A CLASSIFICATION RESULT FOR SIMPLE LIMITS OF CIRCLE ALGEBRAS WITH DIMENSION DROPS

JESPER MYGIND

Abstract. We prove that the simple unital inductive limits of finite direct sums of C^*-algebras of the form $\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_{d_i},\ i = 1, 2, \dots, N\}$, where $x_1, x_2, \dots, x_N$ are elements of $\mathbb{T}$ and $d_1, d_2, \dots, d_N$ are positive integers dividing n, are classified by the Elliott invariant, if either K_0 is non-cyclic or K_1 contains an element of infinite order.

Contents

1. Introduction

During the last decade there has been considerable progress in the field of classification of simple stably finite C^*-algebras, see e.g [7], [8], [19], [9], [21], [13], [14], [18]. The invariant employed is the Elliott invariant that consists of the K_0-group, the K_1-group, and the tracial state space, together with its pairing with the K_0-group. This paper can be seen as an attempt to unify all these results, except that of [14]. Obviously, the classification results of [7], [8], [19], [9], and [13] are contained in the classification result of either [21] or [18].

In [13] Jiang and Su proved that the Elliott invariant is a complete invariant for the class of simple unital infinite dimensional inductive limits of sequences of finite direct sums of building blocks of the form

$$\{f \in C[0,1] \otimes M_n : f(0) \in M_{d_1},\ f(1) \in M_{d_2}\},$$

Date: March 26, 1999.

where d_1 and d_2 are positive integers dividing n. This was generalised by the author in [18] where building blocks of the form

$$\{f \in C[0,1] \otimes M_n : f(x_i) \in M_{d_i},\ i = 1, 2, \ldots, N\},$$

where $d_1, d_2, \ldots, d_N$ are positive integers dividing n, were considered. Remarkably, Jiang and Su showed that their class contains a simple unital projectionless C^*-algebra $\mathcal{Z}$ with the same Elliott invariant as $\mathbb{C}$, thus explaining their assumption of infinite dimensionality.

In [21] Thomsen proved that the Elliott invariant is a complete invariant for the class of simple unital inductive limits of finite direct sums of building blocks of the form

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_d,\ i = 1, 2, \ldots, N\},$$

where d is a positive integer dividing n and $x_1, x_2, \ldots, x_N \in \mathbb{T}$. By [21, Theorem 9.1] Thomsen's class contains C^*-algebras, for which the K_1-group contains elements of infinite order, that were not included in the class considered in [18]. On the other hand, by [18, Theorem 8.9], the class considered in [18] contains C^*-algebras, for which the K_0-group is cyclic, and which were not included in Thomsen's result. Therefore these results are independant, in the sense that neither of them generalises the other.

In order to construct a class containing both these classes we will in this thesis consider the class of simple unital inductive limits of sequences of finite direct sums of building blocks of the form

$$\{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_{d_i},\ i = 1, 2, \ldots, N\},$$

where $x_1, x_2, \ldots, x_N \in \mathbb{T}$ and $d_1, d_2, \ldots, d_N$ are positive integers dividing n. Unfortunately, we have only been able to give a classification result for a subclass of this class. To be precise, our main result is the following theorem:

Theorem 1.1. *Let A and B be a simple unital inductive limit of a finite direct sum of building blocks such that either $K_0(A)$ is non-cyclic or $K_1(A)$ contains an element of infinite order. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an isomorphism of groups with order units, let $\varphi_1 : K_1(A) \to K_1(B)$ be an isomorphism of groups, and let $\varphi_T : T(B) \to T(A)$ be an affine homeomorphism such that*

$$r_B(\omega)(\varphi_0(x)) = r_A(\varphi_T(\omega)(x)), \quad x \in K_0(A),\ \omega \in T(B).$$

*There exists a *-isomorphism $\varphi : A \to B$ such that $\varphi_* = \varphi_0$ on $K_0(A)$, such that $\varphi_* = \varphi_1$ on $K_1(A)$, and such that $\varphi^* = \varphi_T$ on $T(B)$.*

By [21, Theorem 9.1] this result generalises the classification result of Thomsen [21]. Restricted to the class of non-cyclic K_0-group it generalises the classification result in [18]. However, the question of whether those C^*-algebras with cyclic K_0-group, contained in the class considered in [18], are uniquely determined by their Elliott invariant, in the class considered in this thesis, is left open. In particular, the question of whether the C^*-algebra $\mathcal{Z}$ is the only infinite dimensional C^*-algebra in this class with the same Elliott invariant as $\mathbb{C}$, is left open.

