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PROPERTIES OF D'ALEMBERT FUNCTIONS

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Properties of d'Alembert functions

Henrik Stetkær

Abstract

We study properties of solutions f of d'Alembert's functional equations on a topological group G. For nilpotent groups and for connected, solvable Lie groups G, we prove that f has the form $f(x) = (\gamma(x) + \gamma(x^{-1}))/2$, $x \in G$, where γ is a continuous homomorphism of G into the multiplicative group $\mathbb{C} \setminus \{0\}$. We give conditions on G and/or f for equality in the inclusion $\{u \in G \mid f(xu) = f(x) \text{ for all } x \in G\} \subseteq \{u \in G \mid f(u) = 1\}$.

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1 Introduction and notation

Let G be a topological group. In this paper we will study properties of the solutions $f : G \to \mathbb{C}$ of *d'Alembert's functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) \text{ for all } x, y \in G. \tag{1.1}$$

A continuous solution f such that $f(e) = 1$, e denoting the neutral element of G, is said to be a *d'Alembert function* on G. The condition $f(e) = 1$ may be replaced by the equivalent one that $f \neq 0$.

Our purpose is to investigate the d'Alembert functions on groups that need not be compact or abelian, and to find the properties that such functions possess. The d'Alembert functions are known on abelian groups, where Kannappan [10] characterized them (see Theorem 2.12), and they have also been studied on compact groups by Davison [4] and Yang [14].

Any solution of (1.1) is also a solution of *d'Alembert's long functional equation*

$$f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y), \quad x, y \in G. \tag{1.2}$$

Some of our results are valid for solutions of (1.2).

Three themes about a function $f \in C(G)$ dominate the paper:

(I) The first and longest is about the following three subsets of G and their mutual relations:

$$N(f) := \{u \in G \mid f(xu) = f(x) \text{ for all } x \in G\}, \tag{1.3}$$
$$U(f) := \{u \in G \mid f(u) = f(e)\}, \tag{1.4}$$
$$Z(f) := \{z \in G \mid f(xyz) = f(xzy) \text{ for all } x, y \in G\}. \tag{1.5}$$

In particular we ask when f satisfies Kannappan's condition $Z(f) = G$. In other words, when $f(xyz) = f(xzy)$ for all $x, y, z \in G$. Following Davison [4] we say that a d'Alembert function is *abelian* if it satisfies Kannappan's condition, and non-abelian if it does not.

(II) The second is the range of a possibly unbounded solution of d'Alembert's long functional equation.

(III) The third is about properties that d'Alembert functions have in common with group characters, i.e. properties from abstract harmonic analysis.

Some comments on the themes:

(I) $N(f)$ is the group of (right) periods of the function f. For the cosine function the set of periods $\{u \in \mathbb{R} \mid \cos(x+u) = \cos x \text{ for all } x \in \mathbb{R}\}$ and the set $\{u \in \mathbb{R} \mid \cos u = 1\}$ are the same. This equality persists for d'Alembert functions on nilpotent (and hence also on abelian) groups and on connected, solvable Lie groups, although not on all groups (Proposition 4.1, Theorem 5.2, Remark 2.2).

In [4] Davison calls $N(f)$ the nub of f. In hindsight the nub $N(f)$ of a d'Alembert function was in disguise introduced already in the paper [12] (the group H_1 in [12, Proposition 6.3]), but it was not exploited beyond the study of d'Alembert's functions on step 2 nilpotent groups in [12]. Its role was first recognized by [4]. The discussion in the present paper of its relation to $U(f)$ is new.

d'Alembert functions need not be abelian (The subsections 8.2–8.4 provide examples), but we prove that they are so on groups that are close to being abelian (Theorem 4.2 and Theorem 5.2). Furthermore we find and apply a new criterion for a d'Alembert function f to be abelian: f is abelian iff $f([x, y]) = 1$ for all $x, y \in G$ (Theorem 3.7(b)).

In the paper [10] from 1968 Kannappan proved that if a d'Alembert function f is abelian (in particular if G is abelian) then it can be written in the form $f = (\gamma + \check{\gamma})/2$ where $\gamma : G \to \mathbb{C}^*$ is a continuous homomorphism. Before that d'Alembert functions were only known on special groups like the real line. Sufficient conditions for d'Alembert functions on non-abelian groups to be abelian have later been published by Corovei [2, 3], Friis [7] and the author [11, 12]. Most of these results are generalized here in Sections 4 and 5.

(II) In Section 6 we study solutions of d'Alembert's long functional equation, that may be unbounded. Davison [4] considered the range of bounded d'Alembert functions.

(III) In Section 7 we present two properties from abstract harmonic analysis: Linear independence and orthogonality relations.

Examples are gathered in Section 8.

Throughout we let G denotes a topological group with neutral element e. Some authors, for example Bourbaki, include the Hausdorff property in the definition of locally compact. We do not.

By function we always mean complex-valued function. If F is a function on G we let $\check{F}$ denote the function defined by $\check{F}(x) = F(x^{-1})$, $x \in G$. We let $[x, y] = xyx^{-1}y^{-1}$ denote the commutator between $x \in G$ and $y \in G$. For $A, B \subseteq G$ we let $[A, B]$ denote the smallest subgroup of G containing the commutators $\{[a, b] \mid a \in A, b \in B\}$. $C(G)$ denotes the algebra of all continuous, complex-valued functions on G.

$\mathbb{C}^*$ denotes the multiplicative group of the non-zero complex numbers.

The identity matrix will be denoted I.

2 Preliminary results

We now derive some properties of and relations between the sets $N(f)$, $U(f)$ and $Z(f)$, first for any function $f : G \to \mathbb{C}$, and then for a solution of d'Alembert's long functional equation.

Lemma 2.1. *Let $f \in C(G)$.*

(a) $N(f)$ is a closed subgroup of G.

(b) f is a function on the coset space $G/N(f)$, i.e. f is constant on each coset $xN(f)$, $x \in G$.

(c) $Z(f)$ is a closed, normal subgroup of G.

(d) $Z(f) = \{u \in G \mid [u, G] \subseteq N(f)\}$.

(e) $U(f)$ is a closed subset of G, and $N(f) \subseteq U(f)$.

Proof. (a), (b) and (c) are immediate consequences of the definitions.

(d) Let $u \in Z(f)$. For any $x, y \in G$ we get that $f(x[u, y]) = f(x)$, because any element from $Z(f)$ behaves like an element from the center of G when it occurs in argument for f. This shows that $[u, y] \in N(f)$. Since y is arbitrary, we have proved that $[u, G] \subseteq N(f)$, and so that $Z(f) \subseteq \{u \in G \mid [u, G] \subseteq N(f)\}$.

Assume conversely that u is an element of the right hand side. For any $x, y \in G$ we get that $f(xu^{-1}y) = f(xyu^{-1}[u, y^{-1}]) = f(xyu^{-1})$, which shows that $u^{-1} \in Z(f)$. But then also $u \in Z(f)$, proving the other inclusion.

(e) Put $x = e$ in $f(xu) = f(x)$. □

Remark 2.2. Even if f is a d'Alembert function $U(f)$ need not be a subgroup of G (in contrast to $N(f)$ and $Z(f)$), and the inclusion in Lemma 2.1(e) may be strict. More is needed for $U(f)$ to be a subgroup and for the inclusion to be an equality: For example that f is bounded (Proposition 2.11) or that G is nilpotent (Proposition 4.1). The following example illustrates these phenomena:

Let $G = \mathrm{SL}(2, \mathbb{C})$. By [4, Proposition 4.8] $f(X) := \frac{1}{2}\operatorname{tr} X$, $X \in G$, is a d'Alembert function on G such that $N(f) = \{I\}$. We find that

$$U(f) = \left\{ \begin{pmatrix} 1 + bc & b^2 \\ -c^2 & 1 - bc \end{pmatrix} \,\middle|\, b, c \in \mathbb{C} \right\},$$

so $N(f) \subsetneqq U(f)$. Furthermore $U(f)$ is not a subgroup of G:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in U(f), \text{ but } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \notin U(f).$$

Any algebraic combination of homomorphisms of G into $\mathbb{C}^*$ satisfies Kannappan's condition. To take an example, $f = (\gamma + \check{\gamma})/2$ is an abelian d'Alembert function on G, if $\gamma : G \to \mathbb{C}^*$ is a continuous homomorphism.

