology of every nef line bundle (i.e. a line bundle $\mathcal{L}$ such that $\mathcal{L} \otimes \mathcal{M}$ is ample when $\mathcal{M}$ is ample) on $\mathcal{X}$ will be zero. One should however notice that we do not claim that X admits a Frobenius splitting along the support of an ample divisor which compatibly Frobenius splits all the G-Schubert varieties. But letting Y denote the minimal $G \times G$-orbit closure containing $\mathcal{X}$ it does follow, that Y admits a Frobenius splitting along the support of an ample divisor which compatibly Frobenius splits $\mathcal{X}$. In particular, the restriction morphism

$$\mathrm{H}^0(Y, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}, \mathcal{L}),$$

for a nef line bundle $\mathcal{L}$ on Y is surjective. We do not know if this is true when Y is substituted by X. However, if $\mathcal{L}$ is ample or X is the wonderful compactification of an adjoint group, then $\mathrm{H}^0(X, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}, \mathcal{L})$ is surjective.

It seems natural to expect that the above described results should have strong implications on the geometry of G-Schubert varieties. It therefore comes as a complete surprise that these subvarieties are, in general, not even normal. We only provide a single example of this phenomenon (for the wonderful compactification of a group of type G_2), but expect that this absence of normality is the general picture.

This paper is organized in the following way. In Section 2 we briefly define Frobenius splitting and explain its fundamental ideas. Section 3 is devoted to some results on linearized sheaves which should all be well known. In Section 4 we study the Frobenius splitting of varieties of the form $G \times_P X$ for a variety X with an action by a parabolic subgroup P. The main idea is to decompose the Frobenius morphism on $G \times_P X$ into maps associated to the Frobenius morphism on the base G/P and the fiber X of the natural morphism $G \times_P X \to G/P$. In Section 5 we introduce canonical Frobenius splittings and in Section 6 we study the obtained Frobenius splitting relative to effective divisors. Section 7 contains application to general $B \times B$-varieties of the previous sections. In section 8 we define the G-stable pieces and G-Schubert varieties. In Section 9 we apply the material of the previous sections to the class of equivariant embeddings and obtain Frobenius splitting as well as cohomology vanishing results for G-Schubert varieties. Section 10 contains an example of a non-normal G-Schubert variety. Finally Section 11 contains generalizations and variations of the previous sections.

2. The relative Frobenius morphism

By a variety we mean a reduced and separated scheme of finite type over k. In particular, we allow a variety to have multiple components. By definition a variety X comes with an associated morphism

$$p : X \to \mathrm{Spec}(k)$$

of schemes. The Frobenius morphism on $\mathrm{Spec}(k)$ is the morphism of schemes

$$F_k : \mathrm{Spec}(k) \to \mathrm{Spec}(k),$$

which on the level of coordinate rings is defined by $a \mapsto a^p$. As k is assumed to be algebraically closed the morphism F_k is actually an isomorphism and we let F_k^{-1} denote the inverse morphism. Composing p with F_k^{-1} we obtain a new variety

$$p' : X \to \mathrm{Spec}(k),$$

with underlying scheme X. In the following we suppress the morphism p from the notation and simply use X as the notation for the variety defined by p. The variety defined by p' is then denoted by X'.

The relative Frobenius morphism on X is then the morphism of varieties :

$$F_X : X \to X'$$

which as a morphism of schemes is the identity map on the level of points and where the associated map of sheaves

$$F_X^\sharp : \mathcal{O}_{X'} \to (F_X)_* \mathcal{O}_X$$

is the p-th power map. A key property of the Frobenius morphism is the relation $(F_X)^* \mathcal{N}' \simeq \mathcal{N}^p$ which is satisfied for every line bundle $\mathcal{N}$ on X (here $\mathcal{N}'$ denotes the corresponding line bundle on X').

2.1. **Frobenius splitting.** A variety X is said to be Frobenius split if the $\mathcal{O}_{X'}$-linear map of sheaves :

$$F_X^\sharp : \mathcal{O}_{X'} \to (F_X)_* \mathcal{O}_X$$

has a section; i.e. if there exists an element

$$s \in \mathrm{Hom}_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{O}_{X'}\big),$$

such that the composition $s \circ F_X^\sharp$ is the identity endomorphism of $\mathcal{O}_{X'}$. The section s will be called a Frobenius splitting of X. In the following we will use the notation $\mathcal{E}\mathrm{nd}_F(X)$ to denote the sheaf

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{O}_{X'}\big),$$

while $\mathrm{End}_F(X)$ will denote the global sections of this sheaf. The subvectorspace of $\mathrm{End}_F(X)$ consisting of the elements which maps the constant function 1 on X to a constant function on X' will be denoted by $\mathrm{End}_F(X)_c$. In particular, any Frobenius splitting of X will lie in $\mathrm{End}_F(X)_c$.

2.2. **Compatibly Frobenius splitting.** Let Y denote a closed subvariety of X defined by the sheaf of ideals $\mathcal{I}_Y$. Let Y' denote the associated closed subvariety of X' with sheaf of ideals $\mathcal{I}_{Y'}$, and let $i'_Y : Y' \to X'$ denote the inclusion. The kernel of the morphism

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}\big((F_X)_* \mathcal{O}_X, \mathcal{O}_{X'}\big) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}\big((F_X)_* I_Y, (i'_Y)_* \mathcal{O}_{Y'}\big)$$

induced by the inclusion $\mathcal{I}_Y \subset \mathcal{O}_X$ and the projection $\mathcal{O}_{X'} \to (i'_Y)_* \mathcal{O}_{Y'}$, will be denoted by $\mathcal{E}\mathrm{nd}_F(X, Y)$. The set of global sections of $\mathcal{E}\mathrm{nd}_F(X, Y)$ will be denoted by $\mathrm{End}_F(X, Y)$, and consists of the elements s in $\mathrm{End}_F(X)$ satisfying

$$s\big((F_X)_* \mathcal{I}_Y\big) \subset \mathcal{I}_{Y'}.$$

Thus such an element defines an element in $\mathrm{End}_F(Y)$ and we say that s is compatible with Y. If s moreover is a Frobenius splitting of X then we say that Y is compatibly Frobenius split by s.

If $Y_1, Y_2, \dots, Y_m$ is a collection of closed subvarieties of X then notation $\mathcal{E}\mathrm{nd}_F(X, Y_1, \dots, Y_m)$ (or sometimes $\mathcal{E}\mathrm{nd}_F(X, \{Y_i\}_i)$) will denote the intersection of the subsheaves $\mathcal{E}\mathrm{nd}_F(X, Y_i)$ for $i = 1, \dots, m$. The set of global sections of the sheaf $\mathcal{E}\mathrm{nd}_F(X, Y_1, \dots, Y_m)$ will be denoted by $\mathrm{End}_F(X, Y_1, \dots, Y_m)$. Finally we use the notation $\mathrm{End}_F(X, Y)_c$ to denote the intersection of $\mathrm{End}_F(X)_c$ with $\mathrm{End}_F(X, Y)$. The notation $\mathrm{End}_F(X, Y_1, \dots, Y_n)_c$ is then defined similarly.

2.3. **Frobenius D-splittings.** Let $\mathcal{L}$ denote a line bundle on X and let σ denote a global section of $\mathcal{L}$. Then σ induces a map

$$\mathrm{Hom}_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{L}, \mathcal{O}_{X'}\big) \to \mathrm{End}_F(X).$$

When Y is a closed subvariety of X then an element s of the vectorspace

$$\mathrm{Hom}_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{L}, \mathcal{O}_{X'}\big)$$

is said to be compatible with Y if

$$s\big((F_X)_*(\mathcal{I}_Y \otimes \mathcal{L})\big) \subset \mathcal{I}_{Y'}$$

The following is then an easy consequence of the definition

Lemma 2.1. *Assume that s is compatible with closed subvarieties Y and Z of X. Then*

(1) *s is compatible with every irreducible component of Y.*
(2) *If the scheme theoretic intersection $Z \cap Y$ is reduced then s is compatible with $Z \cap Y$.*

Proof. Let Y_1 denote an irreducible component of Y and let

$$\mathcal{J} = s\big((F_X)_*(\mathcal{I}_{Y_1} \otimes \mathcal{L})\big) \subset \mathcal{O}_{X'}.$$

Let U denote the open complement (in X') of the irreducible components of Y' which are different from Y_1'. Then $\mathcal{I}_{Y_1'}$ coincides with $\mathcal{I}_{Y'}$ on U and consequently $\mathcal{J}_{|U} \subset (\mathcal{I}_{Y'})_{|U}$ as s is compatible with Y. In particular, $\mathcal{J}_{|U} \subset (\mathcal{I}_{Y_1'})_{|U}$. As $U \cap Y_1'$ is dense in Y_1' and Y_1' is reduced it follows from the relation $\mathcal{J} \subset \mathcal{O}_{X'}$ that $\mathcal{J} \subset \mathcal{I}_{Y_1'}$. This proves that s is compatible with Y_1. The second claim follows as the sheaf of ideals of the intersection $Z \cap Y$ is $\mathcal{I}_Y + \mathcal{I}_Z$. □

Consider the situation where $\mathcal{L}$ is the line bundle $\mathcal{O}_X(D)$ associated to an effective Cartier divisor D on X. Let $\sigma = \sigma_D$ denote the associated global section. When the image of s in $\mathrm{End}_F(X)$ is a Frobenius splitting $\tilde{s}$ of X then we say that $\tilde{s}$ is a Frobenius D-splitting of X. The following result assures that, in this case, the compatibility condition with closed subvarieties agrees with the classical definition [R, Defn.1.2].

Lemma 2.2. *Assume that s defines a Frobenius D-splitting of X. Then s is compatible with Y if and only if (i) $\tilde{s}$ compatibly Frobenius splits Y and (ii) the support of D does not contain any irreducible component of Y.*

Proof. The *if* part of the statement follows from [R, Prop.1.4]. So assume that s is compatible with Y. Then s induces a morphism

$$\overline{s} : (F_Y)_*\mathcal{O}_X(D)_{|Y} \to \mathcal{O}_{Y'},$$

satisfying $\overline{s}((\sigma_D)_{|Y})$ is the constant function 1 on Y'. As a consequence $(\sigma_D)_{|Y}$ does not vanish on any of the irreducible components of Y. This proves part (ii) of the statement. Part (i) is clearly satisfied. □

It follows that if s is compatible with Y and, moreover, defines a Frobenius D-splitting of X then $D \cap Y$ makes sense as an effective Cartier divisor on Y and s induces a Frobenius $D \cap Y$-splitting of Y.

2.4. **Stable Frobenius splittings along divisors.** Generalizing the ideas above we may consider the morphism

$$\mathrm{Hom}_{\mathcal{O}_{X^{(n)}}}\big((F_X)^n_*\mathcal{O}_X(D), \mathcal{O}_{X^{(n)}}\big) \to \mathrm{Hom}_{\mathcal{O}_{X^{(n)}}}\big((F_X)^n_*\mathcal{O}_X, \mathcal{O}_{X^{(n)}}\big),$$

induced by σ_D, where $X^{(n)}$ denotes the n-th iterated Frobenius twist of X. Then the image of an element s in $\mathrm{Hom}_{\mathcal{O}_{X^{(n)}}}\big((F_X)^n_*\mathcal{O}_X(D), \mathcal{O}_{X^{(n)}}\big)$ is said to be a stable Frobenius splitting of X along D if it maps the global section σ_D of $(F_X)^n_*\mathcal{O}_X(D)$ to the global section 1 of $\mathcal{O}_{X'}$. In this case a closed subvariety Y of X is said to be compatibly with the stable Frobenius splitting if

$$s\big((F_X)^n_*(\mathcal{I}_Y \otimes \mathcal{O}_X(D))\big) \subset \mathcal{I}_{Y^{(n)}}.$$

The following is well known (see e.g. [B-T, Lem.3.1])

Lemma 2.3. *Let D_1 and D_2 denote effective divisors on X. Then X admits stable Frobenius splittings along D_1 and D_2 if and only if X admits a stable Frobenius splitting along $D_1 + D_2$.*

The following result explains one of the main applications of (stable) Frobenius splitting.

Proposition 2.4. *Assume that X admits a stable Frobenius splitting along an effective Cartier divisor D. Then there exists an n such that for each line bundle $\mathcal{L}$ on X we have an inclusion of abelian groups*

$$\mathrm{H}^i(X, \mathcal{L}) \subset \mathrm{H}^i(X, \mathcal{L}^{p^n} \otimes \mathcal{O}_X(D)).$$

In particular, if D is ample and $\mathcal{L}$ is nef, then $\mathrm{H}^i(X, \mathcal{L}) = 0$ for $i > 0$. Moreover, if Y is compatibly Frobenius split, D is ample and $\mathcal{L}$ is nef then the restriction morphism

$$\mathrm{H}^0(X, \mathcal{L}) \to \mathrm{H}^0(Y, \mathcal{L}),$$

is surjective.

Proof. Argue as in the proof [R, Prop.1.13(i)]. □

2.5. **Duality for F_X.** By duality (see [Har2, Ex.III.6.10]) for the finite morphism F_X we may to each quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}$ associate an $\mathcal{O}_X$-module denoted by $(F_X)^!\mathcal{F}$ and satisfying

$$(F_X)_*(F_X)^!\mathcal{F} = \mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{O}_X, \mathcal{F}\big).$$

Actually, as F_X is the identity on the level of points we may define $(F_X)^!\mathcal{F}$ as the sheaf of abelian groups

$$\mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{O}_X, \mathcal{F}\big),$$

with $\mathcal{O}_X$-module structure defined by

$$(g \cdot \phi)(f) = \phi(gf),$$

for $g, f \in \mathcal{O}_X$ and $\phi \in \mathcal{H}om_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{O}_X, \mathcal{F}\big)$. When $\mathcal{F} = \mathcal{O}_X$ we will also use the notation $\mathcal{E}nd^!_F(X)$ for $(F_X)^!\mathcal{O}_X$. If $Y_1, Y_2, \dots, Y_m$ is a collection of closed subvarieties of X then $\mathcal{E}nd^!_F(X, Y_1, \dots, Y_m)$ (or $\mathcal{E}nd^!_F(X, \{Y_i\}_{i=1}^m)$) will denote the subsheaf of $\mathcal{E}nd^!_F(X)$ consisting of elements compatible with Y_i for $i = 1, \dots, m$.