I would like to thank Henning Haahr Andersen who put me on the track of the proof of Lemma 3.1, and Ebbe Thue Poulsen, who found the proof of Lemma 6.2. Thanks are also due to Jesper Villadsen for some useful discussions. But most of all, I am grateful to my advisor Klaus Thomsen for giving this problem to me and for his inspiring and enthusiastic supervision of me during the last four years.

2. Building blocks

Let $\mathbb{T}$ denote the unit circle in the complex plane. We will equip $\mathbb{T}$ with the metric

$$\rho(e^{2\pi is}, e^{2\pi it}) = \min\{\, |s-t+q| : q \in \mathbb{Z} \,\}$$

which is easily seen to be equivalent to the usual metric on $\mathbb{T}$ inherited from $\mathbb{C}$.

Recall some definitions from [19]. A tuple $(a_1, a_2, \ldots, a_L)$ of elements from $\mathbb{T}$ is cyclically numbered if there exist numbers $s_1, s_2, \ldots, s_L \in \mathbb{R}$ such that $a_j = e^{2\pi i s_j}$, $j = 1, 2, \ldots, L$, and

$$s_1 \leq s_2 \leq \cdots \leq s_L \leq s_1 + 1.$$

If furthermore $s_1, s_2, \ldots, s_L \in [0,1[$ we say that the tuple is naturally numbered.

By a building block we will understand a C^*-algebra of the form

$$A(n, d_1, d_2, \ldots, d_N) = \{f \in C(\mathbb{T}) \otimes M_n : f(x_i) \in M_{d_i},\ i = 1, 2, \ldots, N\},$$

where $N \geq 2$, $x_1, x_2, \ldots, x_N$ are (mutually different) elements of $\mathbb{T}$, $d_i | n$ for $i = 1, 2, \ldots, N$, and M_{d_i} is embedded unitally into M_n, e.g via the *-homomorphism $a \mapsto (a, a, \ldots, a)$. The points $x_1, x_2, \ldots, x_N$ will be called the exceptional points of A. We allow $d_i = n$ such that the circle algebra $C(\mathbb{T}) \otimes M_n = A(n, n, n)$ is an example of a building block. It will be convenient to always assume that $(x_1, x_2, \ldots, x_N)$ is a naturally numbered tuple and that 1 is not an exceptional point.

Building blocks will sometimes (but not always) be called circle building blocks, in order to distinguish them from interval building blocks. An interval building block is a C^*-algebra of the form

$$A(n, d_1, d_2, \ldots, d_N) = \{f \in C[0,1] \otimes M_n : f(x_i) \in M_{d_i},\ i = 1, 2, \ldots, N\},$$

where $x_1, x_2, \ldots, x_N \in [0,1]$ and $d_1, d_2, \ldots, d_N$ are positive integers dividing n. Simple unital inductive limits of finite direct sums of interval building blocks were studied by the author in [18].

For every $i = 1, 2, \ldots, N$, evaluation at x_i induces a *-homomorphism from A to M_{d_i}. This *-homomorphism will be denoted by Λ_i or sometimes Λ_i^A. The representation $\Lambda_i \oplus \Lambda_i \oplus \cdots \oplus \Lambda_i$ of A on M_{sd_i} is denoted by Λ_i^s.

The following lemmas are left as exercises.

Lemma 2.1. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. The irreducible representations of A are $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$, together with evaluation at non-exceptional points.*

Lemma 2.2. *Let I be a closed two-sided ideal in A. There is a closed set $F \subseteq \mathbb{T}$ such that*

$$I = \{f \in A : f(x) = 0 \text{ for all } x \in F\}.$$

Let $T(A)$ denote the compact convex set of tracial states on the C^*-algebra A. Let $\operatorname{Aff} T(A)$ denote the order unit space of all continuous real-valued affine functions on $T(A)$.

Lemma 2.3. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block and let $\omega \in T(A)$. There exists a Borel probability measure μ on $\mathbb{T}$ such that*

$$\omega(f) = \int_{\mathbb{T}} \tau(f(x))\, d\mu(x),$$

where τ denotes the (normalised) trace on M_n. Hence $\mathit{Aff}\,T(A)$ and $C_{\mathbb{R}}(\mathbb{T})$ are isomorphic as order unit spaces via the map

$$f \mapsto \big(\omega \mapsto \omega(f \otimes 1)\big), \quad \omega \in T(A), \ f \in C_{\mathbb{R}}(\mathbb{T}).$$

Theorem 2.4. *Let A be a finite direct sum of circle and interval building blocks. Then $A = C^*\langle G|\mathcal{R}\rangle$ for a finite set of stable relations $\mathcal{R}$ in a finite number of indeterminates.*

Proof. First note that A is a one-dimensional non-commutative CW complex, as defined in [6]. Hence A is semiprojective by [6, Theorem 6.2.2] and finitely generated by [6, Lemma 2.4.3]. Thus by [16, Theorem 14.1.4] and [6, Lemma 2.2.5] we get the desired conclusion. □

In the following we let gcd denote the greatest common divisor and lcm the least common multiple of a set of positive integers.