Lemma 2.3. *Let $f : G \to \mathbb{C}$ be a function on G. Then the following three conditions are equivalent*

(a) f satisfies Kannappan's condition.

(b) f is a function on $G/[G, G]$.

(c) $[G, G] \subseteq N(f)$.

If $f : G \to \mathbb{C}$ satisfies Kannappan's condition, then $f([x, y]) = f(e)$ for all $x, y \in G$. We derive in Theorem 3.7(b) a converse under the assumption that f is a d'Alembert function.

Lemma 2.4. *Let G_1 be a group and let $\pi : G_1 \to G$ be a surjective homomorphism. Let f be a function on G. Then $f \circ \pi$ satisfies Kannappan's condition if and only if f does.*

Definition 2.5. We say that a function f on G is *basic*, if $N(f) = \{e\}$.

Lemma 2.6. *If $f : G \to \mathbb{C}$ assumes the value $f(e)$ only at $x = e$, then f is basic.*

Proof. If $u \in N(f)$ then $f(xu) = f(x)$ for all $x \in G$. Taking $x = e$ we find that $f(u) = f(e)$. By our hypothesis we infer that $u = e$. This means that $N(f) = \{e\}$. □

Remark 2.7. Lemma 2.6 is one direction of the last statement of [4, Corollary 4.11]. Compactness of G is not needed for this simple fact. We shall later present a converse for bounded d'Alembert functions (Corollary 3.8(a)).

Lemma 2.8. *Let $f \in C(G)$ be basic. Then $Z(f) = Z(G)$, so f satisfies Kannappan's condition if and only if G is an abelian group.*

Proof. Follows from Lemma 2.1(d). □

Definition 2.9. The pair (G, f) is said to be a *d'Alembert group*, if f is a basic d'Alembert function on G.

From now on we no longer discuss a general function f on G. In the remainder of the present section we restrict our attention to a solution f of d'Alembert's long functional equation (1.2) such that $f(e) = 1$. In that case $U(f) = \{u \in G \mid f(u) = 1\}$. The special case of a d'Alembert function will be studied in the following sections.

Proposition 2.10. *Let $f : G \to \mathbb{C}$ be a bounded solution of d'Alembert's long functional equation. Then $f(G) \subseteq [-1, 1]$.*

Proof. Davison proves the proposition for d'Alembert functions [4, Proposition 6.1], but his proof works also for solutions of d'Alembert's long functional equation. $\square$

Lemma 2.11. *If $f \in C(G)$ is a bounded solution of d'Alembert's long functional equation, then $U(f)$ is a subgroup of G.*

Proof. We may assume that $f(e) = 1$, because otherwise $f = 0$. If $u, v \in U(f) = \{u \in G \mid f(u) = 1\}$, then

$$\tfrac{1}{4}\{f(uv) + f(vu) + f(uv^{-1}) + f(v^{-1}u)\} = f(u)f(v) = 1.$$

f is a bounded solution of d'Alembert's long functional equation, so $-1 \leq f(x) \leq 1$ for all $x \in G$ by Proposition 2.10. Due to the strict convexity of the unit interval we get that $f(uv) = f(vu) = f(uv^{-1}) = f(v^{-1}u) = 1$, so $uv^{-1} \in U(f)$. Thus $U(f)$ is a subgroup of G, because $U(f) \neq \emptyset$. $\square$

Theorem 2.12 (Kannappan's theorem). *Let $f \in C(G)$ be a solution of d'Alembert's long functional equation such that $f(e) = 1$. Then f satisfies Kannappan's condition if and only if there exists a continuous homomorphism $\gamma : G \to \mathbb{C}^*$ such that $f = (\gamma + \check{\gamma})/2$.*

Kannappan's theorem is useful, because (1) it gives an explicit expression for the structure of the abelian d'Alembert functions, and (2) it can be applied to the study of a d'Alembert function f even on a non-abelian group G: For any $x_0 \in G$ the restriction of f to the subgroup $\langle x_0 \rangle$ of G generated by x_0 satisfies Kannappan's condition, because $\langle x_0 \rangle$ is abelian, so Kannappan's theorem can be applied to $\langle x_0 \rangle$. We use the idea (2) in Section 6.

3 Properties of d'Alembert functions

From now on we concentrate on d'Alembert functions. In the present section we continue the discussion from Section 2 of the relations between the sets $N(f)$, $U(f)$ and $Z(f)$. We use it to derive a new criterion for a d'Alembert function to be abelian (Theorem 3.7(b)). We also derive some identities that are valid for any d'Alembert function.

The following Lemma 3.1 describes two important properties of d'Alembert functions. It is well known.

Lemma 3.1. *Any solution f of d'Alembert's functional equation* (1.1) *is* central, *i.e. $f(xy) = f(yx)$ for all $x, y \in G$, and even, i.e. $\check{f} = f$.*

The identity (3.1) in the following Lemma 3.2 is useful in the study of bounded d'Alembert functions.

Lemma 3.2. *Let $f \in C(G)$ be a d'Alembert function on G, and let $u \in U(f)$. Then $f(u^n) = 1$ for all $n \in \mathbb{Z}$. More generally,*

$$f(xu^n) - f(x) = n[f(xu) - f(x)] \quad \text{for all } x \in G \text{ and } n \in \mathbb{Z}. \tag{3.1}$$

Proof. For $n \geq 0$ the proof goes by induction on n. For $n < 0$ we write $n = -m$, where $m \in \mathbb{N}$. Using that $f(u^m) = 1$ (take $x = e$ in (3.1)) we get

$$\begin{aligned}
f(xu^n) - f(x) &= f(xu^{-m}) - f(x) = f(xu^m) + f(xu^{-m}) - f(xu^m) - f(x) \\
&= 2f(x)f(u^m) - f(xu^m) - f(x) = 2f(x) - f(xu^m) - f(x) \\
&= -[f(xu^m) - f(x)] = -[m(f(xu) - f(x))] = n[f(xu) - f(x)].
\end{aligned}$$

□

Remark 3.3. Lemma 3.2 was derived under the hypothesis that $u \in U(f)$, i.e. that $f(u) = 1$. If $f(u) = \pm 1$ we get the more general formula

$$f(xu^n) - f(x)f(u)^n = nf(u)^{n-1}[f(xu) - f(x)f(u)], \ \forall x \in G, \ \forall n \in \mathbb{Z}.$$

Given a function f on G we let $f_x(y) := f(xy) - f(x)f(y)$, $x, y \in G$, and define

$$\Delta(x, y) := f_x(x)f_y(y) - f_x(y)^2, \quad x, y \in G. \tag{3.2}$$

The function $\Delta : G \times G \to \mathbb{C}$ was introduced by Davison in [5]. The formula (3.6) below shows how Δ is related to the values of f on commutators.