Later we will consider $\mathcal{O}_X$-linear morphisms of the form

$$\phi : \mathcal{M} \to \mathcal{E}nd^!_F(X),$$

where $\mathcal{M}$ is a line bundle on X. Notice, that a morphism of this form is equivalent to a global section s of the sheaf

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{M}, \mathcal{O}_{X'}\big).$$

Moreover, the image $\phi(\sigma)$ of a global section σ of $\mathcal{M}$ will factor as

$$(F_X)_*\mathcal{O}_X \xrightarrow{(F_X)_*\sigma} (F_X)_*\mathcal{M} \xrightarrow{s} \mathcal{O}_{X'}.$$

It follows

Lemma 2.5. *Let Y denote a closed subvariety of X. Then the image of ϕ is contained in $End^!_F(X,Y)$ if and only if s is compatible with Y.*

A similar result is true for a collection of closed subvarieties of X. We will also need the following remark

Lemma 2.6. *Let D denote a reduced effective Cartier divisor on X and $\mathcal{L}$ denote a line bundle on X. Let $\mathcal{M} = \mathcal{O}_X((p-1)D) \otimes \mathcal{L}$ and assume that we have a morphism $\phi : \mathcal{M} \to \mathcal{E}\mathrm{nd}^!_F(X)$, as above. Let σ_D denote the canonical section of $\mathcal{O}_X(D)$ and consider the map*

$$\phi_D : \mathcal{L} \to \mathcal{E}\mathrm{nd}^!_F(X),$$

induced by σ_D^{p-1}. Then the element

$$s_D \in \mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}\big((F_X)_*\mathcal{L}, \mathcal{O}_{X'}\big),$$

induced by ϕ_D is compatible with the variety associated with D. In particular, the image of ϕ_D is contained in $\mathcal{E}\mathrm{nd}^!_F(X,D)$.

Proof. Notice that s_D is the composition

$$s_D : (F_X)_*\mathcal{L} \xrightarrow{(F_X)_*\sigma_D^{p-1}} (F_X)_*\mathcal{M} \xrightarrow{s} \mathcal{O}_{X'}.$$

Hence, the restriction of s_D to $\mathcal{L} \otimes \mathcal{O}_X(-D)$ coincides with the map

$$(F_X)_*\big(\mathcal{L} \otimes \mathcal{O}_X(-D)\big) \xrightarrow{(F_X)_*\sigma_D^{p}} (F_X)_*\mathcal{M} \xrightarrow{s} \mathcal{O}_{X'}.$$

But the restriction of s to

$$(F_X)_*\big(\mathcal{O}_X(-pD) \otimes \mathcal{M}\big) \simeq \mathcal{O}_{X'}(-D') \otimes (F_X)_*\mathcal{M},$$

maps by linearity into $\mathcal{O}_{X'}(-D')$. This ends the proof. $\square$

When X is a smooth variety then $\mathcal{E}\mathrm{nd}^!_F(X)$ coincides with the line bundle ω_X^{1-p}, where ω_X denotes the dualizing sheaf of X (see e.g. [B-K, Sect.1.3]).

2.6. **Push-forward operation.** Assume that $f : X \to Z$ is a morphism of varieties satisfying that the associated map $f^\sharp : \mathcal{O}_Z \to f_*\mathcal{O}_X$ is an isomorphism. Let $f' : X' \to Y'$ denote the associated morphism. Then f'_* induces a morphism

$$f'_*\,\mathcal{E}\mathrm{nd}_F(X) \to \mathcal{E}\mathrm{nd}_F(Z).$$

If $Y \subset X$ is a closed subset then this map maps $f'_*\,\mathcal{E}\mathrm{nd}_F(X,Y)$ to $\mathcal{E}\mathrm{nd}_F(X, \overline{f(Y)})$ where $\overline{f(Y)}$ denotes the variety associated to the closure of the image of Y. On the level of global sections this means that every Frobenius splitting s of X induces a Frobenius splitting f'_*s of Z such that when s compatibly Frobenius splits Y then f'_*s compatibly Frobenius splits $\overline{f(Y)}$. Likewise

Lemma 2.7. *With notation as above let $\mathcal{L}$ denote a line bundle on Z and let s be an element of*

$$\mathrm{Hom}_{\mathcal{O}_{X'}}\big((F_X)_* f^*\mathcal{L}, \mathcal{O}_{X'}\big).$$

Then $f'_* s$ is an element of

$$\mathrm{Hom}_{\mathcal{O}_{Z'}}\big((F_Z)_*\mathcal{L}, \mathcal{O}_{Z'}\big).$$

Moreover, if s is compatible with a closed subvariety Y of X then $f'_* s$ is compatible with $\overline{f(Y)}$.

Proof. This follows easily from the fact that the sheaf of ideals of $\overline{f(Y)}$ coincides with $f_*\mathcal{I}_Y$ [B-K, Lem.1.1.8]. □

3. Linearized sheaves

Let H denote a linear algebraic group over the field k and let X denote a H-variety with H-action defined by $\sigma : H \times X \to X$. We let $p_2 : H \times X \to X$ denote projection on the second coordinate. A *H-linearization* of a quasi-coherent sheaf $\mathcal{F}$ on X is an $\mathcal{O}_{H\times X}$-linear isomorphism

$$\phi : \sigma^*\mathcal{F} \to p_2^*\mathcal{F},$$

satisfying the relation

$$(\mu \times \mathbf{1}_X)^*\phi = p_{23}^*\phi \circ (\mathbf{1}_H \times \sigma)^*\phi \qquad (1)$$

as morphisms of sheaves on $H \times H \times X$. Here $\mu : H \times H \to H$ (resp. $p_{23} : H \times H \times X \to H \times X$) denotes the multiplication on H (resp. the projection on the second and third coordinate).

A morphism $\psi : \mathcal{F} \to \mathcal{F}'$ of H-linearized sheaves is a morphism of $\mathcal{O}_X$-modules commuting with the linearizations ϕ and ϕ' of $\mathcal{F}$ and $\mathcal{F}'$, i.e. $\phi' \circ \sigma^*(\psi) = p_2^*(\psi) \circ \phi$.

Linearized sheaves form an abelian category which we denote by $\mathrm{Sh}_H(X)$.

3.1. **Quotients and linearizations.** Assume that the quotient $q : X \to X/H$ exists and q is a locally trivial principal H-bundle. Then for $\mathcal{G} \in \mathrm{Sh}(X/H)$, $q^*\mathcal{G}$ is naturally a H-linearized sheaf on X. This defines a functor $q^* : \mathrm{Sh}(X/H) \to \mathrm{Sh}_H(X)$. On the other hand, for $\mathcal{F} \in \mathrm{Sh}_H(X)$, $q_*\mathcal{F}$ has a natural action of H. Define a functor $q_*^H : \mathrm{Sh}_H(X) \to \mathrm{Sh}(X/H)$ by $q_*^H(\mathcal{F}) = (q_*\mathcal{F})^H$ the subsheaf of H-invariants of $q_*\mathcal{F}$. It is known that the functor $q^* : \mathrm{Sh}(X/H) \to \mathrm{Sh}_H(X)$ is an equivalence of category and the inverse functor is $q_*^H : \mathrm{Sh}_H(X) \to \mathrm{Sh}(X/H)$.

In general, if H is a closed normal subgroup of G and X is a G-variety such that H acts freely on X, then X/H is a G/H-variety and the functor $q^* : \mathrm{Sh}_{G/H}(X/H) \to \mathrm{Sh}_G(X)$ is an equivalence of category and the inverse functor is $q_*^H : \mathrm{Sh}_G(X) \to \mathrm{Sh}_{G/H}(X/H)$.

3.2. **Induction equivalence.** Consider now a connected linear algebraic group G and a parabolic subgroup P in here. Let X denote a P-variety. Then $Y = G \times X$ is a $G \times P$-variety by

$$(g, p)(h, x) = (ghp^{-1}, px)$$

for $g, h \in G$, $p \in P$ and $x \in X$. Then P acts freely on $G \times X$ and we denote by $Z = G \times_P X$ the quotient space and $\pi : Y \to Z$ the quotient map. The quotient of Y by G also exists and may be identified with the projection $p_2 : G \times X \to X$. In particular, we may apply the above consideration to obtain an equivalences between the categories of $\mathrm{Sh}_P(X)$, $\mathrm{Sh}_{G\times P}(Y)$ and $\mathrm{Sh}_G(Z)$. Notice that under this equivalence a P-linearized sheaf $\mathcal{F}$ on X corresponds to the G-linearized sheaf $\mathcal{I}\mathrm{nd}_P^G(\mathcal{F}) = (\pi_* p_2^* \mathcal{F})^P$. In particular, the global sections of $\mathcal{I}\mathrm{nd}_P^G(\mathcal{F})$ equals

$$\mathcal{I}\mathrm{nd}_P^G(\mathcal{F})(Z) = \big(p_2^*\mathcal{F}(G \times X)\big)^P = \big(k[G] \otimes_k \mathcal{F}(X)\big)^P = \mathrm{Ind}_P^G(\mathcal{F}(X)), \tag{2}$$

where the second equality follows by the Künneth formula. This also explains the notation $\mathcal{I}\mathrm{nd}_P^G(\mathcal{F})$. Similarly, starting with a G-linearized sheaf $\mathcal{G}$ on $G \times_P X$ then the associated P-linearized line bundle on X equals $\mathcal{G}' = ((p_2)_* \pi^* \mathcal{G})^G$. However, by [Bri, Lemma 2(1)] the latter also equals the more simple pull back $i^*\mathcal{G}$ by the P-equivariant map

$$i : X \to G \times_P X,$$

sending x to $\pi(1, x)$. In particular, we conclude that the functor $i^* : \mathrm{Sh}_G(Z) \to \mathrm{Sh}_P(X)$ is an equivalence of categories and the inverse functor is $\mathcal{I}\mathrm{nd}_P^G$. Notice also that the global sections of $\mathcal{G}$ is then G-equivariantly isomorphic to

$$\mathcal{G}(Z) = \mathrm{Ind}_P^G\big((i^*\mathcal{G})(X)\big),$$

which follows by (2) above.

4. Frobenius splitting of $G \times_P X$

Let G denote a linear algebraic group over an algebraically closed field k of positive characteristic $p > 0$. Let P denote a parabolic subgroup of G and let X denote a P-variety. In this section we want to consider Frobenius splittings of the quotient $Z = G \times_P X$ of $G \times X$ by P. We let $\pi : Z \to {}^G/_P$ denote the morphism induced by the projection of $G \times X$ on the first coordinate. When $g \in G$ and $x \in X$ we use the notation $[g, x]$ to denote the element in Z represented by (g, x).

4.1. **Decomposing the Frobenius morphism.** The Frobenius morphism F_Z admits a decomposition $F_Z = F_b \circ F_f$ where F_b (resp. F_f) is related to the Frobenius morphism on the base (resp. fiber) of π. More precisely, define $\hat{Z}$ and the morphisms $\hat{\pi}$, F_b as part of the fiber product diagram

$$\begin{tikzcd} \hat{Z} \arrow[r, "F_b"] \arrow[d, "\hat{\pi}"'] & Z' \arrow[d, "\pi'"] \\ {}^G/_P \arrow[r, "F_{G/P}"] & ({}^G/_P)' \end{tikzcd} \tag{3}$$

A local calculation shows that we may identify $\hat{Z}$ with the quotient $G \times_P X'$, where the P-action on the Frobenius twist X' of X is the natural one. With this identification $\hat{\pi} : G \times_P X' \to {}^G/_P$ is just the map $[g, x'] \mapsto gP$. It also follows that the natural morphism (induced by the Frobenius morphism on X)

$$F_f : G \times_P X \to G \times_P X'$$

makes the following diagram commutative

$$\begin{tikzcd} Z \arrow[dr, "F_f"] \arrow[drr, bend left, "F_Z"] \arrow[ddr, bend right, "\pi"'] & & \\ & \hat{Z} \arrow[r, "F_b"] \arrow[d, "\hat{\pi}"'] & Z' \arrow[d, "\pi'"] \\ & {}^G/_P \arrow[r, "F_{G/P}"] & ({}^G/_P)' \end{tikzcd}$$

For a given $\mathcal{O}_{\hat{Z}}$-module $\mathcal{F}$ we now introduce the following notation

$$\mathcal{E}\mathrm{nd}_F^{\mathcal{F}}(Z)_b = \mathcal{H}\mathrm{om}_{\mathcal{O}_{Z'}}\big((F_b)_*\mathcal{F}, \mathcal{O}_{Z'}\big),$$
$$\mathcal{E}\mathrm{nd}_F^{\mathcal{F}}(Z)_f = \mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{O}_Z, \mathcal{F}\big).$$

When $\mathcal{F} = \mathcal{O}_{\hat{Z}}$ we also write $\mathcal{E}\mathrm{nd}_F(Z)_b$ and $\mathcal{E}\mathrm{nd}_F(Z)_f$ respectively. Then using the decomposition $F_Z = F_b \circ F_f$ we obtain a map

$$\Phi_{\mathcal{F}} : \mathcal{E}\mathrm{nd}_F^{\mathcal{F}}(Z)_b \otimes (F_b)_*\mathcal{E}\mathrm{nd}_F^{\mathcal{F}}(Z)_f \to \mathcal{E}\mathrm{nd}_F(Z),$$

induced by composition of morphisms. Taking global sections of the considered sheaves we arrive at a map

$$\Phi_{\mathcal{F}}(Z') : \mathrm{End}_F^{\mathcal{F}}(Z)_b \otimes_k \mathrm{End}_F^{\mathcal{F}}(Z)_f \to \mathrm{End}_F(Z),$$

where $\mathrm{End}_F^{\mathcal{F}}(Z)_b$ (resp. $\mathrm{End}_F^{\mathcal{F}}(Z)_f$) denotes the global sections of sheaf $\mathcal{E}\mathrm{nd}_F^{\mathcal{F}}(Z)_b$ (resp. $\mathcal{E}\mathrm{nd}_F^{\mathcal{F}}(Z)_f$).

4.2. **An equivariant setup.** From now on we assume that $\mathcal{F} = \hat{\mathcal{L}}$ is the pull back $\hat{\pi}^*\mathcal{L}$ of a G-linearized line bundle $\mathcal{L}$ on ${}^{G}/_{P}$. The restriction of $\mathcal{L}$ to the point $eP \in {}^{G}/_{P}$ is then a trivial line bundle with global sections k. We let λ denote the P-character defining the P-action on k.