Let $A = A(n, d_1, d_2, \dots, d_N)$. We define

$$\begin{aligned} s(A) &= \min(d_1, d_2, \dots, d_N), \\ d(A) &= gcd(d_1, d_2, \dots, d_N). \end{aligned}$$

If $A = A_1 \oplus A_2 \oplus \cdots \oplus A_L$ is a finite direct sum of building blocks we set

$$\begin{aligned} s(A) &= \min_j s(A_j), \\ d(A) &= \min_j d(A_j). \end{aligned}$$

3. K-theory

The purpose of this section is to calculate and interpret the K-theory of a building block. We start out with the following lemma, which will be used to calculate the K_1-group.

Lemma 3.1. *Let $N \geq 2$ and let $a_1, a_2, \dots, a_N$ be positive integers. Define a group homomorphism $\varphi : \mathbb{Z}^N \to \mathbb{Z}^N$ to be multiplication with the $N \times N$ matrix*

$$C = \begin{pmatrix} a_1 & -a_2 & & & \\ & a_2 & -a_3 & & \\ & & a_3 & \ddots & \\ & & & \ddots & -a_N \\ -a_1 & & & & a_N \end{pmatrix}.$$

For $i = 1, 2, \dots, N-1$, set

$$s_i = lcm(a_1, a_2, \dots, a_i)$$

and

$$r_i = gcd(s_i, a_{i+1}) = gcd(lcm(a_1, a_2, \dots, a_i), a_{i+1}).$$

Choose integers α_i and β_i such that

$$r_i = \alpha_i s_i + \beta_i a_{i+1}, \quad i = 1, 2, \dots, N-1.$$

Then

$$coker(\varphi) \cong \mathbb{Z} \oplus \mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2} \oplus \cdots \oplus \mathbb{Z}_{r_{N-1}}.$$

This isomorphism can be chosen such that for $i = 1, 2, \ldots, N-2$, a generator of the direct summand $\mathbb{Z}_{r_i}$ is mapped to the coset

$$(\underbrace{0, 0, \ldots, 0}_{i-1\ times}, 1, -\frac{\beta_i a_{i+1}}{r_i}, \underbrace{0, 0, \ldots, 0}_{N-i-2\ times}, -\frac{\alpha_i s_i}{r_i}) + im(\varphi),$$

such that a generator of the direct summand $\mathbb{Z}_{r_{N-1}}$ is mapped to the coset

$$(0, 0, \ldots, 0, 1, -1) + im(\varphi),$$

and such that a generator of the direct summand $\mathbb{Z}$ is mapped to the coset

$$(0, 0, \ldots, 0, 1) + im(\varphi).$$

Proof. Let I_k denote the $k \times k$ identity matrix for any non-negative integer k. For each $i = 1, 2, \ldots, N-2$, define an integer matrix of size $N \times N$ by

$$A_i = \begin{pmatrix} I_{i-1} & & & & & & \\ & 1 & & & & & \\ & -\frac{\alpha_i s_i}{r_i} & 1 & & & & \\ & -\frac{\alpha_i s_i}{r_i} & & 1 & & & \\ & \vdots & & & \ddots & & \\ & -\frac{\alpha_i s_i}{r_i} & & & & 1 & \\ & 0 & & & & & 1 \end{pmatrix}.$$

Let D_i denote the 2×2 matrix

$$\begin{pmatrix} \alpha_i & \frac{a_{i+1}}{r_i} \\ -\beta_i & \frac{s_i}{r_i} \end{pmatrix},$$

and define for $i = 1, 2, \ldots, N-1$, an integer matrix of size $N \times N$ by

$$B_i = \begin{pmatrix} I_{i-1} & & \\ & D_i & \\ & & I_{N-i-1} \end{pmatrix}.$$

For $i = 0, 1, 2, \ldots, N-2$, define yet another $N \times N$ matrix by

$$X_i = \begin{pmatrix} r_1 & & & & & & & & \\ & r_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & r_i & & & & & \\ & & & & s_{i+1} & -a_{i+2} & & & \\ & & & & s_{i+1} & & -a_{i+3} & & \\ & & & & \vdots & & & \ddots & \\ & & & & s_{i+1} & & & & -a_N \\ & & & & 0 & & & & 0 \end{pmatrix}.$$