Proposition 3.4. *If f is a d'Alembert function on G and $x, y \in G$ then*

$$(f(xy) - 1)(f(xy^{-1}) - 1) = (f(x) - f(y))^2 + \frac{1 - f([x, y])}{2}, \tag{3.3}$$

$$f(xy)f(xy^{-1}) = f(x)^2 + f(y)^2 - \frac{1 + f([x, y])}{2}, \tag{3.4}$$

$$\begin{aligned}
\left[\frac{f(xy) - f(xy^{-1})}{2}\right]^2 &= [f(xy) - f(x)f(y)]^2 \\
&= [f(x)^2 - 1][f(y)^2 - 1] - \frac{1 - f([x, y])}{2},
\end{aligned} \tag{3.5}$$

$$\Delta(x, y) = \frac{1 - f([x, y])}{2}. \tag{3.6}$$

Proof. Using that f is central, we find that

$$\begin{aligned}
(f(xy) - 1)(f(xy^{-1}) - 1) &= f(xy)f(xy^{-1}) - [f(xy) + f(xy^{-1})] + 1 \\
&= \tfrac{1}{2}[f(xyxy^{-1}) + f(xy^2x^{-1})] - 2f(x)f(y) + 1 \\
&= \tfrac{1}{2}[f(xyxy^{-1}) + f(y^2)] - 2f(x)f(y) + 1 \\
&= \tfrac{1}{2}[f(xyxy^{-1}) + f(xyx^{-1}y^{-1})] - \tfrac{1}{2}f(xyx^{-1}y^{-1}) + \tfrac{1}{2}f(y^2) - 2f(x)f(y) + 1 \\
&= f(x)f(yxy^{-1}) - \tfrac{1}{2}f([x, y]) + f(y)^2 - \tfrac{1}{2} - 2f(x)f(y) + 1 \\
&= f(x)^2 - \tfrac{1}{2}f([x, y]) + f(y)^2 - 2f(x)f(y) + \tfrac{1}{2} \\
&= \tfrac{1}{2}(1 - f([x, y])) + [f(x) - f(y)]^2.
\end{aligned}$$

(3.4) follows from (3.3) by simple computations. For the proof of (3.5) we first note that $f(xy) - f(xy^{-1}) = 2f(xy) - 2f(x)f(y)$, which gives us the first equality sign. We get the remainder of (3.5) by substituting (3.4) into the identity

$$\begin{aligned}[f(xy) - f(xy^{-1})]^2 &= [f(xy) + f(xy^{-1})]^2 - 4f(xy)f(xy^{-1}) \\ &= [2f(x)f(y)]^2 - 4f(xy)f(xy^{-1}).\end{aligned}$$

The proof of (3.6) consists of the following computation

$$\begin{aligned}\Delta(x, y) &= f_x(x)f_y(y) - f_x(y)^2 \\ &= (f(x)^2 - 1)(f(y)^2 - 1) - (f(xy) - f(x)f(y))^2 \\ &= -f(x)^2 - f(y)^2 + 1 - f(xy)^2 + 2f(xy)f(x)f(y) \\ &= 1 - f(x)^2 - f(y)^2 - f(xy)^2 + f(xy)[f(xy) + f(xy^{-1})] \\ &= 1 - f(x)^2 - f(y)^2 + f(xy)f(xy^{-1}),\end{aligned}$$

combined with the formula (3.4). □

If x and y commute, or just if $f([x, y]) = 1$, then (3.3) reduces to

$$(f(xy) - 1)(f(xy^{-1}) - 1) = (f(x) - f(y))^2, \quad \forall\, x, y \in G, \tag{3.7}$$

from which [4, Proposition 2.2] follows, and (3.5) reduces to

$$[f(xy) - f(x)f(y)]^2 = [f(x)^2 - 1]\,[f(y)^2 - 1],$$

which is [2, Lemma 1].

Proposition 3.5. *Let f be a d'Alembert function on the group G. Then*

(a) $N(f)$ is a closed, normal subgroup of G.

(b) $N(f) = Z(f) \cap U(f)$.

(c) $Z(f) \supseteq \{u \in G \mid u^2 \in N(f)\}$.

(d) If f is non-abelian, then $Z(f) = \{u \in G \mid u^2 \in N(f)\}$.

Proof. During the proof we shall without explicit mentioning use that f as a d'Alembert function is central and even.

(a) This was noted in Lemma 2.1(a) except for the normality, which follows from f being central.

(b) For any $u_0 \in N(f)$ and $x, y \in G$ we get $f(xu_0y) = f(yxu_0) = f(yx) = f(xy) = f(xyu_0)$, showing that $u_0 \in Z(f)$. Using Lemma 2.1(e) we get that $u_0 \in U(f)$ as well. Hence $N(f) \subseteq Z(f) \cap U(f)$. Assume conversely that $u_0 \in Z(f) \cap U(f)$. It follows from [12, Lemma 6.2(c)] that $f(xu_0) = f(x)f(u_0) = f(x)$ for all $x \in G$, i.e. that $u_0 \in N(f)$.

(c) Let $u^2 \in N(f)$. We first note that $f(u^2) = f(e) = 1$ by Lemma 2.1(e). From $f(u^2) = 2f(u)^2 - 1$ we infer that $f(u) = \pm 1$. In particular that $f(u) \neq 0$.

That $u^2 \in N(f)$ means that $f(xu^2) = f(x)$ for all $x \in G$. Replacing x by xu^{-1} here we find that $f(xu) = f(xu^{-1})$ for all $x \in G$. Since f is even we get similarly that

$f(ux) = f(u^{-1}x)$ for all $x \in G$. Using this when taking $y = u$ in d'Alembert's long functional equation (1.2) we find that

$$f(xu) + f(ux) = 2f(x)f(u) \text{ for all } x \in G. \tag{3.8}$$

We replace x in (3.8) by uxu^{-1} and note that $(u^2)^{-1} \in N(f)$ [because $N(f)$ is a group] to get (starting with the right hand side of (3.8)) that

$$\begin{aligned} 2f(uxu^{-1})f(u) &= f(uxu^{-1}u) + f(uuxu^{-1}) = f(ux) + f(u^2xu^{-1}) \\ &= f(ux) + f(ux^{-1})(u^2)^{-1}) = f(ux) + f(ux^{-1})(u^2)^{-1}) \\ &= f(ux) + f(ux^{-1})) = f(ux) + f(xu^{-1}) = 2f(x)f(u). \end{aligned}$$

Since $f(u) \neq 0$ we infer from the computations just made that $f(uxu^{-1}) = f(x)$. It now follows from (3.8) that $f(xu) = f(x)f(u)$. Finally, for any $x, y \in G$ we get that $f(xuy) = f(yxu) = f(yx)f(u) = f(xy)f(u) = f(xyu)$, which shows that $u \in Z(f)$. Thus $Z(f) \supseteq \{u \in G \mid u^2 \in N(f)\}$.

(d) For the other inclusion we first note that $Z(f) \neq G$, since f is non-abelian. By [12, Lemma 6.2(d)] we then get that $f(xu) = f(x)f(u)$ for all $x \in G$ and $u \in Z(f)$, and that $f(u) = \pm 1$ for all $u \in Z(f)$.

If $u \in Z(f)$ we find that $u^2 \in Z(f)$ and that $f(u^2) = f(uu) = f(u)^2 = 1$, so that $u^2 \in Z(f) \cap U(f)$. But according to (b) the right hand side is $N(f)$. □

Remark 3.6. Proposition 3.5(d) is closely related to [4, Proposition 5.2] (via Lemma 2.8).

We have in Lemma 2.11 seen that $U(f)$ is a group, if f is a bounded non-zero solution of d'Alembert's long functional equation. Theorem 3.7 continues the study of $U(f)$, now for a possibly unbounded d'Alembert function. (b) gives a necessary and sufficient condition for a d'Alembert function to be abelian, which is used in the proof of Proposition 5.1.

Theorem 3.7. *Let f be a d'Alembert function on G.*

(a) The following four conditions are equivalent:

(i) $U(f)$ is a subgroup of G

(ii) $f([a, b]) = 1$ for all $a, b \in U(f)$

(iii) $U(f) = N(f)$

(iv) $U(f) \subseteq Z(f)$

(b) f is abelian if and only if $f([x, y]) = 1$ for all $x, y \in G$.

(c) f is abelian if and only if $\Delta(x, y) = 0$ for all $x, y \in G$, where Δ is defined by (3.2).

Proof. (a) If $U(f)$ is a subgroup of G, then $[a, b] = aba^{-1}b^{-1} \in U(f)$ for all $a, b \in U(f)$, so $f([a, b]) = 1$ for all $a, b \in U(f)$.