4.2.1. *A description of* $\mathrm{End}_F^{\hat{\mathcal{L}}}(Z)_f$. Now $\mathcal{E}\mathrm{nd}_F^{\hat{\mathcal{L}}}(Z)_f$ is a G-linearized sheaf on $G \times_P X'$. Let $Y \subset X$ denote a P-stable subvariety of X and let $Z_Y = G \times_P Y$ denote the associated subvariety of Z with sheaf of ideals $\mathcal{I}_{Z_Y} \subset \mathcal{O}_Z$. Let $\hat{Z}_Y$ denote the subset $G \times_P Y'$ of $G \times_P X'$. Then there is a natural morphism of G-linearized sheaves

$$\mathcal{E}\mathrm{nd}_F(Z)_f = \mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{O}_Z, \mathcal{O}_{\hat{Z}}\big) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{I}_{Z_Y}, \mathcal{O}_{\hat{Z}_Y}\big),$$

induced by the inclusion $\mathcal{I}_{Z_Y} \subset \mathcal{O}_Z$ and the projection $\mathcal{O}_{\hat{Z}} \to \mathcal{O}_{\hat{Z}_Y}$. We let $\mathcal{E}\mathrm{nd}_F(Z, Z_Y)_f$ denote the G-linearized kernel of the above map and arrive at a left exact sequence of G-linearized sheaves

$$0 \to \mathcal{E}\mathrm{nd}_F(Z, Z_Y)_f \to \mathcal{E}\mathrm{nd}_F(Z)_f \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{I}_{Z_Y}, \mathcal{O}_{\hat{Z}_Y}\big)$$

and consequently also

$$0 \to \mathcal{E}\mathrm{nd}_F(Z, Z_Y)_f \otimes \hat{\mathcal{L}} \to \mathcal{E}\mathrm{nd}_F^{\hat{\mathcal{L}}}(Z)_f \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{I}_{Z_Y}, \hat{\mathcal{L}}_{|\hat{Z}_y}\big).$$

Taking global section we may identify global sections of $\mathcal{E}\mathrm{nd}_F(Z, Z_Y)_f \otimes \hat{\mathcal{L}}$ with the set of elements in $\mathrm{End}_F^{\hat{\mathcal{L}}}(Z)_f$ which maps $(F_f)_*\mathcal{I}_{Z_Y}$ to $(\mathcal{I}_{\hat{Z}_Y} \otimes \hat{\mathcal{L}}) \subset \hat{\mathcal{L}}$.

Using the observations in Section 3.2 we will now give another description of the global sections of $\mathcal{E}\mathrm{nd}_F(Z, Z_Y)_f \otimes \hat{\mathcal{L}}$. Let $i' : X' \to G \times_P X'$ denote the morphism $i'(x) = [1, x]$. Then, as noticed in Section 3.2, the functor i' is exact on the category of G-linearized sheaves. We want to use this fact on the left exact sequence above : notice first that

$$(i')^*\mathcal{E}\mathrm{nd}_F^{\hat{\mathcal{L}}}(Z)_f = \mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}\big((i')^*(F_f)_*\mathcal{O}_Z, (i')^*\hat{\mathcal{L}}\big)$$

where, moreover, $(i')^*\hat{\mathcal{L}} = \mathcal{O}_{X'} \otimes k_\lambda$ and $(i')^*(F_f)_*\mathcal{O}_Z = (F_X)_*\mathcal{O}_X$. Thus

$$(i')^*\mathcal{E}\mathrm{nd}_F^{\hat{\mathcal{L}}}(Z)_f = \mathcal{E}\mathrm{nd}_F(X) \otimes_k k_\lambda.$$

Similarly,

$$(i')^*\mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{I}_{Z_Y}, \hat{\mathcal{L}}_{|\hat{Z}_y}\big) = \mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}((F_X)_*\mathcal{I}_Y, \mathcal{O}_{Y'}) \otimes k_\lambda$$

where $\mathcal{I}_Y$ denotes the sheaf of ideals defining Y in X. In particular, we see that the P-linearized sheaf on X' corresponding to $\mathcal{E}\mathrm{nd}_F(Z, Z_Y)_f \otimes \hat{\mathcal{L}}$ equals the kernel of the natural map

$$\mathcal{E}\mathrm{nd}_F(X) \otimes_k k_\lambda \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}((F_X)_* \mathcal{I}_Y, \mathcal{O}_{Y'}) \otimes k_\lambda,$$

i.e. it equals $\mathcal{E}\mathrm{nd}_F(X, Y) \otimes_k k_\lambda$. By Section 3.2 the global sections of $\mathcal{E}\mathrm{nd}_F(Z, Z_Y)_f \otimes \hat{\mathcal{L}}$ then identities G-equivariantly with

$$\mathrm{Ind}_P^G\big(\mathrm{End}_F(X, Y) \otimes k_\lambda\big).$$

In conclusion we find

Proposition 4.1. *With notation above,*

$$\mathrm{End}_F^{\hat{\mathcal{L}}}(Z)_f = \mathrm{Ind}_P^G\big(\mathrm{End}_F(X) \otimes k_\lambda\big).$$

Moreover, when Y is a closed P-stable subvariety of X then the set of elements in $\mathrm{End}_F^{\hat{\mathcal{L}}}(Z)_f$ *which maps $(F_f)_* \mathcal{I}_{Z_Y}$ to $(\mathcal{I}_{\hat{Z}_Y} \otimes \hat{\mathcal{L}}) \subset \hat{\mathcal{L}}$ coincides with the set*

$$\mathrm{Ind}_P^G\big(\mathrm{End}_F(X, Y) \otimes k_\lambda\big).$$

4.2.2. *A description of* $\mathrm{End}_F^{\hat{\mathcal{L}}}(Z)_b$. As π' in the fiber-diagram (3) is flat the natural morphism $(\pi')^*(F_{G/P})_* \mathcal{L} \to (F_b)_* \hat{\pi}^* \mathcal{L}$ is an isomorphism ([Har2, Prop.III.9.3]). Thus there is a natural isomorphism of G-linearized sheaves

$$\mathcal{E}\mathrm{nd}_F^{\hat{\mathcal{L}}}(Z)_b \simeq (\pi')^* \mathcal{H}\mathrm{om}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_* \mathcal{L}, \mathcal{O}_{(G/P)'}\big).$$

Let V denote a closed subset of ${}^{G}/_{P}$ and let $\mathcal{I}_V \subset \mathcal{O}_{G/P}$ denote the associated sheaf of ideals. Let $\mathcal{K}_V$ denote the kernel of the natural map (which is not G-linearized)

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_* \mathcal{L}, \mathcal{O}_{(G/P)'}\big) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L}), \mathcal{O}_{\mathcal{O}_{V'}}\big),$$

i.e. $\mathcal{K}_V$ is the subsheaf of $\mathcal{H}\mathrm{om}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_* \mathcal{L}, \mathcal{O}_{(G/P)'}\big)$ consisting of elements mapping $(F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L})$ to $\mathcal{I}_{V'}$.

Let $p : G \to {}^{G}/_{P}$ denote the quotient map. Then $\hat{\pi}^{-1}(V)$ identifies with the quotient $p^{-1}(V) \times_P X'$. Moreover, as π' is locally trivial it follows that $\hat{\pi}^*(\mathcal{I}_V) = \mathcal{I}_{p^{-1}(V) \times_P X'}$. In particular,

$$(\pi')^*(F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L}) \simeq (F_b)_* \hat{\pi}^*(\mathcal{I}_V \otimes \mathcal{L}) \simeq (F_b)_*(\mathcal{I}_{p^{-1}(V) \times_P X'} \otimes \hat{\mathcal{L}})$$

and thus the sheaf

$$(\pi')^* \mathcal{H}\mathrm{om}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L}), \mathcal{O}_{\mathcal{O}_{V'}}\big)$$

is isomorphic to

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{Z'}}\big((F_b)_*(\mathcal{I}_{p^{-1}(V) \times_P X'} \otimes \hat{\mathcal{L}}), \mathcal{O}_{(p^{-1}(V) \times_P X)'}\big).$$

As π' is a flat morphism we conclude that $(\pi')^* \mathcal{K}_V$, as a subsheaf of $\mathcal{E}\mathrm{nd}_F^{\hat{\mathcal{L}}}(Z)_b$, consists of the elements which maps $(F_b)_*(\mathcal{I}_{p^{-1}(V) \times_P X'} \otimes \hat{\mathcal{L}})$ to $\mathcal{I}_{(p^{-1}(V) \times_P X)'}$. In conclusion

Proposition 4.2. *There exists a natural G-equivariant morphism*

$$(\pi')^* : \mathrm{Hom}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_* \mathcal{L}, \mathcal{O}_{(G/P)'}\big) \to \mathrm{End}_F^{\hat{\mathcal{L}}}(Z)_b.$$

Moreover, with V as above, an element in $\mathrm{Hom}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*\mathcal{L}, \mathcal{O}_{(G/P)'}\big)$ *which maps* $(F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L})$ *to* $\mathcal{I}_{V'}$ *is by* $(\pi')^*$ *mapped to an element which maps*

$$(F_b)_*(\mathcal{I}_{p^{-1}(V)\times_P X'} \otimes \hat{\mathcal{L}})$$

to $\mathcal{I}_{(p^{-1}(V)\times_P X)'}$.

From now on we will use the notation $\mathrm{End}_F^{\mathcal{L}}({}^{G}/{}_{P})$ to denote the G-module

$$\mathrm{Hom}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*\mathcal{L}, \mathcal{O}_{(G/P)'}\big),$$

while the subspace consisting of of elements which maps $(F_{G/P})_*(\mathcal{I}_V \otimes \mathcal{L})$ to $\mathcal{I}_{V'}$ will be denoted by $\mathrm{End}_F^{\mathcal{L}}({}^{G}/{}_{P}, V)$.

The following is also useful.

Lemma 4.3. *Let Y denote a closed P-stable subvariety of X and fix notation as above. Then each element of* $\mathrm{End}_F^{\hat{\mathcal{L}}}(Z)_b$ *maps* $(F_b)_*(\mathcal{I}_{\hat{Z}_Y} \otimes \hat{\mathcal{L}})$ *to* $\mathcal{I}_{(Z_Y)'}$.

Proof. It suffices to show that the natural morphism

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{Z'}}\big((F_b)_*\hat{\mathcal{L}}, \mathcal{O}_{Z'}\big) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{Z'}}\big((F_b)_*(\mathcal{I}_{\hat{Z}_Y} \otimes \hat{\mathcal{L}}), \mathcal{O}_{(Z_Y)'}\big)$$

is zero. The latter will follow if the natural morphism

$$\mathcal{I}_{(Z_Y)'} \otimes (F_b)_*\hat{\mathcal{L}} \to (F_b)_*(\mathcal{I}_{\hat{Z}_Y} \otimes \hat{\mathcal{L}})$$

is an isomorphism and this can be checked by local calculation. ☐

4.3. **Conclusions.** Consider the P-equivariant morphism

$$\mathrm{ev}_X : \mathrm{End}_F(X) \to \mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{O}_{X'}, \mathcal{O}_{X'}\big) = \mathcal{O}_{X'}(X'),$$

induced by the morphism $F_X^\sharp$. It follows that there is a morphism

$$\mathrm{Ind}_P^G\big(\mathrm{End}_F(X)_c \otimes k_\lambda\big) \to \mathrm{Ind}_P^G\big(k_\lambda\big) = \mathrm{H}^0({}^{G}/{}_{P}, \mathcal{L}),$$

and thus also an induced morphism

$$\Phi_c : \mathrm{End}_F^{\mathcal{L}}({}^{G}/{}_{P}) \otimes_k \mathrm{Ind}_P^G\big(\mathrm{End}_F(X)_c \otimes k_\lambda\big) \to \mathcal{O}_{(G/P)'}\big(({}^{G}/{}_{P})'\big) = k.$$

We can now state our main technical result.

Theorem 4.4. *Let $\mathcal{L}$ denote an equivariant line bundle on ${}^{G}/{}_{P}$ associated to the P-weight λ. Then there exists a G-equivariant map*

$$\Phi : \mathrm{End}_F^{\mathcal{L}}({}^{G}/{}_{P}) \otimes_k \mathrm{Ind}_P^G\big(\mathrm{End}_F(X) \otimes k_\lambda\big) \to \mathrm{End}_F(Z),$$

satisfying

(1) *When Y is a P-stable closed subset of X then the restriction of Φ to the subspace :*

$$\mathrm{End}_F^{\mathcal{L}}({}^{G}/{}_{P}) \otimes_k \mathrm{Ind}_P^G\big(\mathrm{End}_F(X, Y) \otimes k_\lambda\big),$$

maps into $\mathrm{End}_F(Z, Z_Y)$.

(2) *When V denotes a closed subset of ${}^{G}/{}_{P}$ then the restriction of Φ to the subspace*

$$\mathrm{End}_F^{\mathcal{L}}({}^{G}/{}_{P}, V) \otimes_k \mathrm{Ind}_P^G\big(\mathrm{End}_F(X) \otimes k_\lambda\big),$$

maps into $\mathrm{End}_F(Z, p^{-1}(V) \times_P X)$.