Finally, let P be the $N \times N$ matrix

$$\begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Note that for $i = 1, 2, \ldots, N-2$,

$$s_{i+1} = lcm(s_i, a_{i+1}) = \frac{s_i a_{i+1}}{r_i}.$$

Using this, it is easily seen by induction that

$$A_i A_{i-1} \cdots A_1 PCB_1 B_2 \cdots B_i = X_i \qquad i = 0, 1, 2, \ldots, N-2.$$

From this it follows that

$$A_{N-2} A_{N-1} \cdots A_1 PCB_1 B_2 \cdots B_{N-1} = \begin{pmatrix} r_1 & & & & \\ & r_2 & & & \\ & & \ddots & & \\ & & & r_{N-1} & \\ & & & & 0 \end{pmatrix}.$$

As all the matrices on the left-hand side, except C, are invertible in $M_N(\mathbb{Z})$, we obtain the desired calculation of $\operatorname{coker}(\varphi)$. Finally, it is easily verified that

$$(A_{N-2} A_{N-1} \cdots A_1 P)^{-1} = \begin{pmatrix} 1 & & & & & & \\ -\frac{\beta_1 a_2}{r_1} & 1 & & & & & \\ 0 & -\frac{\beta_2 a_3}{r_2} & 1 & & & & \\ 0 & 0 & -\frac{\beta_3 a_4}{r_3} & 1 & & & \\ \vdots & \vdots & & \ddots & \ddots & & \\ 0 & 0 & & & -\frac{\beta_{N-2} a_{N-1}}{r_{N-2}} & 1 & \\ -\frac{\alpha_1 s_1}{r_1} & -\frac{\alpha_2 s_2}{r_2} & \cdots & \cdots & -\frac{\alpha_{N-2} s_{N-2}}{r_{N-2}} & -1 & 1 \end{pmatrix}.$$

The last part of the lemma follows from this. □

Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block with exceptional points $e^{2\pi i t_i}$, $i = 1, 2, \ldots, N$, where $0 < t_1 < t_2 < \cdots < t_N < 1$. Set $t_{N+1} = t_1 + 1$.

Define continuous functions $\omega_i : \mathbb{T} \to \mathbb{T}$ for $i = 1, 2, \ldots, N$, by

$$\omega_i(e^{2\pi i t}) = \begin{cases} exp(2\pi i \frac{t - t_i}{t_{i+1} - t_i}) & t_i \le t \le t_{i+1}, \\ 1 & t_{i+1} \le t \le t_i + 1. \end{cases}$$

Let U_i be the unitary in A defined by

$$U_i(z) = \operatorname{diag}(\omega_i(z), 1, 1, \ldots, 1), \qquad z \in \mathbb{T}.$$

Theorem 3.2. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. Set for $i = 1, 2, \ldots, N-1$,*

$$s_i = lcm(\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_i}),$$

and

$$r_i = gcd(s_i, \frac{n}{d_{i+1}}) = gcd(lcm(\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_i}), \frac{n}{d_{i+1}}).$$

Choose integers α_i and β_i such that

$$r_i = \alpha_i s_i + \beta_i \frac{n}{d_{i+1}}, \quad i = 1, 2, \ldots, N-1.$$

Then

$$K_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2} \oplus \cdots \oplus \mathbb{Z}_{r_{N-1}}.$$

This isomorphism can be chosen such that for $i = 1, 2, \dots, N-2$, a generator of the direct summand $\mathbb{Z}_{r_i}$ is mapped to

$$[U_i] - \frac{\beta_i n}{r_i d_{i+1}}[U_{i+1}] - \frac{\alpha_i s_i}{r_i}[U_N],$$

such that a generator of the direct summand $\mathbb{Z}_{r_{N-1}}$ is mapped to

$$[U_{N-1}] - [U_N],$$

and such that a generator of the direct summand $\mathbb{Z}$ is mapped to $[U_N]$.