Assume conversely that $f([a, b]) = 1$ for all $a, b \in U(f)$. Since $e \in U(f)$, all we need to prove is that $a, b \in U(f) \Rightarrow ab^{-1} \in U(f)$. So let $a, b \in U(f)$. From the formula (3.3) we read that

$$(f(ab) - 1)(f(ab^{-1}) - 1) = 0,$$

so either $f(ab) = 1$ or $f(ab^{-1}) = 1$. By help of the formula $f(ab) + f(ab^{-1}) = 2f(a)f(b) = 2$ we see that $f(ab) = 1$ implies that $f(ab^{-1}) = 1$. So in any case $f(ab^{-1}) = 1$, which means that $ab^{-1} \in U(f)$.

If $U(f) = N(f)$ then $U(f)$ is a subgroup of G, because so is $N(f)$. So let us conversely assume that $U(f)$ is a subgroup of G. The inclusion $N(f) \subseteq U(f)$ is noted in Lemma 2.1, so left is the reverse inclusion: Let $u \in U(f)$ and let $x \in G$. Replacing a by xu and b by x in the identity (3.3) we get that

$$[f(xux) - 1][f(xux^{-1}) - 1] = [f(xu) - f(x)]^2 + \frac{1 - f([xu, x])}{2},$$

which, because f is central, implies that

$$0 = [f(xu) - f(x)]^2 + \frac{1 - f([xu, x])}{2}. \tag{3.9}$$

Now, $[xu, x] = x\{uxu^{-1}x^{-1}\}x^{-1}$, so, f being central,

$$f([xu, x]) = f(x\{uxu^{-1}x^{-1}\}x^{-1}) = f(uxu^{-1}x^{-1}).$$

Here we note that $U(f)$ is a normal subgroup of G because f is central. Therefore $xu^{-1}x^{-1} \in U(f)$ and hence $u(xu^{-1}x^{-1}) \in U(f)$, i.e. $f(uxu^{-1}x^{-1}) = 1$. Finally we get from (3.9) that $0 = [f(xu) - f(x)]^2$ as desired. Thus $U(f) = N(f)$.

From Proposition 3.5(b) we read that $N(f) = Z(f) \cap U(f)$. This means that $U(f) = N(f)$ if and only if $U(f) \subseteq Z(f)$.

(b) That f is abelian means that it satisfies Kannappan's condition (1.5)). Putting $z = x^{-1}y^{-1}$ in (1.5)) we get that $f([x, y]) = 1$ for all $x, y \in$G.

Let us conversely assume that $f([x, y]) = 1$ for all $x, y \in G$. Then all commutators $[x, y] \in G$ are elements of $U(f)$, and so $U(f)$ is a subgroup of G (by (a)). Hence the subgroup of G generated by the commutators is contained in $U(f)$, i.e. $[G, G] \subseteq U(f)$. Using (a) we have that $[G, G] \subseteq U(f) = N(f)$, which means that Kannappan's condition is satisfied (see Lemma 2.8(c)).

(c) is immediate from (b) and formula (3.6). □

Corollary 3.8. *Let f be a bounded d'Alembert function on G. Then*

(a) f is basic if and only if $f(x) = 1$ only for $x = e$.

(b) If f is basic, then $f|_H$ is basic for any subgroup H of G.

Proof. (a) Let f be basic. We read from Lemma 2.11 that $U(f)$ is a subgroup of G. In that case $U(f) = N(f)$ by Theorem 3.7(a). Thus $U(f) = \{e\}$, so that $f(u) = 1$ only for $u = e$. The converse is Lemma 2.6.

(b) is immediate from (a). □

Remark 3.9. (i) Corollary 3.8 extends [4, Corollary 4.11], in which G is compact.

(ii) Corollary 3.8(a) contains a partial converse of Lemma 2.6.

It is not true in general that the restriction of a basic d'Alembert function to a subgroup is basic, not even if the restriction is bounded. But here is a positive result about restrictions:

Proposition 3.10. *Let f be a d'Alembert function on G. Let H be a compact subgroup of G, or a normal subgroup of G such that $f(H)$ is bounded. Then*

(a) $f(xH)$ is bounded for any fixed $x \in G$.

(b) $H \cap U(f) = H \cap N(f)$.

(c) Let f be basic. Then $f|_H$ is basic. Furthermore $f|_H \neq 1$ if $H \neq \{e\}$.

Proof. (a) Let $x_0 \in G$ be fixed. From d'Alembert's functional equation $f(x_0h) + f(x_0h^{-1}) = 2f(x_0)f(h)$ we see that $\{f(x_0h) + f(x_0h^{-1}) \mid h \in H\}$ is bounded. From the identity (3.5) we see that $\{f(x_0h) - f(x_0h^{-1}) \mid h \in H\}$ is bounded. Hence so is their sum. In particular $\{f(x_0h) \mid h \in H\}$ is bounded.

(b) Let $u \in H \cap U(f)$ and let $x_0 \in G$ be fixed. The left hand side of the formula (3.1), i.e. of $f(x_0u^n) - f(x_0) = n[f(x_0u) - f(x_0)]$, has by (a) a bound independent of n. A glance at the right hand side reveals that in that case $f(x_0u) - f(x_0) = 0$. We have thus shown that $H \cap U(f) \subseteq N(f)$. The rest follows from $N(f) \subseteq U(f)$ (Lemma 2.1(e)).

(c) Using (b) we get that $N(f|_H) \subseteq U(f|_H) = H \cap U(f) = \{e\}$, so f is basic on H. Now $f|_H$ is both bounded and basic, so we may refer to Corollary 3.8(a), in which we replace G by H. □

4 d'Alembert functions on nilpotent groups

If f is an abelian d'Alembert function on an abelian group G, then $U(f) = N(f)$ (This is immediate from Proposition 3.5(b)). It is known more generally to hold on step 2 nilpotent groups ([12, Lemma 7.1(b)]). The following Proposition 4.1 finishes the considerations for nilpotent groups by removing the step 2 condition.

Proposition 4.1. *Let f be a d'Alembert function on a nilpotent group G. Then $N(f) = U(f)$.*

Proof. $N(f) \subseteq U(f)$ by Lemma 2.1. To prove the converse inclusion we let $u \in G$ be such that $f(u) = 1$. We shall prove that $u \in N(f)$.

Let $\pi : G \to \overline{G} := G/N(f)$ be the quotient map. Then $(\overline{G}, \overline{f})$, where $f = \overline{f} \circ \pi$, is a d'Alembert group which is n-nilpotent for some $n \in \mathbb{N} \cup \{0\}$. For any $x \in G$ we use the abbreviation $\overline{x} = \pi(x) \in \overline{G}$.

If $n \leq 1$, then $\overline{G}$ is abelian, so $\overline{f}$ and hence also f is abelian (Lemma 2.4). Then $u \in N(f)$ as observed in the beginning of this subsection.

If $n \geq 2$ we read from [4, Lemma 5.3] that $\overline{u}^{2^n} = \overline{e}$. The formula in Lemma 3.2 tells us that

$$\overline{f}(\overline{x}\,\overline{u}^{2^n}) - \overline{f}(\overline{x}) = 2^n[\overline{f}(\overline{x}\,\overline{u}) - \overline{f}(\overline{x})] \quad \text{for all } x \in G,$$

which (because $\overline{u}^{2^n} = \overline{e}$) implies that $\overline{f}(\overline{x}\,\overline{u}) - \overline{f}(\overline{x}) = 0$, i.e. $f(xu) - f(x) = 0$, or equivalently that $u \in N(f)$. □

Theorem 4.2. *Let G be a nilpotent group that is generated by its squares. Then any d'Alembert function on G is abelian.*

Proof. Let f be a d'Alembert function on G. Possibly replacing G by $G/N(f)$ we may assume that (G, f) is a nilpotent d'Alembert group. If f is abelian we are done, so let us assume that f is not abelian and then lead this assumption to a contradiction. So G is now an n-nilpotent d'Alembert group for some $n \in \mathbb{N}$, where $n \geq 2$, because G is not abelian. With $Z_0(G) := \{e\}$ and inductively $Z_{j+1}(G) := \{x \in G \mid [x, G] \subseteq Z_j(G)\}$ this means that $\{e\} = Z_0(G) \subset Z_1(G) = Z(G) \subset \cdots \subset Z_{n-1}(G) \subset Z_n(G) = G$ with strict inclusions. In particular $Z_{n-1}(G) \neq G$.