(3) *The restriction of* Φ *to*

$$\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P}) \otimes_k \mathrm{Ind}_P^G\big(\mathrm{End}_F(X)_c \otimes k_\lambda\big),$$

maps into $\mathrm{End}_F(Z)_c$*. Moreover, the composition of this restriction with the morphism* ev_Z*, defined similar to* ev_X *above, coincides with* Φ_c*.*

Proof. Set $\Phi = \Phi_{\hat{\mathcal{L}}}(Z') \circ \big((\pi')^* \otimes \mathbf{1}_{\mathrm{Ind}_P^G(\mathrm{End}_F(X)\otimes k_\lambda)}\big)$, where$(\pi')^*$ is defined in Proposition 4.2. The first statement then follows from Proposition 4.1 and Lemma 4.3. The second statement follows from Proposition 4.2 and Lemma 4.5. It remains to prove the third statement. Consider the natural morphism

$$\mathcal{H}\mathrm{om}(F_f^\sharp, *) : \mathcal{E}\mathrm{nd}_F^{\hat{\mathcal{L}}}(Z)_f \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big(\mathcal{O}_{\hat{Z}}, \hat{\mathcal{L}}\big) = \hat{\mathcal{L}}$$

induced by $F_f^\sharp : \mathcal{O}_{\hat{Z}} \to (F_f)_*\mathcal{O}_Z$ defined by F_f. Applying the functor $(i')^*$, as in Section 4.2.1, we obtain a morphism

$$(i')^*\mathcal{H}\mathrm{om}(F_f^\sharp, *) : \mathcal{E}\mathrm{nd}_F(X) \otimes_k k_\lambda \to \mathcal{O}_{X'} \otimes k_\lambda.$$

It follows that $\mathcal{H}\mathrm{om}(F_f^\sharp, *)$ on the level of global section

$$\mathrm{Ind}_P^G\big(\,\mathrm{End}_F(X) \otimes k_\lambda\big) \to \mathrm{Ind}_P^G(\mathcal{O}_{X'}(X') \otimes k_\lambda)$$

is the map induced by ev_X. By this observation and the definition of Φ the result now easily follows. $\square$

Lemma 4.5. *Let* V *denote a closed subset of* ${}^{G}\!/\!{}_{P}$*. Then every element of* $\mathcal{E}\mathrm{nd}_F^{\hat{\mathcal{L}}}(Z)_f$ *will map* $(F_f)_*\mathcal{I}_{\pi^{-1}(V)}$ *to* $\mathcal{I}_{(\pi')^{-1}(V)} \otimes \hat{\mathcal{L}}$*.*

Proof. It suffices to prove that the natural morphism

$$\mathcal{I}_{(\hat{\pi})^{-1}(V)} \otimes (F_f)_*\mathcal{O}_Z \to (F_f)_*\mathcal{I}_{\pi^{-1}(V)},$$

is an isomorphism which can be checked by a local calculation. $\square$

5. Canonical Frobenius splittings

In this section we assume that G is a connected linear algebraic group. By duality for the Frobenius morphism $F_{G/P}$ there is an isomorphism

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*\mathcal{L}, \mathcal{O}_{(G/P)'}\big) \simeq (F_{G/P})_*\big(\omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1}\big),$$

where $\omega_{G/P}$-denotes the dualizing sheaf on ${}^{G}\!/\!{}_{P}$. This way we obtain a G-equivariant identification of $\mathrm{End}_F^{\mathcal{L}}({}^{G}\!/\!{}_{P})$ with the global sections of the line bundle $\omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1}$. Let χ denote the P-character associated to the G-linearized line bundle $\omega_{G/P}^{-1}$. Then multiplication defines a G-equivariant map

$$\mathrm{Ind}_P^G\big((p-1)\chi - \lambda\big) \otimes_k \mathrm{Ind}_P^G\big(\lambda\big) \to \mathrm{Ind}_P^G\big((p-1)\chi\big) \simeq \mathrm{End}_F\,\big({}^{G}\!/\!{}_{P}\big).$$

Actually the above multiplication map is surjective if the domain is nonzero [R-R, Thm.3], i.e. if $\mathcal{L}$ and $\omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1}$ are effective line bundles. Moreover, by construction the multiplication map makes the following diagram commutative

$$\begin{tikzcd}
\mathrm{End}_F^{\mathcal{L}}\big({}^{G}\!/\!{}_{P}\big) \otimes_k \mathrm{Ind}_P^G\big(\mathrm{End}_F(X)_c \otimes k_\lambda\big) \arrow[r, "\Phi"] \arrow[d, "1\otimes \mathrm{Ind}_P^G(\mathrm{ev}_X \otimes 1)"'] & \mathrm{End}_F(Z)_c \arrow[dd, "\mathrm{ev}_Z"] \\
\mathrm{Ind}_P^G\big((p-1)\chi - \lambda\big) \otimes_k \mathrm{Ind}_P^G\big(\lambda\big) \arrow[d] & \\
\mathrm{End}_F\,\big({}^{G}\!/\!{}_{P}\big) \arrow[r, "\mathrm{ev}_{G/P}"'] & k
\end{tikzcd}$$

In particular, the following statement is now an easy consequence.

Corollary 5.1. *Assume that $\mathcal{L}$ and $\omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1}$ are effective line bundles on ${}^{G}\!/\!{}_{P}$ and that the G-equivariant morphism*

$$\mathrm{Ind}_P^G(\mathrm{ev}_X \otimes 1) : \mathrm{Ind}_P^G\big(\mathrm{End}_F(X)_c \otimes k_\lambda\big) \to \mathrm{Ind}_P^G\big(\lambda\big),$$

is surjective. Then $G \times_P X$ admits a Frobenius splitting.

We then define

Definition 5.2. With notation as above, a λ-*canonical Frobenius splitting* of X is a P-equivariant morphism

$$\phi_\lambda : \mathrm{Ind}_P^G(\lambda) \otimes k_{-\lambda} \to \mathrm{End}_F(X)_c,$$

such that the composed map $\mathrm{ev}_{\mathrm{X}} \circ \phi_\lambda$ is nonzero; or equivalently, the image of ϕ_λ contains a Frobenius splitting of X.

Notice that a λ-canonical Frobenius splitting of X defines a composed surjective morphism

$$\mathrm{Ind}_P^G(\lambda) \xrightarrow{\phi_\lambda \otimes k_\lambda} \mathrm{End}_F(X)_c \otimes k_\lambda \xrightarrow{\mathrm{ev}_{\mathrm{X}} \otimes 1} k_\lambda.$$

But the set of P-equivariant morphisms between $\mathrm{Ind}_P^G(\lambda)$ and k_λ is 1-dimensional. In particular, Frobenius reciprocity implies that the induced map

$$\mathrm{Ind}_P^G(\lambda) \xrightarrow{\phi_\lambda \otimes k_\lambda} \mathrm{Ind}_P^G\big(\mathrm{End}_F(X)_c \otimes k_\lambda\big) \xrightarrow{\mathrm{Ind}_P^G(\mathrm{ev}_{\mathrm{X}} \otimes 1)} \mathrm{Ind}_P^G(\lambda),$$

is surjective. Thus also the map $\mathrm{Ind}_P^G(\mathrm{ev}_{\mathrm{X}} \otimes 1)$ is surjective. It follows

Proposition 5.3. *Assume that X admits a λ-canonical Frobenius splitting. Then ϕ_λ and Φ induces a G-equivariant morphism*

$$\Phi_\lambda : \mathrm{Ind}_P^G\big((p-1)\chi - \lambda\big) \otimes_k \mathrm{Ind}_P^G\big(\lambda\big) \to \mathrm{End}_F(Z)_c$$

such that the diagram

$$\begin{tikzcd}
\mathrm{Ind}_P^G\big((p-1)\chi - \lambda\big) \otimes_k \mathrm{Ind}_P^G\big(\lambda\big) \arrow[r, "\Phi_\lambda"] \arrow[d] & \mathrm{End}_F(Z)_c \arrow[d, "\mathrm{ev}_Z"] \\
\mathrm{End}_F\,\big({}^{G}\!/\!{}_{P}\big) \arrow[r, "\mathrm{ev}_{G/P}"'] & k
\end{tikzcd}$$

is commutative. In particular, if $\omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1}$ is an effective line bundle then the image of Φ_λ contains a Frobenius splitting of Z.

The notion of a λ-canonical Frobenius splitting generalizes the existing definition of a B-canonical Frobenius splitting (see the subsection below) . Actually we are only going to apply the notion in the case of a B-canonical Frobenius splittings, but as the general notion seems so natural we have decided also to include it in this paper.

5.1. **B-Canonical Frobenius splitting.** Assume for a moment that G is a connected, semisimple and simply connected linear algebraic group. We will consider the situation described above for the special case where P equals a Borel subgroup B of G. In this case, the dualizing sheaf on ${}^{G}/_{B}$ is the square of the linearized sheaf defined by the B-weight ρ. The induced module $\mathrm{Ind}_B^G((1-p)\rho)$ is called the Steinberg module of G and will be denoted by St. It is well known that St is a simple and selfdual G-module and thus, by Frobenius reciprocity, there exists up to a nonzero constant a unique G-equivariant non-degenerate bilinear form

$$\eta : \mathrm{St} \otimes \mathrm{St} \to k.$$

A $(1-p)\rho$-canonical Frobenius splitting of X is then a B-equivariant map

$$\mathrm{St} \otimes k_{(p-1)\rho} \to \mathrm{End}_F(X)_c,$$

such that the image contains a Frobenius splitting of X. Actually it suffices to assume that that the image of the above map is contained in $\mathrm{End}_F(X)$: as the image contains a Frobenius splitting, which has T-weight 0 and is contained in $\mathrm{End}_F(X)_c$, it follows by the simplicity of St that the image will automatically be contained in $\mathrm{End}_F(X)_c$. In particular, this coincides with the condition for X to admit a B-canonical Frobenius splitting as presented e.g. in [B-K, Lemma 4.1.2]. The importance of B-canonical Frobenius splitting was first observed by O. Mathieu in connection with good filtrations of G-modules. From now on a $(1-p)\rho$-canonical Frobenius splitting in the above setting will be called a B-canonical Frobenius splitting.

Corollary 5.4. *Let $\phi_X : \mathrm{St} \otimes k_{(p-1)\rho} \to \mathrm{End}_F(X)_c$ denote a B-canonical Frobenius splitting of X. Then there is an induced morphism*

$$\Phi_X : \mathrm{St} \otimes \mathrm{St} \to \mathrm{End}_F(G \times_B X)_c$$

such that the composed map $\mathrm{ev}_Z \circ \Phi_X$ coincides with the G-equivariant bilinear map η defined above. Moreover,

1. *If the image of ϕ_X is contained in $\mathrm{End}_F(X, Y)_c$ for a B-stable closed subvariety Y of X, then the image of Φ_X is contained in $\mathrm{End}_F(G \times_B X, G \times_B Y)_c$.*
2. *Let f denote an element of $\mathrm{Ind}_B^G(-\rho)$. Consider f as a global section of the line bundle on ${}^{G}/_{B}$ associated with the B-character $-\rho$, and let V denote the zero scheme of f. Then V is a subvariety of ${}^{G}/_{B}$. Furthermore, for any element $w \in \mathrm{St}$ we have*

$$\Phi_X(f^{p-1} \otimes w) \in \mathrm{End}_F(G \times_B X, p^{-1}(V) \times_B X)$$

with $p : G \to {}^{G}/_{B}$ denoting the quotient map.

Proof. The first part of the statement follows from the considerations above and Theorem 4.4(1). Let $\mathcal{L}$ denote the G-linearized line bundle associated with the weight $(1-p)\rho$. To prove the last part of the statement it suffices, by Theorem 4.4(2), to check that V is a variety and that f^{p-1} as an element of $\mathrm{End}_F^{\mathcal{L}}({}^{G}/_{B})$ is contained in $\mathrm{End}_F^{\mathcal{L}}({}^{G}/_{B}, V)$. Both these statements follow from [L-T, Thm.2.3]. $\square$

The first part of the above result is well known (see e.g. [B-K, Ex. 4.1.E(4)]). However, the second part seems to be new.

Although Corollary 5.4 is only stated for connected, semisimple and simply connected groups it also applies in other cases : assume that G is a connected linear algebraic group containing a connected semisimple subgroup H such that the induced map ${}^{H}/_{H\cap B} \to {}^{G}/_{B}$ is an isomorphism. E.g. this is satisfied for any parabolic subgroup of a reductive connected linear algebraic group. Let $q : H_{\rm sc} \to H$ denote a simply connected cover of H. Then X admits an action of the parabolic subgroup $B_{\rm sc} := q^{-1}(B \cap H)$ of $H_{\rm sc}$. Furthermore, the natural morphism

$$H_{\rm sc} \times_{B_{\rm sc}} X \to G \times_B X,$$

is then an isomorphism. We then say that X admits a B-canonical Frobenius splitting if X, as a $B_{\rm sc}$-module, admits a $B_{\rm sc}$-canonical Frobenius splitting. In this case we may apply Corollary 5.4 to obtain Frobenius splitting properties of $G \times_B X$.

5.2. **Restriction to Levi subgroups.** Return to the setup where G is simply connected and let P_J denote the parabolic subgroup of G containing B and associated to a subset J of the set of simple roots. Let L_J denote the Levi subgroup of P_J containing the maximal torus T and let L_J' denote the commutator subgroup of L_J. Then L_J' is a simply connected semisimple linear algebraic group with Borel subgroup $B_J = L_J' \cap B$ and maximal torus $T_J = T \cap L_J'$. We let ${\rm St}_J$ denote the associated Steinberg module. Notice that ${\rm St}_J = {\rm Ind}_{B_J}^{L_J'}(k_{(p-1)\rho_J})$ where ρ_J denotes the restriction of ρ to T_J. The following should be well known but we do not know a good reference.

Lemma 5.5. *There exists L_J'-equivariant morphism*

$${\rm St}_J \to {\rm St},$$

such that B_J^--invariant line of ${\rm St}_J$ *maps surjectively to the B^--invariant line of* St. *In particular, if X is a G-variety admitting a B-canonical Frobenius splitting then X admits a B_J-canonical Frobenius splitting as a L_J'-variety.*

Proof. Let M denote the T-stable complement to the B-stable line in St. Then M is B^--invariant and thus also B_J^--invariant. Let w_0^J denote the longest element in the Weyl group of T_J and let $\dot{w}_0^J$ denote a representative of this element in ${\rm N}_{L_J'}(T_J)$. Then the translate $\dot{w}_0^J M$ is invariant under B_J. In particular, we obtain a B_J-equivariant morphism

$${\rm St} \to {\rm St}/(\dot{w}_0^J M) \simeq k_{(1-p)\rho_J}.$$

By Frobenius reciprocity this defines a L_J'-equivariant map ${\rm St} \to {\rm St}_J$ such that the B-stable line of St maps onto the B_J-stable line of ${\rm St}_J$. Now apply the selfduality of ${\rm St}_J$ and St to obtain the desired map. This proves the first part of the statement.

The second part follows easily by composing the obtained morphism ${\rm St}_J \to {\rm St}$ with the B-canonical Frobenius splitting

$${\rm St} \to {\rm End}_F(X)_c \otimes k_{(1-p)\rho},$$

of X and noticing that the restriction of ρ to B_J is ρ_J.

□

6. Frobenius D-splittings

In this section we study some D-splittings properties of the Frobenius splittings considered in the previous section. The two results give Frobenius splitting properties relative to divisors along the base scheme ${}^{G}\!/_{P}$ and the fibre scheme X of Z respectively. We use the same notation as in the previous section.

6.1. D-splittings associated to the base.

Proposition 6.1. *Let v denote a nonzero global section of the line bundle $\omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1}$ and let D denote the effective Cartier divisor associated to the pull back $\pi^*(v)$ to Z. Let σ denote the canonical section of $\mathcal{O}_Z(D)$. Then for any w in $\mathrm{Ind}_P^G\big(\mathrm{End}_F(X) \otimes k_\lambda\big)$ the element $\Phi(v \otimes w)$ will factor through the morphism*

$$(F_Z)_*\mathcal{O}_Z \xrightarrow{(F_Z)_*\sigma} (F_Z)_*\mathcal{O}_Z(D),$$

induced by σ. In particular, any Frobenius splitting of Z of the form $\Phi(v \otimes w)$ is a D-splitting.