Proof. Define a *-homomorphism $\pi : A \to M_{d_1} \oplus M_{d_2} \oplus \cdots \oplus M_{d_N}$ by

$$\pi(f) = (\Lambda_1(f), \Lambda_2(f), \dots, \Lambda_N(f)).$$

Under the identification $SM_n \cong C_0(0,1) \otimes M_n$ we define a *-homomorphism $\iota : (SM_n)^N \to A$ by

$$\iota(f_1, f_2, \dots, f_N)(e^{2\pi i t}) = f_i(\frac{t - t_i}{t_{i+1} - t_i}), \qquad t_i \leq t \leq t_{i+1}.$$

The short exact sequence

$$0 \longrightarrow (SM_n)^N \xrightarrow{\iota} A \xrightarrow{\pi} M_{d_1} \oplus M_{d_2} \oplus \cdots \oplus M_{d_N} \longrightarrow 0$$

induces a six-term exact sequence

$$\begin{tikzcd} K_0((SM_n)^N) \arrow[r, "\iota_*"] & K_0(A) \arrow[r, "\pi_*"] & K_0(M_{d_1} \oplus \cdots \oplus M_{d_N}) \arrow[d, "\delta"] \\ K_1(M_{d_1} \oplus \cdots \oplus M_{d_N}) \arrow[u] & K_1(A) \arrow[l, "\pi_*"] & K_1((SM_n)^N) \arrow[l, "\iota_*"] \end{tikzcd}$$

where δ denotes the exponential map.

By Bott periodicity $K_1((SM_n)^N) \cong \mathbb{Z}^N$ is generated by $[V_1], [V_2], \dots, [V_N]$, where

$$V_j(t) = (1, 1, \dots, 1, \underbrace{\operatorname{diag}(e^{2\pi i t}, 1, \dots, 1)}_{\text{coordinate } j}, 1, 1, \dots, 1), \qquad t \in [0,1].$$

Note that $\tilde{\iota}(V_j) = U_j$. As $K_1(M_{d_1} \oplus \cdots \oplus M_{d_N}) = 0$ it follows that $K_1(A)$ is generated by $[U_1], \dots, [U_N]$, and that ι_* gives rise to an isomorphism between the cokernel of δ and $K_1(A)$.

Let $\{e^k_{ij}\}$ denote the standard matrix units in $M_{d_1} \oplus \cdots \oplus M_{d_N}$. Recall that $K_0(M_{d_1} \oplus \cdots \oplus M_{d_N}) \cong \mathbb{Z}^N$ is generated by $[e^1_{11}], [e^2_{11}], \dots, [e^N_{11}]$. We leave it with the reader to check that

$$\delta([e^1_{11}]) = -\frac{n}{d_1}[V_N] + \frac{n}{d_1}[V_1],$$

and for $i = 2, 3, \dots, N$,

$$\delta([e^i_{11}]) = -\frac{n}{d_i}[V_{i-1}] + \frac{n}{d_i}[V_i].$$

The conclusion follows from Lemma 3.1. □

For a building block A as above, choose a continuous function $\alpha : \mathbb{T} \to \mathbb{R}$ such that

$$Det(U_N(z)) = z \exp(2\pi i \alpha(z)), \quad z \in \mathbb{T}.$$

Define the canonical unitary V_A in A by

$$V_A(z) = U_N(z)\exp(-2\pi i\frac{\alpha(z)}{n}), \quad z \in \mathbb{T}.$$

By the above theorem $[V_A]$ generates the direct summand $\mathbb{Z}$ of $K_1(A)$. Note that $Det(V_A(z)) = z$, $z \in \mathbb{T}$.

Lemma 3.3. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block and let $U \in A$ be a unitary. Assume that*

$$Det(\Lambda_i(U)) = 1, \quad i = 1, 2, \ldots, N,$$
$$Det(U(z)) = \exp(2\pi i\gamma(z)), \quad z \in \mathbb{T},$$

where $\gamma : \mathbb{T} \to \mathbb{R}$ is a continuous function that equals 0 at all the exceptional points of A. Then U is trivial in $K_1(A)$.

Proof. First note that if W is a unitary in a circle algebra B, if $Det(W(\cdot))$ has winding number 0, and if $W(z) = 1$ on some arc $I \subseteq \mathbb{T}$, then W is homotopic to 1 via a path $(W_t)_{t\in[0,1]}$ of unitaries in B with $W_t(z) = 1$ for $z \in I$.

Let $x_1, x_2, \ldots, x_N$ denote the exceptional points of A. As the group of unitaries in M_{d_i} with determinant 1 is arcwise connected, there exist unitaries $V_i \in C(\mathbb{T}) \otimes M_{d_i} \subseteq C(\mathbb{T}) \otimes M_n$ such that $Det(V_i(z)) = 1$, $z \in \mathbb{T}$, such that $V_i(x_i) = U(x_i)$, and such that V_i equals 1 on some arc I_i that contains the remaining exceptional points of A. By the above remark, V_i is homotopic to 1 in $C(\mathbb{T}) \otimes M_{d_i}$ via a path of unitaries that equal 1 on I_i. Hence V_i is homotopic to 1 within the unitary group of A. Set

$$V = UV_1^*V_2^* \ldots V_N^*.$$

Then $Det(V(z)) = \exp(2\pi i\gamma(z))$, $z \in \mathbb{T}$, $V(x_i) = 1$, $i = 1, 2, \ldots, N$, and $[U] = [V]$ in $K_1(A)$.