Let $x \in G$ be arbitrary. Since G is generated by its squares, x may be written in the form $x = x_1^2 x_2^2 \ldots x_s^2$, where $x_1, x_2, \ldots, x_s \in G$. From the proof of [4, Lemma 5.3] we read in particular that $y \in G = Z_n(G) \Rightarrow y^2 \in Z_{n-1}(G)$. Hence $x \in Z_{n-1}(G)$. It follows that $G \subseteq Z_{n-1}(G)$, which implies the desired contradiction $Z_{n-1}(G) = G$. $\square$

Theorem 4.2 has as corollaries

1. [4, Proposition 5.4], who imposes the stronger condition on the nilpotent group G that it shall be 2-divisible. Our proof of Theorem 4.2 is a modification of Davison's procedure in [4].

2. [2, Theorem 2] who deals with a nilpotent group, all of whose elements have (finite) odd order.

3. [7, Theorem 2.6] in which G is a connected nilpotent Lie group. More precisely only in the case of d'Alembert functions, because Friis' result holds for solutions of d'Alembert's long functional equation.

The assumption in Theorem 4.2 that the group is generated by its squares cannot be omitted: (Q_8, g_0) from subsection 8.4 is a counter-example.

$\frac{1}{2}$ tr is a non-abelian d'Alembert function on $G = \mathrm{SL}(2, \mathbb{R})$. Being a connected Lie group G is generated by its squares, so we see that the assumption about nilpotency in Theorem 4.2 cannot be deleted either.

Nilpotent groups are members of the larger class of solvable groups, for which we obtain a special result in the next section (Theorem 5.2).

5 d'Alembert functions on solvable Lie groups

d'Alembert's functional equation has been studied on various classes of groups: Abelian, nilpotent, compact etc. However, on solvable groups it has only been studied in the special instance of the $(ax + b)$-group [6, Example 3.14], where it was shown that the d'Alembert functions are abelian. We will in Theorem 5.2 below extend this special result by proving that all d'Alembert functions on solvable, connected Lie groups are abelian.

First a result that can be applied when G is a semi-direct product $G = N \times_s H$ of two subgroups N and H:

Proposition 5.1. *Let N and H be two subgroups of G such that $G = NH = \{nh \mid n \in N, h \in H\}$. We assume that*

(a) N is normal,

(b) H is abelian, and

(c) H is connected, or the subgroup generated by the squares $\{h^2 \mid h \in H\}$ is dense in H, or the subgroup generated by the squares $\{x^2 \mid x \in G\}$ is dense in G.

Let $f \in C(G)$ be a d'Alembert function on G such that $f|_N$ is abelian. Then f is abelian.

Proof. By Theorem 3.7(b) it suffices to prove that $f([x,y]) = 1$ for all $x, y \in G$.

$f|_N$ being abelian there exists by Kannappan's theorem (Theorem 2.12) a continuous homomorphism $\gamma : N \to \mathbb{C}^*$ such that $f|_N = \frac{1}{2}(\gamma + \check{\gamma})$. For any $x \in G$ we let $x \cdot \gamma : N \to \mathbb{C}^*$ be the homomorphism defined by $(x \cdot \gamma)(n) = \gamma(x^{-1}nx)$, $n \in N$. Since f is central we find for any fixed $x \in G$ that $\frac{1}{2}(\gamma + \check{\gamma}) = f|_N = \frac{1}{2}(x \cdot \gamma + x \cdot \check{\gamma})$. This implies that either $x \cdot \gamma = \gamma$ or $x \cdot \gamma = \check{\gamma}$, because the set of homomorphisms of G into $\mathbb{C}^*$ is linearly independent in the vector space of complex-valued functions on G ([9, Lemma 29.41]).

- Here we assume that H is connected. $H_+ := \{h \in H \mid h \cdot \gamma = \gamma\}$, and $H_- := \{h \in H \mid h \cdot \gamma = \check{\gamma}\}$ are closed subsets of H, such that $H = H_+ \cup H_-$. Furthermore $e \in H_+$, so H_+ is not empty. Let $h \in H$. If $\gamma = \check{\gamma}$, then $h \cdot \gamma = \gamma$. If $\gamma \neq \check{\gamma}$, then H_+ and H_- are disjoint. By the connectedness of H we get that $H = H_+$, so that $h \cdot \gamma = \gamma$. So in both cases $h \cdot \gamma = \gamma$ for all $h \in H$.

- Here we assume that H is generated by its squares. Let $x \in H$.

  (1) If $x \cdot \gamma = \gamma$, then $(x^2) \cdot \gamma = x \cdot (x \cdot \gamma) = x \cdot \gamma = \gamma$.

  (2) If $x \cdot \gamma = \check{\gamma}$, then $(x^2) \cdot \gamma = x \cdot (x \cdot \gamma) = x \cdot \check{\gamma} = (x \cdot \gamma)^\vee = (\check{\gamma})^\vee = \gamma$.

  Since H is generated by its squares we get that $h \cdot \gamma = \gamma$ for all $h \in H$.

- Here we assume that G is generated by its squares. Arguing as in the previous point we get that $x \cdot \gamma = \gamma$ for all $x \in G$. In particular that $h \cdot \gamma = \gamma$ for all $h \in H$.

Let $x = n_1h_1$ and $y = n_2h_2$, where $n_1, n_2 \in N$ and $h_1, h_2 \in H$. Then

$$[x,y] = [n_1h_1, n_2h_2] = n_1(h_1n_2h_1^{-1})(h_1h_2h_1^{-1}n_1^{-1}(h_1h_2h_1^{-1})^{-1})[h_1,h_2]n_2^{-1}.$$

Since H is abelian, so that the factor $[h_1,h_2] = e$ vanishes and $h_1h_2h_1^{-1} = h_2$, this expression reduces to $[x,y] = n_1(h_1n_2h_1^{-1})(h_2n_1^{-1}h_2^{-1})n_2^{-1}$. Each factor belongs to N, so also $[x,y] \in N$. Now $\gamma : N \to \mathbb{C}^*$ is a homomorphism, so

$$\begin{aligned} \gamma([x,y]) &= \gamma(n_1)\,\gamma(h_1n_2h_1^{-1})\,\gamma(h_2n_1^{-1}h_2^{-1})\,\gamma(n_2^{-1}) \\ &= \gamma(n_1)\,(h_1^{-1} \cdot \gamma)(n_2)\,(h_2^{-1} \cdot \gamma)(n_1^{-1})\,\gamma(n_2^{-1}) \\ &= \gamma(n_1)\,\gamma(n_2)\,\gamma(n_1^{-1})\,\gamma(n_2^{-1}) = 1. \end{aligned}$$

Hence $f([x,y]) = \frac{1}{2}(\gamma + \check{\gamma})([x,y]) = \frac{1}{2}(1+1) = 1$. $\square$

Theorem 5.2. *Any d'Alembert function on a connected, solvable Lie group is abelian.*

Proof. The proof goes by induction on the dimension n of the group. The theorem is clearly true if $n \leq 1$, because the group then is abelian.

Assume that the theorem is true for all groups of dimension $\leq n-1$, where $n \geq 2$. Let f be a d'Alembert function on a connected, solvable Lie group G of dimension n. We shall prove that f is abelian. Possibly replacing G by its universal covering group we may (according to Lemma 2.4) assume that G is simply connected.