Proof. By the discussion in Section 5 we may consider v as an element in

$$\mathrm{Hom}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*\mathcal{L}, \mathcal{O}_{(G/P)'}\big).$$

Actually v then identifies (up to a nonzero constant) with the composed map

$$(F_{G/P})_*\mathcal{L} \xrightarrow{(F_{G/P})_*(v)} (F_{G/P})_*\omega_{G/P}^{1-p} \to \mathcal{O}_{(G/P)'},$$

where the second map denotes a nonzero element in

$$\mathrm{Hom}_{\mathcal{O}_{(G/P)'}}\big((F_{G/P})_*\omega_{G/P}^{1-p}, \mathcal{O}_{(G/P)'}\big) \simeq k.$$

In particular, the element $(\pi')^*(v) = \mathrm{Hom}_{\mathcal{O}_{Z'}}\big((F_b)_*\hat{\mathcal{L}}, \mathcal{O}_{Z'}\big)$ factors through the map

$$(F_b)_*\hat{\mathcal{L}} \xrightarrow{(F_b)_*\hat{\pi}^*(v)} (F_b)_*\big(\hat{\pi}^*(\omega_{G/P}^{1-p})\big).$$

Regard w as an element of $\mathrm{Hom}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{O}_Z, \hat{\mathcal{L}}\big)$ and notice that we have a commutative diagram

$$\begin{tikzcd}
(F_Z)_*\mathcal{O}_Z \arrow[r, "(F_b)_*w"] \arrow[d, "(F_f)_*\sigma"'] & (F_b)_*\hat{\mathcal{L}} \arrow[d, "(F_b)_*\hat{\pi}^*(v)"] \\
(F_Z)_*\mathcal{O}_Z(D) \arrow[r] & (F_b)_*\big(\hat{\pi}^*(\omega_{G/P}^{1-p})\big)
\end{tikzcd}$$

where the lower horizontal map is the tensor product of $(F_b)_*w$ with $(\pi')^*(\omega_{G/P}^{1-p} \otimes \mathcal{L}^{-1})$. Now the statement follows by the definition of Φ. $\square$

6.2. D-splitting associated to the fibre.

Let $\mathcal{M}$ denote a P-linearized line bundle on X. Assume that there is a morphism of P-linearized sheaves

$$\psi : \mathcal{M} \to \mathcal{E}\mathrm{nd}_F^!(X, Y) \otimes k_\lambda.$$

Inducing from P to G this defines a morphism

$$\psi_Z : \mathcal{I}\mathrm{nd}_P^G(\mathcal{M}) \to (F_f)^!\hat{\mathcal{L}},$$

of sheaves on Z. Here $(F_f)^!\hat{\mathcal{L}}$ is defined by duality of the finite morphism F_f such that the relation

$$(F_f)_*(F_f)^!\hat{\mathcal{L}} = \mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{O}_Z, \hat{\mathcal{L}}\big)$$

as $\mathcal{O}_{\hat{Z}}$-modules is satisfied.

As in Section 2.5 the morphism ψ_Z defines a global section s of

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{\hat{Z}}}\big((F_f)_*\mathcal{I}\mathrm{nd}_P^G(\mathcal{M}), \hat{\mathcal{L}}\big).$$

The compatibility of ψ with Y then implies

$$s\big((F_f)_*\big(\mathcal{I}_{Z_Y} \otimes \mathcal{I}\mathrm{nd}_P^G(\mathcal{M})\big)\big) \subset \mathcal{I}_{\hat{Z}_Y} \otimes \hat{\mathcal{L}}.$$

Moreover, let σ denote a global section of the line bundle $\mathcal{I}\mathrm{nd}_P^G(\mathcal{M})$. Then the image $\psi_Z(\sigma)$ will factor as

$$(F_f)_*\mathcal{O}_Z \xrightarrow{(F_f)_*\sigma} (F_f)_*\mathcal{I}\mathrm{nd}_P^G(\mathcal{M}) \xrightarrow{s} \hat{\mathcal{L}}.$$

It follows

Lemma 6.2. *With notation as above let v be an element in $\mathrm{End}_F^{\mathcal{L}}({}^G/_P)$. Then $\Phi\big(v \otimes \psi_Z(\sigma)\big) \in \mathrm{End}_F(Z, Z_Y)$ factors as*

$$(F_Z)_*\mathcal{O}_{Z_Y} \xrightarrow{(F_Z)_*\sigma} (F_Z)_*\mathcal{I}\mathrm{nd}_P^G(\mathcal{M}) \xrightarrow{(F_b)_*s} (F_b)_*\hat{\mathcal{L}} \xrightarrow{(\pi)'^*v} \mathcal{O}_{Z'},$$

where the composition $(\pi)'^*v \circ (F_b)_*s$ is compatible with Z_Y.

Proof. Apply Lemma 4.3 and the remarks above. □

In case X is a G-variety we may identify Z with ${}^G/_P \times X$. Under this identification $\mathcal{I}\mathrm{nd}_P^G(\mathcal{M})$ corresponds to the pull back $(p_2)^*\mathcal{M}$ of $\mathcal{M}$ by projection on the second factor. In particular, the global sections of $\mathcal{M}$ and $\mathcal{I}\mathrm{nd}_P^G(\mathcal{M})$ coincide. Thus, for $v \in \mathrm{End}_F^{\mathcal{L}}({}^G/_P)$, we obtain a map

$$\eta_v : \mathcal{M}(X) \to \mathrm{End}_F(Z, Z_Y).$$

Moreover, any Frobenius splitting $\eta_v(\sigma)$ in the image of η_v will factor through $(F_{Z_Y})_*\mathcal{M}_Z$ where $\mathcal{M}_Z$ denotes the pull back of $\mathcal{M}$ to Z by the morphism $Z \to X$ defined by $[g, x] \mapsto g \cdot x$.

7. Applications to $G \times G$-varieties

In this section we assume that G is a connected linear algebraic group containing a connected semisimple subgroup H such that ${}^H/_{H\cap B} \to {}^G/_B$ is an isomorphism. We define H_{sc} and B_{sc} as in the end of Section 5.1. We will need the following well known result (see e.g. [B-K, Thm.2.3.8 including proof])

Lemma 7.1. *Assume that $G = H_{\mathrm{sc}}$ and apply the notation of Section 5.1. Let $\mathcal{M}$ denote the line bundle on ${}^G/_B \times {}^G/_B$ associated with the weight $(1-p)\rho \boxtimes (1-p)\rho$. Let f denote a nonzero $\mathrm{diag}(G)$-invariant element of $\mathrm{St} \boxtimes \mathrm{St}$. Then the image s of f under the identification*

$$\mathrm{St} \boxtimes \mathrm{St} \simeq \mathrm{Hom}_{\mathcal{O}_{({}^G/_B \times {}^G/_B)'}}\big((F_{{}^G/_B \times {}^G/_B})_*\mathcal{M}, \mathcal{O}_{({}^G/_B \times {}^G/_B)'}\big)$$

is compatible with the diagonal $\mathrm{diag}({}^G/_B)$ in ${}^G/_B \times {}^G/_B$.

We want to apply the results of the preceding sections to the case when the group equals $G \times G$. So let X denote a $B \times B$-variety and assume that X admits a $B_{\rm sc} \times B_{\rm sc}$-canonical Frobenius splitting defined by

$$\phi : (\mathrm{St} \boxtimes \mathrm{St}) \otimes (k_{(p-1)\rho} \boxtimes k_{(p-1)\rho}) \to \mathrm{End}_F(X)_c$$

which compatibly splits certain $B \times B$-stable subvarieties $X_1, \dots, X_m$, i.e. the image of ϕ is contained in $\mathrm{End}_F(X, X_i)$ for all i. Then

Theorem 7.2. *With assumptions as above, the variety $(G \times G) \times_{B \times B} X$ admits a Frobenius splitting which compatibly splits the subvarieties* $\mathrm{diag(G)} \times_{\mathrm{diag}(B)} X$ *and* $(G \times G) \times_{(B \times B)} X_i$ *for all i.*

Proof. It suffices to consider the case where $G = H_{\rm sc}$. By Corollary 5.4 there exists a $G \times G$-equivariant morphism

$$\Phi' : (\mathrm{St} \boxtimes \mathrm{St}) \otimes (\mathrm{St} \boxtimes \mathrm{St}) \to \mathrm{End}_F((G \times G) \times_{B \times B} X),$$

satisfying certain compatibility conditions. Let $f \in \mathrm{St} \boxtimes \mathrm{St}$ denote the element of Lemma 7.1. Let $v \in \mathrm{St} \boxtimes \mathrm{St}$ denote any element such that $\Phi'\big(f \otimes v\big)$ is a Frobenius splitting of $(G \times G) \times_{B \times B} X$. Then by construction of Φ' and Theorem 4.4 the element $\Phi'\big(f \otimes v\big)$ has the desired properties. $\square$

As f, in the proof of the above result, is $\mathrm{diag}(G)$-invariant the map

$$\Phi_\Delta : \mathrm{St} \otimes \mathrm{St} \to \mathrm{End}_F(\mathrm{diag}(G) \times_{\mathrm{diag}(B)} X),$$

given by $\Phi_\Delta(w) = \Phi'(f \otimes w)$, defines a $\mathrm{diag}(B)$-canonical Frobenius splitting of $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} X$. By the general machinery of canonical Frobenius splittings this would also be true if X had a $\mathrm{diag}(B)$-canonical Frobenius splitting (see e.g. [B-K, Prop.4.1.7]). However in the present setup X only admits a $B \times B$-canonical Frobenius splitting which is less restrictive than a $\mathrm{diag}(B)$-canonical Frobenius splitting. Notice however that, in contrast to the situation when X admits a $\mathrm{diag}(B)$-canonical Frobenius splitting , we do not obtain compatibly splitting of subvarieties of the form $p^{-1}(Y) \times_B X$ with Y denoting a Schubert variety of ${}^{G}\!/_{B}$.

8. G-Schubert varieties in equivariant Embeddings

From now on, unless otherwise stated, we assume that G is a connected reductive group. We fix a Borel subgroup B and a maximal torus $T \subset B$. The set of simple roots determined by (B, T) will be denoted by Δ. The Weyl group $W = N_G(T)/T$ is then generated by the simple reflections s_i for $i \in \Delta$. The length of $w \in W$ will be denoted by $l(w)$. For $J \subset \Delta$, let W_J denote the subgroup of W generated by J and W^J (resp. ${}^J W$) denote the set of minimal length coset representatives for W/W_J (resp. $W_J \backslash W$). The unique maximal element in W will be denoted by w_0 and the unique maximal element in W_J will be denoted by w_0^J. For any $w \in W$, let $\dot{w}$ denote a representative of w in $N_G(T)$.

For $J \subset \Delta$, let $P_J \supset B$ denote the corresponding standard parabolic subgroup and $P_J^- \supset B^-$ its opposite. Let $L_J = P_J \cap P_J^-$ be the common Levi subgroup of P_J and P_J^-. Let U (resp. U^-) denote the unipotent radical of B (resp. B^-).

Consider G as a $G \times G$-variety by left and right translation. An equivariant embedding X of G is then a normal irreducible $G \times G$-variety containing an open subset which is $G \times G$-equivariantly isomorphic to G. In particular, we may identify G with an open subset of X. Any equivariant embedding of G is a spherical variety (with respect to the induced $B \times B$-action) and thus X contains finitely may $B \times B$-orbits.

A subvariety of the form $\mathrm{diag}(G) \cdot V$ for some $B \times B$-orbit closure V is called a *G-Schubert variety.* Notice that $\mathrm{diag}(G) \cdot V$ is the image of $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} V$ under the proper map $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} X \to X$. Thus the G-Schubert varieties are closed subvarieties of X.

In the rest of this section, we will define the G-stable pieces in a toridal embedding and show that the G-Schubert varieties are actually the closures of the G-stable pieces. However, we don't know if there is a good notion for G-stable pieces for general equivariant embeddings.

8.1. **G-stable pieces in the wonderful compactification.** In this subsection, we assume furthermore that G is connected semisimple group of adjoint type and $\mathbf{X}$ is the associated wonderful compactification. The boundary $\mathbf{X} \setminus G$ is a union of irreducible divisors $\mathbf{X}_j$, $j \in \Delta$ and they intersect transversally. For a subset $J \subset \Delta$, we denote the intersection $\cap_{j \in J} \mathbf{X}_j$ by $\mathbf{X}_J$. As a $(G \times G)$-variety, $\mathbf{X}_J$ is isomorphic to the variety $(G \times G) \times_{P^-_{\Delta \setminus J} \times P_{\Delta \setminus J}} Y$, where Y denotes the wonderful compactification of the adjoint group of $L_{\Delta \setminus J}$. Here the $P^-_{\Delta \setminus J} \times P_{\Delta \setminus J}$-action on Y is defined by the quotient maps $P_{\Delta \setminus J} \to L_{\Delta \setminus J}$ and $P^-_{\Delta \setminus J} \to L_{\Delta \setminus J}$. Let $\mathbf{h}_J \in \mathbf{X}_J$ denote the image of $(1,1,1) \in (G \times G) \times Y$ under this isomorphism.

For $J \subset \Delta$ and $w \in W^{\Delta \setminus J}$, we let

$$\mathbf{X}_{J,w} = \mathrm{diag}(G)(Bw, 1) \cdot \mathbf{h}_J$$

and call $\mathbf{X}_{J,w}$ a G-stable piece of $\mathbf{X}$. By [L, section 12] and [He, section 2],

$$\mathbf{X} = \bigsqcup_{\substack{J \subset \Delta \\ w \in W^{\Delta \setminus J}}} \mathbf{X}_{J,w}.$$

Moreover, by the proof of [He2, Theorem 4.5], for any $B \times B$-orbit closure V in $\mathbf{X}$, the G-Schubert variety $\mathrm{diag}(G) \cdot V$ is the closure of some G-stable piece and is a finite union of G-stable pieces.

8.2. **G-stable pieces in a toroidal embedding.** Let G_{ad} denote the semisimple group of adjoint type associated to G and let $\mathbf{X}$ denote the wonderful compactification of G_{ad}. The equivariant embedding X is said to be toroidal if the natural map $G \to G_{\mathrm{ad}}$ extends to a morphism $\pi : X \to \mathbf{X}$.