Now let $x_i = e^{2\pi i t_i}$, where $0 < t_1 < t_2 < \cdots < t_N < 1$. Set $t_{N+1} = t_1 + 1$ and let for $i = 1, 2, \ldots, N$,

$$J_i = \{e^{2\pi i t} : t \in [t_i, t_{i+1}]\}.$$

Define a unitary Y_i by

$$Y_i(z) = \begin{cases} V(z) & z \in J_i, \\ 1 & \text{otherwise,} \end{cases}$$

and a continuous function $\gamma_i : \mathbb{T} \to \mathbb{R}$ by

$$\gamma_i(z) = \begin{cases} \gamma(z) & z \in J_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $V = Y_1Y_2 \ldots Y_N$, and $Det(Y_i(z)) = \exp(2\pi i\gamma_i(z))$, $z \in \mathbb{T}$. Again by the above remark, Y_i is homotopic to 1 in $C(\mathbb{T}) \otimes M_n$ via a path of unitaries that equal 1 on $\mathbb{T}\backslash J_i^\circ$. It follows that Y_i is homotopic to 1 within the unitary group of A. Hence $[V] = [1]$ in $K_1(A)$. □

Theorem 3.4. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block with exceptional points $x_1, x_2, \ldots, x_N$. Let $W \in A$ be a unitary such that $Det(\Lambda_i(W)) = 1$, $i = 1, 2, \ldots, N$. Assume that W is trivial in $K_1(A)$. Then $Det(W(z)) = e^{2\pi i\gamma(z)}$ for a continuous function $\gamma : \mathbb{T} \to \mathbb{R}$ that satisfies*

$$\gamma(x_i) \equiv 0 \quad mod\ \frac{n}{d_i}, \qquad i = 1, 2, \ldots, N.$$

Proof. As W is trivial in $K_1(A)$, $Det(W(\cdot))$ is homotopic to a constant loop. Thus there exists a continuous function $\gamma : \mathbb{T} \to \mathbb{R}$ such that $Det(W(z)) = e^{2\pi i \gamma(z)}$, $z \in \mathbb{T}$.

Define s_i, r_i, α_i, and β_i as in Theorem 3.2. Set

$$\begin{aligned} V_i &= U_i \, U_{i+1}^{-\frac{\beta_i n}{r_i d_{i+1}}} \, U_N^{-\frac{\alpha_i s_i}{r_i}}, \qquad i = 1, 2, \ldots, N-2, \\ V_{N-1} &= U_{N-1} U_N^{-1}. \end{aligned}$$

Set

$$\begin{aligned} t_1 &= \gamma(x_2) - \gamma(x_1), \\ t_i &= \gamma(x_{i+1}) - \gamma(x_i) + \frac{\beta_{i-1} n}{r_{i-1} d_i} t_{i-1}, \qquad i = 2, 3, \ldots, N-1. \end{aligned}$$

By induction, we see that for $k = 1, 2, \ldots, N-2$,

$$V_1^{t_1} \, V_2^{t_2} \cdots V_k^{t_k} = U_1^{\gamma(x_2) - \gamma(x_1)} \cdots U_k^{\gamma(x_{k+1}) - \gamma(x_k)} \, U_{k+1}^{-\frac{\beta_k n t_k}{r_k d_{k+1}}} \, U_N^{-\sum_{i=1}^{k} \frac{\alpha_i s_i t_i}{r_i}}.$$

As

$$\sum_{i=1}^{N-1} t_i = \gamma(x_N) - \gamma(x_1) + \sum_{i=1}^{N-2} \frac{\beta_i n}{r_i d_{i+1}} \, t_i$$

we see that

$$\gamma(x_N) - \gamma(x_1) = t_{N-1} + \sum_{i=1}^{N-2} (-\frac{\beta_i n}{r_i d_{i+1}} + 1) \, t_i = t_{N-1} + \sum_{i=1}^{N-2} \frac{\alpha_i s_i t_i}{r_i}.$$

Thus

$$V_1^{t_1} \, V_2^{t_2} \ldots V_{N-1}^{t_{N-1}} = U_1^{\gamma(x_2) - \gamma(x_1)} \cdots U_{N-1}^{\gamma(x_N) - \gamma(x_{N-1})} \, U_N^{\gamma(x_1) - \gamma(x_N)}.$$