Let $\mathfrak{g}$ denote the Lie algebra of G. Choose a subspace $\mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{n} \supseteq [\mathfrak{g}, \mathfrak{g}]$ and $\dim \mathfrak{n} = n-1$. Then $\mathfrak{n}$ is an ideal of $\mathfrak{g}$. Choose a complementary subspace $\mathfrak{h}$ of $\mathfrak{n}$ in $\mathfrak{g}$. Then $\dim \mathfrak{h} = 1$, so $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$. Let N and H be the respective analytic subgroups of G defined by $\mathfrak{n}$ and $\mathfrak{h}$. Then N is a normal subgroup of G, and H is an abelian subgroup of G. According to [13, Lemma 3.18.4] $G = N \times_s H$ is the semidirect product of the two subgroups N and H. By our induction hypothesis $f|_N$ is abelian, so we get from Proposition 5.1 that f is abelian. $\square$

6 On the range of a d'Alembert function

If f is a bounded solution of d'Alembert's long functional equation, then $f(G) \subseteq [-1, 1]$ (Lemma 2.10). We shall in this section study

(A) the range of solutions of d'Alembert's long functional equation under weaker boundedness assumptions. For example assuming only that the real part $\Re f$ of the solution f is bounded.

(B) relations between a solution of d'Alembert's long functional equation on G and its restriction to the identity component G_0 of G.

Unless otherwise specified $f : G \to \mathbb{C}$ will throughout this Section 6 denote a non-zero solution of d'Alembert's long functional equation on G. That $f \neq 0$ is equivalent to $f(e) = 1$.

As illustrations the reader may have in mind the d'Alembert functions

1. $\cos x$, $\cosh x$ and $\cos(1+i)x = \cos x \cosh x - i \sin x \sinh x$ on $G = \mathbb{R}$. $\cosh x$ is an unbounded real-valued d'Alembert function, which is bounded from below by 1. The range of $\cos x$ is all of $[-1, 1]$.

2. $\frac{1}{2} \operatorname{tr} x$, $x \in \mathrm{SL}(2, \mathbb{R})$. This is a real-valued d'Alembert function, which is unbounded both from above and from below.

3. $f(x) = (x + x^{-1})/2$, $x \in G = \mathbb{Z}_3 = \{\exp(\frac{2\pi i}{3} n) \mid n = 0, 1, 2\} \subseteq \mathbb{C}^*$. Here $f(G) = \{-\frac{1}{2}, 1\}$.

None of the other sections depend on this section.

Let $x_0 \in G$ be arbitrary. Kannappan's theorem applied to the subgroup $\langle x_0 \rangle$ of G says that there exists a homomorphism $\gamma : \langle x_0 \rangle \to \mathbb{C}^*$ such that $f(x) = (\gamma(x) + \gamma(x^{-1}))/2$ for all $x \in \langle x_0 \rangle$. Let us write $\gamma(x_0) = r_0 e^{i\theta_0}$, where $r_0 > 0$ and $\theta_0 \in]-\pi, \pi]$.

Possibly replacing x_0 by x_0^{-1} we may assume that $r_0 \geq 1$. By elementary computations we find for any $n \in \mathbb{Z}$ we find that

$$f(x_0^n) = \frac{r_0^n + r_0^{-n}}{2} \cos(n\theta_0) + i\,\frac{r_0^n - r_0^{-n}}{2} \sin(n\theta_0), \text{ so that} \tag{6.1}$$

$$\Re f(x_0^n) = \frac{r_0^n + r_0^{-n}}{2} \cos(n\theta_0), \text{ and in particular} \tag{6.2}$$

$$\Re f(x_0) = \frac{r_0 + r_0^{-1}}{2} \cos(\theta_0). \tag{6.3}$$

The above notation will be used in the proofs of the Propositions 6.1 and 6.2 whenever an $x_0 \in G$ is given.

Proposition 6.1. *(a) If $\Re f$ is bounded from above, then $f(G) \subseteq [-1,1]$.*

(b) If $\Re f$ is bounded from below, then $\Re f(x) \geq -1$ for all $x \in G$.

(c) If $\Im f$ is bounded, then f is real-valued.

Proof. Writing $f = g + ih$, where $g = \Re f$ and $h = \Im f$, we find for all $x, y \in G$ that

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4\{g(x)g(y) - h(x)h(y)\} \tag{6.4}$$

$$h(xy) + h(yx) + h(xy^{-1}) + h(y^{-1}x) = 4\{g(x)h(y) + h(x)g(y)\}. \tag{6.5}$$

(a) If $g = \Re f$ is also bounded from below, then it follows from (6.4) that $h = \Im f$ is bounded. Hence $f = g + ih$ is bounded, so that we are done by Proposition 2.10.

Thus it suffices to show that $\Re f$ is bounded from below. This we do by contradiction, so we assume that $\Re f$ is not bounded from below. Then there exists an $x_0 \in G$, such that $\Re f(x_0) < -1$. If θ_0 is a rational multiple of π, say $\theta_0 = \frac{p}{q}\pi$, where $p, q \in \mathbb{Z}$ and $q > 0$, then we get for any $m \in \mathbb{N}$ by taking $n = 2mq$ in (6.2) that

$$\Re f(x_0^{2mq}) = \frac{r_0^{2mq} + r_0^{-2mq}}{2} \cos(2mp\pi) = \frac{r_0^{2mq} + r_0^{-2mq}}{2}.$$

Letting $m \to \infty$ we see that $\Re f$ is not bounded from above, contradicting our assumption.

Thus θ_0 is not a rational multiple of π, and so there exist arbitrarily large $n \in \mathbb{N}$ such that $\cos n\theta_0 > 1/2$. Once again we see from (6.2) that $\Re f$ is not bounded from above, contradicting our assumption.

(b) We prove that $\Re f \geq -1$ by contradiction, so we assume that there exists an element $x_0 \in G$ such that $\Re f(x_0) < -1$. We get from (6.3) that $\cos\theta_0 < 0$ and that $r_0 > 1$.

If θ_0 is a rational multiple of π, say $\theta_0 = p/q\,\pi$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, we see from (6.2) for any $n \in \mathbb{N}$ that

$$\Re f(x_0^{1+2qn}) = \frac{r_0^{1+2qn} + r_0^{-1-2qn}}{2} \cos(\theta_0),$$

which converges to $-\infty$ for $n \to \infty$, contradicting that $\Re f$ is bounded from below. If θ_0 is not a rational multiple of π, then $\cos(n\theta_0) < -1/2$ for infinitely many $n \in \mathbb{N}$, which once more via (6.2) leads to a contradiction to $\Re f$ being bounded from below.

(c) Here $h = \Im f$ is bounded. We shall prove that $h = 0$, so we may assume that $h \neq 0$. From (6.5) we see that also g is bounded. But then f is real-valued by (a). $\square$

Proposition 6.2. *(a) If $\Re f(x) \geq -1/2$ for all $x \in G$, then f is real-valued.*

(b) If $\Re f(x) > -1/2$ for all $x \in G$, then $f(x) \geq 1$ for all $x \in G$.

Proof. (a) Letting $x_0 \in G$ be arbitrary we shall prove that $f(x_0) \in \mathbb{R}$.

If $r_0 = 1$, then the last term of (6.1) vanishes, so that $f(x_0) \in \mathbb{R}$. Thus we may from now on assume that $r_0 > 1$. Combining our assumption with (6.2) we find for any $n \in \mathbb{N}$ that

$$\frac{r_0^n + r_0^{-n}}{2} \cos(n\theta_0) \geq -\frac{1}{2}.$$

For any $n \in \mathbb{N}$

$$\frac{r_0^n + r_0^{-n}}{2} \geq \frac{r_0 + r_0^{-1}}{2} > 1, \text{ so}$$

$$\cos(n\theta_0) \geq -\frac{1}{2}\left(\frac{r_0^n + r_0^{-n}}{2}\right)^{-1} \geq -\frac{1}{2}\left(\frac{r_0 + r_0^{-1}}{2}\right)^{-1} > -\frac{1}{2},$$

which implies that $\theta_0 = 0$, because it holds for all $n \in \mathbb{N}$. And then $f(x_0) \in \mathbb{R}$ by (6.1).
(b) We get from (a) that f is real-valued, so that $f(x) > -1/2$ for all $x \in G$. Then $-1/2 < f(x^2) = 2f(x)^2 - 1$, or equivalently $f(x)^2 > 1/4$, so that either $f(x) > 1/2$ or $f(x) < -1/2$. The last possibility must be discarded by our assumption so that $f(x) > 1/2$ for all $x \in G$.