We fix a toroidal embedding X of G. The irreducible components of the boundary $X - G$ will be denoted by $X_1, \dots, X_n$. As G is an affine variety these boundary component all have codimension 1 in X [Har, Prop.3.1]. For each $G \times G$-orbit closure Y in X we then associate the set

$$K_Y = \{i \in \{1, \dots, n\} \mid Y \subset X_i\},$$

where by definition $K_Y = \varnothing$ when $Y = X$. Then by [B-K, Prop.6.2.3], $Y = \cap_{i \in K_Y} X_i$. Moreover, we define

$$\mathcal{I} = \{K_Y \subset \{1, \dots, n\} \mid Y \text{ a } G \times G\text{-orbit closure in } X \},$$

and write $X_K := \cap_{i \in K} X_i$ for $K \in \mathcal{I}$. Then $(X_K)_{K \in \mathcal{I}}$ are the closures of $G \times G$-orbits in X. Then we may define a map $p : \mathcal{I} \to \mathcal{P}(\Delta)$ such that $\pi(X_K) = \mathbf{X}_{p(K)}$. Here $\mathcal{P}(\Delta)$ denotes the set of subsets of Δ.

As in [H-T2, 5.4], for $K \in \mathfrak{I}$ we may choose a base point h_K in the open $G \times G$-orbit of X_K which maps to $\mathbf{h}_{p(K)}$. By [H-T2, Proposition 5.3], X_K is naturally isomorphic to $(G \times G) \times_{P^-_{\Delta \setminus J} \times P_{\Delta \setminus J}} \overline{L_{\Delta \setminus J} \cdot h_K}$, where $J = p(K)$ and $\overline{L_{\Delta \setminus J} \cdot h_K}$ is a toroidal embedding of a quotient $L_{\Delta \setminus J}/H_{\Delta \setminus J}$ by some subgroup $H_{\Delta \setminus J}$ of the center of $L_{\Delta \setminus J}$.

For $K \in \mathfrak{I}$ and $w \in W^{\Delta \setminus p(K)}$, we let

$$X_{K,w} = \operatorname{diag}(G)(Bw, 1) \cdot h_K$$

and call $\mathbf{X}_{J,w}$ a G-stable piece of X. One can show in the same way as in [He2, 4.3] that

$$X = \bigsqcup_{\substack{K \in \mathfrak{I} \\ w \in W^{\Delta \setminus p(K)}}} X_{K,w}.$$

Also similar to the proof of [He2, Theorem 4.5], for any $B \times B$-orbit closure V in X, the G-Schubert variety $\operatorname{diag}(G) \cdot V$ is the closure of some G-stable piece and is a finite union of G-stable pieces.

9. Frobenius splitting of G-Schubert varieties

In this section, we assume that X is an equivariant embedding of G. Let G_{sc} denote a simply connected cover of the semisimple commutator subgroup (G, G) of G. We also fix a Borel subgroup B of G and a compatible Borel subgroup B_{sc} of G_{sc}. Similarly we fix maximal tori $T \subset B$ and $T_{\mathrm{sc}} \subset B_{\mathrm{sc}}$. By a canonical Frobenius splitting of the G-variety X we mean canonical with respect to the induced G_{sc}-action.

Let $X_1, \dots, X_n$ denote the boundary divisors. The closure within X of the $B \times B$-orbit $Bs_j w_0 B \subset G$ will be denoted by D_j. Then D_j is also of codimension 1 in X. The translate $(w_0, w_0)D_j$ of D_j will be denoted by $\tilde{D}_j$.

By earlier work we know

Theorem 9.1. [H-T2, Prop.7.1] *The equivariant embedding X admits a $B \times B$-canonical Frobenius splitting which compatibly Frobenius splits the closure of every $B \times B$-orbit closure.*

As a direct consequence of Theorem 7.2 we then obtain

Corollary 9.2. *Let X denote an equivariant embedding of G. Then the variety $(G \times G) \times_{B \times B} X$ admits a Frobenius splitting which compatibly Frobenius splits the subvarieties $(G \times G) \times_{B \times B} Y$ and* $\operatorname{diag}(G) \times_{\operatorname{diag}(B)} Y$ *for every $B \times B$-orbit closure Y in X. Moreover, $(G \times G) \times_{B \times B} Y$ admits a $B \times B$-canonical Frobenius splitting while* $\operatorname{diag}(G) \times_{\operatorname{diag}(B)} Y$ *admits a* $\operatorname{diag}(B)$*-canonical Frobenius splitting.*

Proposition 9.3. *The equivariant embedding X of G admits a* $\operatorname{diag}(B)$*-canonical Frobenius splitting which compatibly splits all the G-Schubert varieties.*

Proof. By Corollary 9.2 the variety $Z = \operatorname{diag}(G) \times_{\operatorname{diag}(B)} X$ admits a B-canonical Frobenius splitting which compatibly Frobenius splits all subvarieties of the form $\operatorname{diag}(G) \times_{\operatorname{diag}(B)} Y$ with Y denoting a $B \times B$-orbit closure in X. As X is a $\operatorname{diag}(G)$-stable we may identify Z with ${}^{G}/_{B} \times X$ via the isomorphism

$$\operatorname{diag}(G) \times_{\operatorname{diag}(B)} X \to {}^{G}/_{B} \times X,$$
$$[g, x] \mapsto (gB, g \cdot x).$$

In particular, we see that the morphism

$$\pi : Z = \mathrm{diag}(G) \times_{\mathrm{diag}(B)} X \to X,$$

$$[g, x] \mapsto g \cdot x,$$

is projective and that $\pi_*(\mathcal{O}_Z) = \mathcal{O}_X$. As a consequence (see Section 2.6) the Frobenius splitting of Z induces a Frobenius splitting of X which compatibly splits all subsets of the form

$$\pi(\mathrm{diag}(G) \times_{\mathrm{diag}(B)} Y) = \mathrm{diag}(G) \cdot Y.$$

This ends the proof. □

As a direct consequence, we conclude the following vanishing result (see [B-K, Theorem 1.2.8]).

Corollary 9.4. *Let X denote a projective equivariant embedding of G. Let $\mathcal{X}$ denote a G-Schubert variety and let $\mathcal{L}$ denote an ample line bundle on $\mathcal{X}$. Then*

$$\mathrm{H}^i(\mathcal{X}, \mathcal{L}) = 0, i > 0.$$

Moreover, if $\mathcal{X}' \subset \mathcal{X}$ is another G-Schubert variety, then the restriction map

$$\mathrm{H}^0(\mathcal{X}, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}', \mathcal{L})$$

is surjective.

9.1. **F-splittings along ample divisors.** In this subsection we assume that X is toroidal. The following structural properties of toroidal embeddings can all be found in [B-K, Sect.6.2]. Let X_0 denote the complement in X of the union of the subsets $\overline{Bs_iB^-}$ for $i \in \Delta$. Let X' denote the closure of T in G and let $X'_0 = X' \cap X_0$. Then X'_0 is a T-stable subset of X and the morphism

$$U \times U^- \times X'_0 \to X_0,$$

$$(x, y, z) \mapsto (x, y) \cdot z,$$

is an isomorphism. Moreover, every $G \times G$-orbit in X intersects X'_0 in a unique T-orbit. Consequently this intersection is isomorphic to a product of copies of k^*.

Lemma 9.5. *Let X denote a projective toroidal equivariant embedding of G and let $Y = X_K$, $K \in \mathcal{I}$, denote a $G \times G$-orbit closure in X. Then*

$$Y \cap \Big(\bigcup_{j \notin K} X_j \cup \bigcup_{i \in \Delta} (1, w_0) D_i \Big)$$

has pure codimension 1 in Y and contains the support of an ample effective Cartier divisor on Y.

Proof. Let $X^K = \cup_{j \notin K} X_j$. We claim that $Y \setminus X^K$ coincides with the open $G \times G$-orbit Y_0 of Y. Clearly Y_0 is contained in $Y \setminus X^K$. On the other hand, let U be a $G \times G$-orbit in $Y \setminus X^K$. Then X_j contains U if and only if $j \notin K$. But every $G \times G$-orbit closure is the intersection of the X_j's which contain it [B-K, Prop.6.2.3]. It follows that the closure of Y and U coincide and thus $U = Y$.

As X is normal we may choose a $G \times G$-linearized very ample line bundle $\mathcal{L}$ on X. As $\mathrm{H}^0(Y, \mathcal{L})$ is finite dimensional it contains an element v which is $B \times B^-$-invariant up to constants. The support of v is then the union of $B \times B^-$-invariant divisors on

Y. As $Y_0 \cap X_0'$ is a single $T \times T$-orbit it follows that $Y_0 \cap X_0 = U \times U^- \times (X_0' \cap Y_0)$ is a single $B \times B^-$-orbit. In particular, the support of v is contained in

$$Y \setminus (Y_0 \cap X_0) = Y \cap (X^K \cup \bigcup_{i \in \Delta} (1, w_0) D_i).$$

This shows the second part of the statement. The first part follows as $Y_0 \cap X_0$ is affine [Har, Prop.3.1]. □

Let now X denote a projective smooth toroidal embedding of G. In this case there exists an isomorphism [B-K, Prop.6.2.6]

$$\omega_X^{-1} \simeq \mathcal{O}_X\Big(\sum_{i \in \Delta} (D_i + \tilde{D}_i) + \sum_{j=1}^{n} X_j\Big).$$

Let τ_i denote the canonical section of the line bundles $\mathcal{O}_X(D_i)$. Then τ_i is a $B_{\mathrm{sc}} \times B_{\mathrm{sc}}$-eigenvector with weight $\omega_i \boxtimes -w_0\omega_i$, where ω_i denotes the fundamental weight associated to the i-th simple root. This follows from the corresponding statement for $\mathbf{X}$ in [B-K, Prop.6.1.11] as the pull back of $\mathcal{O}_{\mathbf{X}}(\mathbf{D}_i)$ to X is isomorphic to $\mathcal{O}_X(D_i)$.

Let V denote a $B \times B$-orbit closure in X and let K denote the set of elements j in $\{1, \dots, n\}$ such that X_j contains V. Then $Y = \cap_{j \in K} X_j$ is the smallest $G \times G$-stable closed subset of X which contains V. Let $\rho = \sum_{i=1}^{l} \omega_i$. As ω_X^{1-p} is isomorphic to $\mathcal{E}\mathrm{nd}_F^!(X)$ it follows by Lemma 2.6 that we have a morphism of $B_{\mathrm{sc}} \times B_{\mathrm{sc}}$-linearized sheaves

$$\mathcal{M} \to \mathcal{E}\mathrm{nd}_F^! \left(X, \{D_i, X_j\}_{i \in \Delta, j \in K}\right) \otimes k_{(1-p)\rho \boxtimes (1-p)\rho},$$

where $\mathcal{M}$ denotes the line bundle $\mathcal{O}_X\big((p-1)(\sum_{i \in \Delta} \tilde{D}_i + \sum_{j \notin K} X_j)\big)$.

By [H-T2, Prop.6.5] and Lemma 2.1 any element in $\mathcal{E}\mathrm{nd}_F^!(X)$ which is compatibly with the closed subvarieties D_i, $i \in D$, and X_j, $j \in K$, is also compatibly with V and Y. In particular, we find

$$s : \mathcal{M} \to \mathcal{E}\mathrm{nd}_F^!(X, Y, V\big) \otimes k_{(1-p)\rho \boxtimes (1-p)\rho}.$$

Let σ_j denote the canonical section of $\mathcal{O}_X(X_j)$ and consider the global section

$$\sigma = \prod_{i \in \Delta} \tau_i^{p-1} \prod_{j \notin K} \sigma_j^{p-1}$$

of $\mathcal{M}$ (here we use that $\mathcal{O}_X(D_i)$ is isomorphic to $\mathcal{O}_X(\tilde{D}_i)$). Then σ is a $B \times B$-eigenvector of weight $(p-1)\rho \boxtimes (p-1)\rho$. By Frobenius reciprocity and the selfduality of the Steinberg module St associated to G_{sc}, it follows that we have a $B_{\mathrm{sc}} \times B_{\mathrm{sc}}$-equivariant morphism

$$\mathcal{O}_X \otimes (\mathrm{St} \boxtimes \mathrm{St}) \to \mathcal{M} \xrightarrow{s} \mathcal{E}\mathrm{nd}_F^!(X, Y, V) \otimes k_{(1-p)\rho \boxtimes (1-p)\rho}.$$

Moreover, taking global section the induced map

$$\mathrm{St} \boxtimes \mathrm{St} \to \mathrm{End}_F(X, Y, V) \otimes k_{(1-p)\rho \boxtimes (1-p)\rho},$$

defines a canonical Frobenius splitting of X. This follows as the section

$$\Big(\prod_{i \in \Delta} (\tau_i \tilde{\tau}_i) \prod_{j=1}^{n} \sigma_j\Big)^{p-1},$$

with $\tilde{\tau}_i$ denoting the canonical section of $\mathcal{O}_X(\tilde{D}_i)$, defines a Frobenius splitting of X (see e.g. [B-K, proof of Thm.6.2.7]).

Proposition 9.6. *Fix notation as above and let D denote the effective Cartier divisor*

$$(p-1)\Big(\sum_{i\in\Delta}(1,w_0)D_i+\sum_{j\notin K}X_j\Big)$$

on X. Then X admits a Frobenius D-splitting which compatibly Frobenius splits the subvarieties Y and $\mathrm{diag}(G)\cdot V$.

Proof. Above we have constructed a $G_{\mathrm{sc}}\times G_{\mathrm{sc}}$-equivariant map

$$\mathrm{St}\boxtimes\mathrm{St}\to\mathrm{Ind}_{B_{\mathrm{sc}}}^{G_{\mathrm{sc}}}\big(\mathrm{End}_F(X,Y,V)\otimes k_{(1-p)\rho\boxtimes(1-p)\rho}\big).$$

Let $\mathcal{L}$ denote the line bundle on ${G_{\mathrm{sc}}}/{B_{\mathrm{sc}}}\times{G_{\mathrm{sc}}}/{B_{\mathrm{sc}}}$ associated to the weight $(1-p)\rho\boxtimes(1-p)\rho$. Applying the map Φ from Theorem 4.4 we find

$$\Phi':\mathrm{End}_F^{\mathcal{L}}\left({G_{\mathrm{sc}}}/{B_{\mathrm{sc}}}\times{G_{\mathrm{sc}}}/{B_{\mathrm{sc}}}\right)\otimes(\mathrm{St}\boxtimes\mathrm{St})\to\mathrm{End}_F\big(Z,Z_Y,Z_V\big),$$

where $Z=(G_{\mathrm{sc}}\times G_{\mathrm{sc}})\times_{(B_{\mathrm{sc}}\times B_{\mathrm{sc}})}X$, $Z_Y=(G_{\mathrm{sc}}\times G_{\mathrm{sc}})\times_{(B_{\mathrm{sc}}\times B_{\mathrm{sc}})}Y$ and $Z_V=(G_{\mathrm{sc}}\times G_{\mathrm{sc}})\times_{(B_{\mathrm{sc}}\times B_{\mathrm{sc}})}V$. Let

$$v\in\mathrm{St}\boxtimes\mathrm{St}\simeq\mathrm{End}_F^{\mathcal{L}}\left({G_{\mathrm{sc}}}/{B_{\mathrm{sc}}}\times{G_{\mathrm{sc}}}/{B_{\mathrm{sc}}}\right),$$

denote a nonzero $\mathrm{diag}(G)$-invariant element and let

$$w=v_+\boxtimes v_-\in\mathrm{St}\boxtimes\mathrm{St}$$

where v_+ (resp. v_-) denotes a nonzero B (resp. B^-) -eigenvector of St such that $\Phi'(v\boxtimes w)$ defines a Frobenius splitting.