By Lemma 3.3, this unitary equals W in $K_1(A)$. Hence by Theorem 3.2 there exist integers $l_1, l_2, \ldots, l_{N-1}$, such that $t_i = r_i l_i$, $i = 1, 2, \ldots, N-1$. Set

$$q = \gamma(x_1) + \sum_{i=1}^{N-1} \alpha_j l_j s_j.$$

By induction,

$$\gamma(x_i) = \beta_{i-1} l_{i-1} \frac{n}{d_i} - \sum_{j=i}^{N-1} \alpha_j l_j s_j + q, \quad i = 2, 3, \ldots, N.$$

Hence if we substitute $\gamma - q$ for γ we obtain the desired conclusion. □

Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block and set $d = d(A)$. Since $d(A)$ divides d_i for every $i = 1, 2, \ldots, N$, we have an injective and unital *-homomorphism $\epsilon : M_d \to A$ given by $\epsilon(f) = \mathrm{diag}(f, f, \ldots, f)$.

Lemma 3.5. *Let p be a projection in $A = A(n, d_1, d_2, \ldots, d_N)$. Then p is unitarily equivalent to a projection in $im(\epsilon) \subseteq A$. Hence if p has rank $r \in \mathbb{Z}$, $r \neq 0$, then $pAp \cong A(r, \frac{r}{n} d_1, \frac{r}{n} d_2, \ldots, \frac{r}{n} d_N)$.*

Proof. Let $t_1, t_2, \ldots, t_N \in]0,1[$ be numbers such that $e^{2\pi i t_i}$ are the exceptional points of A, $i = 1, 2, \ldots, N$. Set

$$B = \{ f \in C[0,1] \otimes M_n : f(0) = f(1), \ f(t_i) \in M_{d_i}, \ i = 1, 2, \ldots, N \}.$$

Then the map $\varphi : A \to B$, $\varphi(f)(t) = f(e^{2\pi i t})$, is an isomorphism. For convenience, we will prove the lemma for B.

Set $d = d(A)$. As $\frac{n}{d_i}$ divides r for $i = 1, 2, \dots, N$, it follows that $\frac{n}{d}$ also divides r. Hence there is a projection $e \in M_d \subseteq A$ with the same trace as p.

For each $t \in [0, 1]$ there is a unitary $u_t \in M_n$ such that

$$e = u_t p(t) {u_t}^*.$$

We may assume that $u_{x_i} \in M_{d_i}$, $i = 1, 2, \dots, N$, and that $u_0 = u_1$. By compactness, write

$$[0,1] = \bigcup_{k=1}^{L-1} [s_k, s_{k+1}]$$

where $0 = s_1 < s_2 < \cdots < s_L = 1$, $\{x_1, x_2, \dots, x_N\} \subseteq \{s_1, s_2, \dots, s_L\}$, and

$$t \in [s_k, s_{k+1}] \Rightarrow \|u_{s_k}\, p(t)\, {u_{s_k}}^* - e\| < 1.$$

Set $z_k(t) = v_k(t)|v_k(t)|^{-1}$ for $t \in [s_k, s_{k+1}]$, $k = 1, 2, \dots, L-1$, where

$$v_k(t) = 1 - u_{s_k}\, p(t)\, {u_{s_k}}^* - e + 2\, e\, u_{s_k}\, p(t)\, {u_{s_k}}^*.$$

Then $t \mapsto z_k(t)$, $t \in [s_k, s_{k+1}]$, is a path of unitaries in M_n, and by [17, Lemma 6.2.1]

$$e = z_k(t)\, u_{s_k}\, p(t)\, {u_{s_k}}^*\, z_k(t)^*, \qquad t \in [s_k, s_{k+1}].$$

As $U(M_n) \cap \{e\}'$ is path-connected there is, for each $k = 1, 2, \dots, L-1$, a continuous map $\gamma_k : [s_k, s_{k+1}] \to U(M_n) \cap \{e\}'$ such that

$$\gamma_k(s_k) = 1, \qquad \gamma_k(s_{k+1}) = u_{s_{k+1}}\, {u_{s_k}}^*\, z_k(s_{k+1})^*.$$

Since $z_k(s_k) = 1$ for $k = 1, 2, \dots, L-1$, we can define a unitary $u \in B$ by

$$u(t) = \gamma_k(t) z_k(t) u_{s_k}, \qquad t \in [s_k, s_{k+1}].$$

Then $upu^* = e$. □

Corollary 3.6. *The *-homomorphism $\epsilon : M_d \to A$ induces an isomorphism $\epsilon_* : K_0(M_d) \to K_0(A)$ of ordered groups with order unit. In particular,*

$$\big(K_0(A), K_0(A)^+, [1]\big) \cong \big(\mathbb{Z}, \mathbb{Z}^+, d(A)\big).$$