Assume now that $f(x) > a$ for all $x \in G$ for some $a \geq 1/2$. Then $a < f(x^2) = 2f(x)^2 - 1$, so that $f(x)^2 > (1+a)/2$, implying that either $f(x) > \sqrt{(1+a)/2}$ or $f(x) < -\sqrt{(1+a)/2}$. The last possibility does not occur (because $f(x) > a \geq 1/2 > 0$), so $f(x) > \sqrt{(1+a)/2}$ for all $x \in G$. Since $\sqrt{(1+a)/2} \geq 1/2$ we may continue the process. We obtain a sequence $a_1 = 1/2, a_2, \ldots,$ such that $1/2 \leq a_n \leq f(x)$ and $a_{n+1} = \sqrt{(1+a_n)/2}$. It follows by induction that the sequence $\{a_n\}$ is increasing. It is bounded from above by $f(x)$, so it converges. Let $a_0 = \lim_{n\to\infty} a_n$. Then $a_0 \leq f(x)$ for all $x \in G$. When we let $n \to \infty$ in $a_{n+1} = \sqrt{(1+a_n)/2}$ we get that $a_0 = \sqrt{(1+a_0)/2}$, which implies that $a_0 = 1$. $\square$

Remark 6.3. The condition $\Re f(x) > -1/2$ in Proposition 6.2(b) cannot be strengthened to $\Re f \geq -1/2$ as shown by the example $G = \mathbb{Z}_3$ from the beginning of this section.

Proposition 6.4. *Let G be connected. Then*

(a) $f(x) \geq 1$ for all $x \in G$ or $\Re f(G) \supseteq [-1, 1]$.

(b) If furthermore f is bounded, then $f(G) = \{1\}$ or $f(G) = [-1, 1]$.

Proof. (a) Due to Proposition 6.2(b) we may assume that $\inf \Re f(G) \leq -1/2$. By connectedness $\Re f(G) \supseteq [-1/2, f(e)] = [-1/2, 1]$, so there exists an $a \in G$ such that $\Re f(a) = 0$. Then $\Re f(a^2) = 2\Re(f(a)^2) - 1 \leq 2(\Re f(a))^2 - 1 = 0 - 1 = -1$. By connectedness $\Re f(G) \supseteq [-1, 1]$.

(b) Combine Proposition 2.10 and (a). $\square$

Many of the ingredients of the following Lemma 6.5 are present in [4, Proposition 6.6]. However, we deal with a locally compact group, while [4] only has a compact one.

Lemma 6.5. *A totally disconnected, locally compact d'Alembert group is discrete.*

Proof. Let the d'Alembert group be denoted (G, f). We get from [8, Theorem 7.7] that every neighborhood of e contains a compact open subgroup of G. In particular $P = \{x \in G \mid \Re f(x) > 0\}$ contains a compact open subgroup of G, say $H \subseteq P$. It suffices to prove that $H = \{e\}$, because then $\{e\}$ is open, and so G is discrete.

We note that $f(H) \subseteq [-1, 1]$, because H is compact so that f is bounded (Proposition 2.10).

Furthermore on H the function f assumes the value 1 only at the point e (Corollary 3.8(a)). If $H \neq \{e\}$ we get from Proposition 6.2(b) with G replaced by H that there exists an $h \in H$ such that $f(h) \leq -1/2$. But $h \in H \subseteq P$, so that contradicts the definition of P. Hence $H = \{e\}$. □

Proposition 6.6. *Let f be a d'Alembert function on a locally compact group G. Assume that $f = 1$ on the identity component G_0 of G. Then $N(f)$ is open and $G_0 \subseteq N(f)$.*

Proof. The quotient map $\pi : G \to G/N(f)$ is an open and continuous homomorphism of G onto $G/N(f)$. Define $\tilde{f} : G/N(f) \to \mathbb{C}$ by $f = \tilde{f} \circ \pi$. Then $(G/N(f), \tilde{f})$ is a d'Alembert group.

Let G_0 and $(G/N(f))_0$ denote the identity components of G and $G/N(f)$ respectively. $\overline{\pi(G_0)} = (G/N(f))_0$ according to [8, Theorem 7.12].

Now $\tilde{f} = 1$ on $\pi(G_0)$ and hence also on $\overline{\pi(G_0)} = (G/N(f))_0$. Since $\tilde{f}$ is basic we read from Corollary 3.10(c) that $(G/N(f))_0 = \{eN(f)\}$. Then $G/N(f)$ is totally disconnected, and hence discrete by Lemma 6.5. But then $N(f)$ is an open, and hence also closed, subgroup of G. By connectedness $G_0 \subseteq N(f)$. □

7 Relations to harmonic analysis

The following two results describe properties of d'Alembert functions that they have in common with group characters (cf. [9, Lemma 29.41] and the orthogonality relations) and more generally with spherical functions.

None of the other sections depend on this section.

Proposition 7.1. *The non-zero solutions of d'Alembert's long functional equation on a group G are linearly independent in the vector space of all complex-valued functions on G.*

Proof. We shall show that any finite set of non-zero solutions of d'Alembert's long functional equation is linearly independent. We do this by induction on the number n of elements in the set. It is true, if $n = 1$, because the solutions are non-zero. Assuming that any set of n non-zero solutions of d'Alembert's long functional equation is linearly independent, we shall prove it is also true for any set of $n + 1$ elements. So let $f_1, \dots, f_n, f_{n+1}$ be $n+1$ different non-zero solutions of d'Alembert's long functional equation, and let $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \mathbb{C}$ be such that

$$\alpha_1 f_1 + \cdots + \alpha_n f_n + \alpha_{n+1} f_{n+1} = 0. \tag{7.1}$$

We shall prove that $\alpha_1 = \cdots = \alpha_n = \alpha_{n+1} = 0$.

Let $x, y \in G$ be arbitrary. Evaluating (7.1) at xy^{-1}, xy^{-1}, yx and $y^{-1}x$ and adding the resulting four identities we obtain from d'Alembert's long functional equation that

$$\alpha_1 f_1(x) f_1(y) + \cdots + \alpha_n f_n(x) f_n(y) + \alpha_{n+1} f_{n+1}(x) f_{n+1}(y) = 0. \tag{7.2}$$

Multiplying (7.1) by $f_{n+1}(y)$ and subtracting the result from (7.2) we get

$$\alpha_1 f_1(x)[f_1(y) - f_{n+1}(y)] + \cdots + \alpha_n f_n(x)[f_n(y) - f_{n+1}(y)] = 0.$$

By our induction hypothesis $\{f_1, \dots, f_n\}$ is a linearly independent set, so

$$\alpha_1[f_1(y) - f_{n+1}(y)] = \cdots = \alpha_n[f_n(y) - f_{n+1}(y)] = 0.$$

Since $f_{n+1} \neq f_k$ for $k = 1, \dots, n$, we get that $\alpha_1 = \cdots = \alpha_n = 0$. By (7.1) then also $\alpha_{n+1} = 0$, so all the coefficients vanish. $\square$

Next the orthogonality relations:

Proposition 7.2. *Let $g_1, g_2 \in C(G)$ be two different d'Alembert functions on a compact group G. Then*

$$\int_G g_1(x) g_2(x)\, dx = 0,$$

where dx denotes a Haar measure on G.

Since bounded d'Alembert functions are real (Proposition 2.10) the integral in the proposition is the usual inner product in $L^2(G)$ between g_1 and g_2. The orthogonality relations of course imply the linear independence of the set of d'Alembert functions, but only under the assumption that G is compact. Proposition 7.1 works on any group.