Let $\mathcal{M}_Z$ denote the pull back of $\mathcal{M}$ to Z by the map

$$\eta:(G_{\mathrm{sc}}\times G_{\mathrm{sc}})\times_{(B_{\mathrm{sc}}\times B_{\mathrm{sc}})}X\to X$$

$$[(g,h),x]\mapsto(g,h)\cdot x.$$

Let σ denote the global section of $\mathcal{M}_Z$ defined as the pull back of the image σ_w of w under the morphism $\mathrm{St}\boxtimes\mathrm{St}\to\mathcal{M}(X)$ defined above. Then by Lemma 6.2 the Frobenius splitting $\Phi'(v\boxtimes w)$ will factor as

$$(F_Z)_*\mathcal{O}_Z\xrightarrow{(F_Z)_*\sigma}(F_Z)_*\mathcal{M}_Z\xrightarrow{(F_b)_*s}(F_b)_*\hat{\mathcal{L}}\xrightarrow{(\pi')^*v}\mathcal{O}_{Z'}.$$

where s is some map $(F_f)_*\mathcal{M}_Z\to\hat{\mathcal{L}}$. Moreover

$$t=(\pi')^*v\circ(F_b)_*s:(F_Z)_*\mathcal{M}_Z\to\mathcal{O}_{Z'}$$

is compatible with Z_Y and Z_V, and by Proposition 4.2, Lemma 4.5 and Lemma 7.1, also with $Z_\Delta=\mathrm{diag}(G)\times_{\mathrm{diag}(B)}X$. In particular, t is also compatible with the intersection $Z_{V,\Delta}=\mathrm{diag}(G)\times_{\mathrm{diag}(B)}V$ of Z_Δ and Z_V.

Notice that the natural morphism $\eta^\sharp:\mathcal{O}_X\to\eta_*\mathcal{O}_Z$ is an isomorphism. Thus, by Lemma 2.7 the push forward

$$\eta_*t:(F_X)_*\mathcal{M}\to\mathcal{O}_{X'}$$

is compatible with $Y=\eta(Z_Y)$ and $\mathrm{diag}(G)\cdot V=\eta(Z_{V,\Delta})$. Moreover, the composition of η_*t with

$$\eta_*(F_Z)_*\sigma=(F_X)_*\sigma_w:(F_X)_*\mathcal{O}_X\to(F_X)_*\mathcal{M},$$

is, by construction, a Frobenius splitting of X. It follows that X admits a Frobenius D-splitting which compatibly splits $Y=\eta(Z_Y)$ and $\mathrm{diag}(G)\cdot V=\eta(Z_{V,\Delta})$, where D

is the effective Cartier divisor associated to σ_w. But by the remarks preceding this proposition it follows that D, by construction, equals

$$(p-1)\Big(\sum_{j\notin K} X_j + \sum_{i\in\Delta}(1,w_0)D_i\Big).$$

This ends the proof. $\square$

Definition 9.7. A morphism $f : X \to Y$ is a called a *rational morphism* if the induced map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism and $\mathrm{R}^i f_*\mathcal{O}_X = 0$, $i > 0$.

The following criteria for a morphism to be rational will be very useful ([R, Lem.2.11]).

Lemma 9.8. *Let $f : X \to Y$ denote a projective morphism of irreducible varieties and let $\hat{X}$ denote a closed irreducible subvariety of X. Consider the image $\hat{Y} = f(\hat{X})$ as a closed subvariety of Y. Let $\mathcal{L}$ denote an ample line bundle on Y and assume*

(1) $f_*\mathcal{O}_X = \mathcal{O}_Y$.
(2) $\mathrm{H}^i(X, f^*\mathcal{L}^n) = \mathrm{H}^i(\hat{X}, f^*\mathcal{L}^n) = 0$, *for $i > 0$ and $n \gg 0$.*
(3) *The restriction map $\mathrm{H}^0(X, f^*\mathcal{L}^n) \to \mathrm{H}^0(\hat{X}, f^*\mathcal{L}^n)$ is surjective for $n \gg 0$.*

Then the induced map $f' : \hat{X} \to \hat{Y}$ is a rational morphism.

Lemma 9.9. *Let X denote a projective embedding of a reductive group G and let Y denote a $G\times G$-orbit closure of X. Then there exists a smooth toroidal embedding $\hat{X}$ of G and a projective morphism $f : \hat{X} \to X$ extending the identity map on G. Moreover, we may also assume that $\hat{X}$ contains a $G\times G$-orbit closure $\hat{Y}$ with $f(\hat{Y}) = Y$ and such that the induced morphism $f : \hat{Y} \to Y$ is a rational morphism.*

Proof. Assume first that X is toroidal. Remember that the closure of T in any toroidal embedding of G is a a toric variety and that this defines (see [B-K, Sect.6.2]) a correspondence between certain toric varieties and the set of toroidal equivariant embeddings of G. In particular, if we let $\bar{T}$ denote the closure of T in X, then there exists a toroidal embedding $\hat{X}$ whose associated toric variety $\hat{T}$ is a resolution of singularities of $\bar{T}$. We may assume that $\hat{T}$ is constructed by a refinement of the fan associated to $\bar{T}$ as discussed in [Ful, Sect.2.6]. Thus it follows that any T-orbit closure V of $\bar{T}$ is the birational image of a T-orbit closure $\hat{V}$ in $\hat{T}$ (see e.g. the discussion at the end of Chapter 5 in [Ful]).

By [B-K, Prop.6.2.3] T-orbit closures of $\bar{T}$ correspond to $G\times G$-orbit closures in X. So let V be the T-orbit closure associated to Y. Choose $\hat{V}$ as above and let $\hat{Y}$ denote the corresponding $G\times G$-orbit closure of $\hat{X}$. Then by the discussion in [B-K, Sect.6.2] there is an induced birational map $f : \hat{Y} \to Y$. By [H-T2, Cor.8.4] the orbit closure Y is normal and thus, by Zariski's main theorem, we conclude $f_*\mathcal{O}_{\hat{Y}} = \mathcal{O}_Y$. By Lemma 9.8 (used with $X = \hat{X}$ and $Y = \hat{Y}$) it now suffices to prove that

$$\mathrm{H}^i(\hat{Y}, f^*\mathcal{L}) = 0,\ i > 0$$

for a very ample line bundle $\mathcal{L}$ on Y. This follows from [H-T2, Prop.7.2] and ends the proof in the case when X is toroidal.

Consider now an arbitrary projective equivariant embedding X of G. Let $\hat{X}$ denote the normalization of the closure of the image of the natural $G\times G$-equivariant embedding

$$G \to X \times \mathbf{X}.$$

Then $\hat{X}$ is a toroidal embedding of G with an induced equivariant morphism $f : \hat{X} \to X$. Let $\hat{Y}$ denote any $G \times G$-orbit closure in $\hat{X}$. Then $f : \hat{Y} \to f(\hat{Y})$ is a rational morphism [H-T2, Lem.8.3]. In particular, we may find a $G \times G$-orbit closure $\hat{Y}$ of $\hat{X}$ with an induced rational morphism $f : \hat{Y} \to Y$. Finally we may apply the first part of the proof to $\hat{Y}$ and use that a composition of rational morphisms is a rational morphism. □

Corollary 9.10. *Let X denote a projective embedding of a reductive group G and let $\mathcal{X}$ denote a G-Schubert variety in X. Let Y denote the minimal $G \times G$-orbit closure of X containing $\mathcal{X}$. When $\mathcal{L}$ is a nef line bundle on $\mathcal{X}$ then*

$$\mathrm{H}^i(\mathcal{X}, \mathcal{L}) = 0, \ i > 0.$$

Moreover, when $\mathcal{L}$ is a nef line bundle on Y then the restriction morphism

$$\mathrm{H}^0(Y, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}, \mathcal{L}),$$

is surjective.

Proof. Assume first that X is smooth and toroidal. Then by Proposition 9.6 and Lemma 9.5 the variety Y admits a stable Frobenius splitting along an ample divisor which compatibly Frobenius splits $\mathcal{X}$. Thus the statement follows in this case by Proposition 2.4.

Let now X denote an arbitrary projective equivariant embedding of G. Choose a projective toroidal embedding $\hat{X}$ as in Lemma 9.9 and let $f : \hat{Y} \to Y$ denote the induced rational morphism. Let V denote a $B \times B$-orbit in Y such that $\mathcal{X} = \mathrm{diag}(G) \cdot \bar{V}$. As Y is the minimal $G \times G$-orbit closure containing $\mathcal{X}$ it follows that V is contained in the open $G \times G$-orbit of Y. In particular, there exists a $B \times B$-orbit $\hat{V}$ in $\hat{X}$, contained in the open $G \times G$-orbit of $\hat{Y}$, which maps to V. In particular,

$$\hat{\mathcal{X}} := \mathrm{diag}(G) \cdot \bar{\hat{V}}$$

is a G-Schubert variety in $\hat{X}$ which by f maps onto $\mathcal{X}$. Moreover, $\hat{Y}$ is the minimal $G \times G$-orbit closure containing $\hat{\mathcal{X}}$. Applying Lemma 9.8 and the part of the statement which is already proved, it follows that $f : \hat{\mathcal{X}} \to \mathcal{X}$ is a rational morphism. In particular,

$$\mathrm{H}^i(\mathcal{X}, \mathcal{L}) = \mathrm{H}^i(\hat{\mathcal{X}}, f^*\mathcal{L})$$
$$\mathrm{H}^i(Y, \mathcal{L}) = \mathrm{H}^i(\hat{Y}, f^*\mathcal{L})$$

for all i and all line bundles $\mathcal{L}$ on $\mathcal{X}$ or Y. Now apply the part of statement which is already proved. □

By the proof of the above result we also find that any G-Schubert variety $\mathcal{X}$ a projective equivariant embedding of G, will admit a G-equivariant rational morphism $f : \hat{\mathcal{X}} \to \mathcal{X}$ by the closure $\hat{\mathcal{X}}$ of some G-stable piece of some smooth projective toroidal embedding of G.

Remark 9.11. *When $X = \mathbf{X}$ is the wonderful compactification of a group G of adjoint type and $\mathcal{L}$ is a nef line bundle on $\mathbf{X}$, then the restriction morphism*

$$\mathrm{H}^0(\mathbf{X}, \mathcal{L}) \to \mathrm{H}^0(Y, \mathcal{L}),$$

to any $G \times G$-stable irreducible subvariety Y of $\mathbf{X}$ is surjective. In particular, also the restriction morphism

$$\mathrm{H}^0(\mathbf{X}, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}, \mathcal{L}),$$

to any G-Schubert variety $\mathfrak{X}$ is surjective by the above result. We do not know if the latter is true for arbitrary equivariant embedding.

10. Normality questions

It is natural to expect that the Frobenius splitting properties of closures of G-stable pieces together with the cohomology vanishing results in Corollary 9.10 will have strong implications on the geometry of G-Schubert varieties. However, below will see that there exists an example of a G-Schubert variety in a wonderful compactification which is not even normal. In fact, it seems that there are plenty of such examples.

10.1. **Some general theory.** We keep the notations in 8.1. For $J \subset \Delta$ and $w \in W^{\Delta \setminus J}$, We let $\overline{\mathbf{X}_{J,w}}$ denote the closure of $\mathbf{X}_{J,w}$ in $\mathbf{X}$. Let

$$K = \max\{K' \subset \Delta \setminus J; wK' \subset K'\}.$$

By [He2, Prop. 1.12], we have a diag(G)-equivariant isomorphism

$$\mathrm{diag}(G) \times_{\mathrm{diag}(P_K)} (P_K w, P_K)\mathbf{h}_J \simeq \mathbf{X}_{J,w}$$

induced by the inclusion of $(P_K w, P_K)\mathbf{h}_J$ in $\mathbf{X}$. Let V denote the closure of $(P_K w, P_K)h_J$ within $\mathbf{X}$. Then V is the closure of a $B \times B$-orbit and we find that the induced map

$$f : \mathrm{diag}(G) \times_{\mathrm{diag}(P_K)} V \simeq \overline{\mathbf{X}_{J,w}}$$

is a birational and projective morphism. By the results in [H-T2] the $B \times B$-orbit closure V is normal. Thus a necessary condition for $\overline{\mathbf{X}_{J,w}}$ to be normal is that the fibers of f are connected.

10.2. **An example of a non-normal closure.** Let now, furthermore, G be a group of type G_2. Let α_1 denote the short simple root and α_2 denote the long simple root. The associated simple reflections are denoted by s_1 and s_2. Let J be the subset of Δ defined by α_2 and let $w = s_1 s_2 \in W^{\Delta \setminus J}$. In this case $K = \emptyset$ and we obtain a birational map

$$f : \mathrm{diag}(G) \times_{\mathrm{diag}(B)} V \simeq \overline{\mathbf{X}_{J,w}}$$

where V is the closure of $(Bw, B)\mathbf{h}_J$. By [Sp, Prop. 2.4], the part of V which intersect the open $G \times G$-orbit of X_J equals

$$\bigcup_{w \leq w'} (Bw', B)\mathbf{h}_J \cup \bigcup_{ws_1 \leq w'} (Bw', Bs_1)\mathbf{h}_J. \tag{a}$$

In particular, $x := (\dot{v}, 1)\mathbf{h}_J$ is an element of V, where $v = s_2 s_1 s_2$. We claim that the fiber of f over x is not connected. To see this let y denote a point in the fiber over x. Then we may find $g \in G$ and $\tilde{x} \in V$ such that

$$y = [g, \tilde{x}].$$

By (a), $\tilde{x} = (bw', p)\mathbf{h}_J$ for some $b \in B$, $p \in P_{\Delta \setminus J}$ and $w' \geq w$. Then

$$(gbw', gp)\mathbf{h}_J = (\dot{v}, 1)\mathbf{h}_J.$$

It follows that $(\dot{v}^{-1} gbw', gp)$ lies in the stabilizer of $\mathbf{h}_J$. In particular, $gp \in P_{\Delta \setminus J}$ and thus also $g \in P_{\Delta \setminus J}$. If $g \in B$ then $y = [1, x]$. So assume that $g = u_1(t)\dot{s}_1$ where u_1 is the root homomorphism associated to α_1. Assume that $t \neq 0$. Then we may find $b_1 \in B$ and $s \in k$ such that $g = u_{-1}(s)b_1$ where u_{-1} is the root homomorphism associated to $-\alpha_1$. Thus

$$\begin{aligned}\tilde{x} &= (g^{-1}, g^{-1})(\dot{v}, 1)\mathbf{h}_J \\ &= (b_1^{-1}u_{-1}(-s)\dot{v}, g^{-1})\mathbf{h}_J \\ &= (b_1^{-1}\dot{v}, g^{-1})\mathbf{h}_J \\ &\in (Bv, Bs_1)\mathbf{h}_J\end{aligned}$$

where the third equality follows as $\dot{v}^{-1}u_{-1}(-s)\dot{v}$ is contained in the unipotent radical of $P^-_{\Delta\setminus J}$. But $(Bv, Bs_1)\mathbf{h}_J$ has empty intersection with V which contradicts the assumption that $t \neq 0$. It follows that the only possibilities for y are $[1, v]$ and $[\dot{s}_1, (\dot{s}_1\dot{v}, \dot{s}_1)\mathbf{h}_J]$. As $(\dot{s}_1\dot{v}, \dot{s}_1)\mathbf{h}_J \in V$ we conclude that the fiber of f over x consists of 2 points; in particular the fiber is not connected and thus $\overline{\mathbf{X}_{J,w}}$ is not normal.

10.3. **Normalization of G-Schubert varieties.** The example above shows that the G-Schubert varieties within wonderful compactifications are, in general, not normal. Now we turn our attention towards the normalization of G-Schubert varieties which we expect to have nice singularities.

Let $\mathbf{X}_{J,w}$ be a G-stable piece and let $\mathcal{Z}_{J,w}$ denote the normalization of the closure of $\mathbf{X}_{J,w}$. Then the birational morphism f factors through $\mathcal{Z}_{J,w}$. In particular, there is an induced birational and projective morphism

$$f' : \mathrm{diag}(G) \times_{\mathrm{diag}(B)} V \to \mathcal{Z}_{J,w}.$$

By the results in [H-T2] the $B \times B$-orbit closure V is globally F-regular. Thus $\mathrm{diag}(G) \times_{\mathrm{diag}(B)} V$ is locally strongly F-regular. As $f'_*\mathcal{O}_{\mathrm{diag}(G)\times_{\mathrm{diag}(B)}V} = \mathcal{O}_{\mathcal{Z}_{J,w}}$ one could hope that a similar result was true for $\mathcal{Z}_{J,w}$. Moreover, using Proposition 9.6 and Lemma 9.5 one may conclude that $\mathcal{Z}_{J,w}$ admits a stable Frobenius splitting along an ample divisor. In particular, if the above hope was true then $\mathcal{Z}_{J,w}$ would be globally F-regular. At the moment we do not know if $\mathcal{Z}_{J,w}$ is strongly F-regular. We refer to [S] for an introduction to global F-regularity.

11. Generalizations

An admissible triple of $G\times G$ is by definition a quadruple $\mathcal{C} = (J_1, J_2, \theta_\delta)$ consisting of $J_1, J_2 \subset \Delta$, an isomorphism $\delta : W_{J_1} \to W_{J_2}$ with $\delta(J_1) = J_2$ and an isomorphism $\theta_\delta : L_{J_1} \to L_{J_2}$ that maps T to T and the root subgroup U_{α_i} to the root subgroup $U_{\alpha_{\delta(i)}}$ for $i \in J_1$. To each admissible triple $\mathcal{C} = (J_1, J_2, \theta_\delta)$, we associate the subgroup $\mathcal{R}_\mathcal{C}$ of $G \times G$ defined by

$$\mathcal{R}_\mathcal{C} = \{(p, q) : p \in P_{J_1}, q \in P_{J_2}, \theta_\delta(\pi_{J_1}(p)) = \pi_{J_2}(q)\},$$

where $\pi_J : P_J \to L_J$, for a subset $J \subset \Delta$, denotes the natural quotient map.

Let G_{sc} denote the simply connected cover of the commutator subgroup of G and let $\mathcal{C} = (J_1, J_2, \theta_\delta)$ denote an admissible triple on $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$. By definition in [L-Y, section 7], a $\mathcal{R}_\mathcal{C}$-stable piece in the wonderful compactification $\mathbf{X}$ of G_{ad} is a subvariety of the form $\mathcal{R}_\mathcal{C} \cdot Y$, where $Y = (Bv_1, Bv_2) \cdot \mathbf{h}_J$ for some $J \subset \Delta$, $v_1 \in W^J$ and $v_2 \in {}^{J_2}W$. Notice that when $J_1 = J_2 = \Delta$ and θ_δ is the identity map then a $\mathcal{R}_\mathcal{C}$-stable piece is the same as a G-stable piece. On the other hand, when $J_1 = J_2 = \emptyset$, then a $\mathcal{R}_\mathcal{C}$-stable piece is the same as a $B \times B$-orbit. Moreover, for any $B \times B$-orbit closure V in $\mathbf{X}$, $\mathcal{R}_\mathcal{C} \cdot V$ is the closure of some $\mathcal{R}_\mathcal{C}$-stable piece and is a finite union of $\mathcal{R}_\mathcal{C}$-stable pieces [L-Y, Section 7]. We call $\mathcal{R}_\mathcal{C} \cdot V$ a $\mathcal{R}_\mathcal{C}$-Schubert variety of $\mathbf{X}$.

The following is a generalization of Proposition 9.3 and Proposition 9.6.

Proposition 11.1. *Every equivariant embedding X of G admits a Frobenius splitting which compatibly splits all closed subsets of the form $\mathcal{R}_{\mathcal{C}} \cdot V$, where V denotes a $B \times B$-orbit closure of X. If, moreover, X is a smooth projective toroidal embedding and $Y = X_K$ is the minimal $G \times G$-orbit closure containing a $B \times B$-orbit closure V, then X admits a Frobenius splitting along the Cartier divisor*

$$D = (p-1)\Big(\sum_{i \in \Delta} (w_0^{J_1}, 1)\tilde{D}_i + \sum_{j \notin K} X_j\Big),$$

which is compatibly with Y and $\mathcal{R}_{\mathcal{C}} \cdot \bar{V}$.

Proof. In the following L_J, for a subset $J \subset \Delta$, denotes the Levi subgroup in G_{sc} associated to J. The corresponding commutator subgroup is denoted by L_J'. Define $X_{\mathcal{C}}$ to be the $L_{J_1}' \times L_{J_1}'$-variety which as a variety is X but where the action is twisted by the morphism

$$L_{J_1}' \times L_{J_1}' \xrightarrow{\mathbf{1} \times \theta_\delta} L_{J_1}' \times L_{J_2}'.$$

Then the $(L_{J_1}' \cap B_{\mathrm{sc}}) \times (L_{J_2}' \cap B_{\mathrm{sc}})$-canonical Frobenius splitting of X defined by Theorem 9.1 and Lemma 5.5 induces a $(L_{J_1}' \cap B_{\mathrm{sc}}) \times (L_{J_1}' \cap B_{\mathrm{sc}})$-canonical Frobenius splitting of $X_{\mathcal{C}}$. In particular, all subvarieties of $X_{\mathcal{C}}$ which corresponds to $B \times B$-orbit closures in X will be compatibly Frobenius split by this canonical Frobenius splitting. Now apply an argument as in the proof of Proposition 9.3 and use the idenfication of $\mathcal{R}_{\mathcal{C}} \cdot V \subset X$ with $\mathrm{diag}(L_{J_1}) \cdot V \subset X_{\mathcal{C}}$. This ends the proof of the first statement.

Assume now that X is a smooth projective toroidal embedding and consider the morphisms

$$\mathcal{O}_X \otimes (\mathrm{St} \boxtimes \mathrm{St}) \to \mathcal{M} \xrightarrow{s} \mathcal{E}\mathrm{nd}_F^!(X, Y, V) \otimes k_{(1-p)\rho \boxtimes (1-p)\rho}.$$

of the discussion above Proposition 9.6 in Section 9. Let $Y_{\mathcal{C}}$ and $V_{\mathcal{C}}$ be defined similar to $X_{\mathcal{C}}$. Applying Lemma 5.5 we obtain

$$\mathcal{O}_{X_{\mathcal{C}}} \otimes (\mathrm{St}_{\mathrm{J}_1} \boxtimes \mathrm{St}_{\mathrm{J}_1}) \to \mathcal{M} \xrightarrow{s} \mathcal{E}\mathrm{nd}_F^!(X_{\mathcal{C}}, Y_{\mathcal{C}}, V_{\mathcal{C}}) \otimes k_{(1-p)\rho_{J_1} \boxtimes (1-p)\rho_{J_1}}.$$

with notation as in Section 5.2. Let v_- (resp. v_+) denote a lowest (resp. highest) weight vector in $(\mathrm{St}_{\mathrm{J}_1} \boxtimes \mathrm{St}_{\mathrm{J}_1})$ and let σ be the global section of $\mathcal{M}$ which is the image of $v_+ \boxtimes v_-$ under the map

$$\mathrm{St}_{\mathrm{J}_1} \boxtimes \mathrm{St}_{\mathrm{J}_1} \to \mathcal{M}(X).$$

Let D denote the zero divisor of σ. Arguing as in the proof of Proposition 9.6 we then obtain a D-splitting of $X_{\mathcal{C}}$ which is compatible with the subvarieties $Y_{\mathcal{C}}$ and $\mathrm{diag}(L_{J_1}) \cdot V_{\mathcal{C}}$. Notice finally that D equals the divisor

$$(p-1)\Big(\sum_{i \in \Delta} (w_0^{J_1}, 1)\tilde{D}_i + \sum_{j \notin K} X_j\Big).$$

This ends the proof. □

In the case where $k = \mathbb{C}$ and X is the wonderful compactification, the subvarieties $(w_0^{J_1}, 1)\tilde{D}_i$, X_j and all the $\mathcal{R}_{\mathcal{C}}$-Schubert varieties are Poisson subvarieties with respect to the Poisson structure on X corresponding to the splitting

$$\mathrm{Lie}(G) \oplus \mathrm{Lie}(G) = l_1 \oplus l_2,$$

where $l_1 = \mathrm{Lie}(\mathcal{R}_{\mathcal{C}})$ and l_2 is certain subalgebra of $\mathrm{Ad}(w_0^{J_1})\,\mathrm{Lie}(B^-) \oplus \mathrm{Lie}(B^-)$. See [L-Y2, 4.5].

We may also argue as in Corollary 9.10 to obtain

Corollary 11.2. *Let X denote a projective embedding of a reductive group G and let V denote the closure of a $B \times B$-orbit in X. Let Y denote the minimal $G \times G$-orbit closure of X containing $\mathcal{X}_{\mathcal{C}} = \mathcal{R}_{\mathcal{C}} \cdot V$. When $\mathcal{L}$ is a nef line bundle on $\mathcal{X}_{\mathcal{C}}$ then*

$$\mathrm{H}^i(\mathcal{X}_{\mathcal{C}}, \mathcal{L}) = 0, \ i > 0.$$

Moreover, when $\mathcal{L}$ is a nef line bundle on Y then the restriction morphism

$$\mathrm{H}^0(Y, \mathcal{L}) \to \mathrm{H}^0(\mathcal{X}_{\mathcal{C}}, \mathcal{L}),$$

is surjective.

11.1. **Further variations.** Define an To an admissible triple $\mathcal{R}_{\mathcal{C}}$ we may associate the variety $Z_{\mathcal{C}} = (G \times G)/\mathcal{R}_{\mathcal{C}}$ which can also be identified with the variety

$$(G \times G) \times_{P_{J_1} \times P_{J_2}} L_{J_1},$$

where the action of $P_{J_1} \times P_{J_2}$ on $G \times G \times L_{J_1}$ is defined by

$$(p_1, p_2) \cdot (g_1, g_2, l) = \big(g_1 p_1, g_2 p_2, \pi_{J_1}(p_1) l \theta_\delta^{-1}(\pi_{J_2}(p_2)^{-1})\big).$$

Then

Lemma 11.3. *There is a $B \times B$-canonical splitting on $Z_{\mathcal{C}}$ that compatibly splits all the $B \times B$-orbit closures.*

Proof. By [B-K, Thm.6.2.7], there exists a $B \times B$-canonical splitting on L_{J_1} that compatibly splits all the $B \times B$-orbit closures. Then by [B-K, Proposition 4.1.17 & Exercise 4.1.E(4)], there exists a $B \times B$-canonical splitting on $(G \times G) \times_{B \times B} L_{J_1}$ that compatibly splits all the $B \times B$-orbit closures. By a push forward argument this implies that $Z_{\mathcal{C}} = (G \times G) \times_{P_{J_1} \times P_{J_2}} L_{J_1}$ admits a $B \times B$-canonical Frobenius splitting which compatibly Frobenius splits all $B \times B$-orbits closures. □

Now let h be the element $[1, 1, 1] \in (G \times G) \times_{P_{J_1} \times P_{J_2}} L_{J_1} = Z_{\mathcal{C}}$ and let $\mathcal{C}' = (J_1', J_2', \theta_{\delta'})$ be another admissible triple. A $\mathcal{R}_{\mathcal{C}'}$-stable piece of $Z_{\mathcal{C}}$ is then a subset of the form $\mathcal{R}_{\mathcal{C}'} \cdot Y$ where $Y = (Bv_1, Bv_2) \cdot h$ for some $v_1 \in W^{J_1}$ and $v_2 \in {}^{J_2'}W$. Similar to the proof of the first part of Proposition 11.1 we may then prove

Proposition 11.4. *The variety $Z_{\mathcal{C}}$ admits a Frobenius splitting which compatibly Frobenius splits all closures of $\mathcal{R}_{\mathcal{C}'}$-stable pieces.*

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