Proof. $(g_1 * g_2)(x) = \int_G g_1(y) g_2(y^{-1}x)\, dy$ for $x \in G$. g_1 being even we can change variables to get $(g_1 * g_2)(x) = \int_G g_1(y) g_2(yx)\, dy$. Adding the two identities gives that $g_1 * g_2 = \{\int_G g_1(y) g_2(y)\, dy\}\, g_2$.

g_1 is central (Lemma 3.1), which implies that $g_1 * f = f * g_1$ for any $f \in C(G)$. In particular $g_1 * g_2 = g_2 * g_1$, so that

$$\left\{\int_G g_1(y) g_2(y)\, dy\right\} g_2 = \left\{\int_G g_2(y) g_1(y)\, dy\right\} g_1.$$

If $\int_G g_1(y) g_2(y)\, dy \neq 0$ we get the contradiction $g_1 = g_2$. $\square$

8 Examples

8.1 The group $\mathbb{R}$

The d'Alembert functions on $G = \mathbb{R}$ are the functions $f_\alpha(x) = \cos(\alpha x)$, where the parameter α ranges over $\mathbb{C}$. The d'Alembert functions for which $\alpha \in \mathbb{C} \setminus \mathbb{R}$ are basic, while the remaining ones are not.

8.2 The special linear groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$

Theorem 8.1. *$g_1 = 1$, $g_2 = \frac{1}{2}\operatorname{tr}$ and $g_3 = \frac{1}{2}\overline{\operatorname{tr}}$ (i.e. the complex conjugate of g_2) are the only d'Alembert functions on the group $\mathrm{SL}(2,\mathbb{C})$. g_1 is the only abelian d'Alembert function, while the functions $g_2 = \frac{1}{2}\operatorname{tr}$ and $g_3 = \frac{1}{2}\overline{\operatorname{tr}}$ are basic and non-abelian d'Alembert functions.*

We see that there may well exist two different basic, non-abelian d'Alembert functions on a group.

For a proof that the functions g_1, g_2 and g_3 are d'Alembert functions, and that g_2 and g_3 are basic see [4, Proposition 4.8]. $g = 1$ is the only abelian solution of d'Alembert's functional equation on $G = \mathrm{SL}(2,\mathbb{C})$, i.e. the only solution of the form $g(A) = (\gamma(A) + \gamma(A^{-1})/2$, $A \in \mathrm{SL}(2,\mathbb{C})$, where $\gamma : G \to \mathbb{C}^*$ is a homomorphism. Indeed, as is well known $G = [G,G]$ (see for example [13, Corollary 3.18.10]). Being a homomorphism γ is 1 on any commutator $[A,B] = ABA^{-1}B^{-1}$, so it follows that γ is identically 1 on G (We could also apply Theorem 3.7(b)).

g_2 and g_3 are not abelian: Indeed, abelian d'Alembert functions are 1 on commutators, but g_2 and g_3 take the value 3/2 on the commutator

$$\left[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right] = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}3 & -1\\ 1 & 0\end{pmatrix}$$

We skip the details in our long proof of the uniqueness statement of Theorem 8.1. The main ingredient is that d'Alembert functions, being central, are constant on the orbits of inner automorphisms.

By similar considerations to the ones for $\mathrm{SL}(2,\mathbb{C})$ we derive

Theorem 8.2. *$g = 1$ and $g = \frac{1}{2}\operatorname{tr}$ are the only d'Alembert functions on the group $\mathrm{SL}(2,\mathbb{R})$. g_1 is the only abelian d'Alembert function, while the function $g = \frac{1}{2}\operatorname{tr}$ is basic and non-abelian.*

8.3 The special unitary group $\mathrm{SU}(2)$

There are only two d'Alembert functions on the compact group $\mathrm{SU}(2)$, viz. the constant function 1 and the function $\operatorname{tr}/2$ [14, Theorem 2]. That $\operatorname{tr}/2$ is a d'Alembert function is clear, because it is the restriction to $\mathrm{SU}(2)$ of the d'Alembert function g_2 on $\mathrm{SL}(2,\mathbb{C})$. It is non-abelian by the argument above for $\mathrm{SL}(2,\mathbb{C})$, applied to

$$\left[\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right] = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

Finally $\operatorname{tr}/2$ is basic by Corollary 3.10(c), because it is basic on $\mathrm{SL}(2,\mathbb{C})$.

8.4 The quaternion group

The first non-abelian d'Alembert functions that were discovered, were related to the quaternions. Below we point out how they are related to the quaternions and to one another.

As is well known, the map

$$a + bi + cj + dk \mapsto \begin{bmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{bmatrix}, \quad a, b, c, d \in \mathbb{R},$$

is an injective $\mathbb{R}$-linear multiplicative homomorphism of the quaternions into the complex 2×2 matrices. The restriction h to the group $Q = \{a+ib+cj+dk \mid a^2+b^2+c^2+d^2 = 1\}$ of unit quaternions is a topological group isomorphism between Q and $\mathrm{SU}(2)$ (See [9, 29.54]).

Since h is a topological isomorphism, the functions 1 and $g = \frac{1}{2}\operatorname{tr} \circ h : Q \to \mathbb{C}$ are the only d'Alembert functions on Q by the results described in subsection 8.3. g is basic and non-abelian, because so is $\frac{1}{2}\operatorname{tr}$ on $\mathrm{SU}(2)$. A simple computation reveals that $g(a + ib + cj + dk) = a$ for $a + ib + cj + dk \in Q$. Ng observed in [1, Remark] that g is a non-abelian d'Alembert function on Q.

Corovei had earlier in [2, p. 105–106] (see also [3, Example]) considered the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and found that the function $g_0 : Q_8 \to \mathbb{C}$ defined by $g_0(\pm 1) = \pm 1$, $g_0(\pm i) = g_0(\pm j) = g_0(\pm i) = 0$, is a non-abelian d'Alembert function on Q_8. g_0 is the restriction of Ng's function g from Q to Q_8, so it is clearly a d'Alembert function. That g_0 is non-abelian, is immediate from the computations $g_0(ijk) = g_0(k^2) = g_0(-1) = -1$ and $g_0(ikj) = g_0(-ijk) = g_0(1) = 1$. It is basic by Corollary 3.8(b). Actually g_0 is the only basic d'Alembert function on Q_8: Let f_0 be any basic d'Alembert function on Q_8. Then f_0 is non-abelian by Lemma 2.8. But there is only one non-abelian d'Alembert function on Q_8, viz g_0 (according to [12, Example 7.4]). Hence $f_0 = g_0$.

8.5 On existence of non-trivial d'Alembert functions

There are both non-compact and compact groups on which $g = 1$ is the only d'Alembert function:

Consider the group $G = P\mathrm{SL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$. Let $\pi : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$ be the quotient map. If g is a d'Alembert function on G, then $g \circ \pi$ is a d'Alembert function on $\mathrm{SL}(2, \mathbb{R})$. By Theorem 8.2 we either have $g \circ \pi = 1$ or $g \circ \pi = \frac{1}{2}\operatorname{tr}$. But the latter case can not occur, because it leads to a contradiction: $(g \circ \pi)(-I) = g(\pi(-I)) = g(e) = 1$, but $\frac{1}{2}\operatorname{tr}(-I) = -1$. Hence $g \circ \pi = 1$, and so $g = 1$ is the only d'Alembert function on $\mathrm{PSL}(2, \mathbb{R})$.

$g = 1$ is the only d'Alembert function on $\mathrm{SU}(n)$ for $n \geq 3$ [14, Proposition 1]. See [14] for examples of d'Alembert functions on connected, compact groups.

By similar arguments as for $G = \mathrm{PSL}(2, \mathbb{R})$ you see that $g = 1$ is the only d'Alembert function on the special orthogonal group $G = \mathrm{SO}(3)$. Indeed, $\mathrm{SU}(2)$ is a 2-fold covering group of $\mathrm{SO}(3)$ (actually its universal covering group) and $g = \frac{1}{2}\operatorname{tr}$ and $g = 1$ are the only d'Alembert functions on $\mathrm{SU}(2)$.

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