Lemma 3.7. *$A(n, d_1, d_2, \dots, d_N)$ is unital projectionless if and only if $d(A) = 1$.*

Proof. As in the proof of Lemma 3.5 we see that there exists a projection $p \in A$ of rank r if and only if $\frac{n}{d(A)}$ divides r. The conclusion follows. □

Lemma 3.8. *Let d and K be positive integers and let H be a finite abelian group. There exists a building block A with $(K_0(A), K_0(A)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$, $K_1(A) \cong \mathbb{Z} \oplus H$, and $s(A) \geq K$.*

Proof. Let

$$H \cong \mathbb{Z}_{p_1^{k_1}} \oplus \mathbb{Z}_{p_2^{k_2}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{k_m}},$$

where m is a positive integer, $k_1, \dots, k_m$ are non-negative integers, and $p_1, \dots, p_m$ are prime numbers. Let $q_1, q_2, \dots, q_{m+1} \geq K$ be prime numbers, mutually different as well as different from $p_1, p_2, \ldots, p_m$. Define integers n and $d_1, d_2, \ldots, d_{m+1}$ by

$$\begin{aligned} n &= d\, p_1^{k_1}\, p_2^{k_2} \ldots p_m^{k_m}\, q_1\, q_2 \ldots q_{m+1}, \\ d_1 &= d\, q_2\, q_3 \ldots q_{m+1}, \\ d_i &= d\, \frac{p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}}{p_{i-1}^{k_{i-1}}}\, \frac{q_1 q_2 \ldots q_{m+1}}{q_i}, \quad 2 \leq i \leq m+1. \end{aligned}$$

Set $A = A(n, d_1, d_2, \ldots, d_{m+1})$. Then $K_1(A) \cong \mathbb{Z} \oplus H$ by Theorem 3.2. And by Corollary 3.6 we have that $(K_0(A), K_0(A)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$. □

4. KK-THEORY

KK-theory seems to be an indispensable tool when it comes to classification of simple inductive limits of C^*-algebras with dimension drops, cf [9], [13], [21]. This paper is no exception.

Recall a few facts about KK-theory. KK is a homotopy invariant bifunctor from the category of C^*-algebras to the category of abelian groups that is contravariant in the first variable and covariant in the second. A *-homomorphism $\varphi : A \to M_n(B)$ defines an element $[\varphi] \in KK(A, B)$. We have an associative map $KK(B, C) \times KK(A, B) \to KK(A, C)$, the Kasparov product, that generalises composition of *-homomorphisms.

We will need the K-homology groups $K^0(A) = KK(A, \mathbb{C})$. If $\varphi : A \to B$ is a *-homomorphism, φ induces a group homomorphism $\varphi^* : K^0(B) \to K^0(A)$ via the Kasparov product. $K^0(M_n) \cong \mathbb{Z}$ is generated by the class of the identity map on M_n.

Lemma 4.1. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be an interval building block. Then $K^0(A)$ is generated by $[\Lambda_1], [\Lambda_2], \ldots, [\Lambda_N]$. For $a_1, a_2, \ldots, a_N \in \mathbb{Z}$ we have that*

$$\begin{aligned} a_1[\Lambda_1] + a_2[\Lambda_2] + \cdots + a_N[\Lambda_N] = 0 \ \Leftrightarrow \ & \exists b_1, \ldots, b_{N-1} \in \mathbb{Z} : \\ & a_1 = -b_1 \frac{n}{d_1}, \\ & \ldots \\ & a_{N-1} = -b_{N-1} \frac{n}{d_{N-1}}, \\ & a_N = (b_1 + \cdots + b_{N-1}) \frac{n}{d_N}. \end{aligned}$$

Finally, $K^0(A) \cong \mathbb{Z} \oplus K_1(A)$.

Proof. See [18, Lemma 3.4]. □

Lemma 4.2. *Let $A = A(n, d_1, \ldots, d_N)$ be a building block with exceptional points $x_1, x_2, \ldots, x_N \in \mathbb{T}$. Choose $t_i \in]0, 1[$ such that $e^{2\pi i t_i} = x_i$, $i = 1, 2, \ldots, N$. Let*

$$B = \{f \in C[0, 1] \otimes M_n : f(t_i) \in M_{d_i},\ i = 1, 2, \ldots, N\}.$$

*The *-homomorphism $\iota : A \to B$ defined by $\iota(f)(t) = f(e^{2\pi i t})$, $t \in [0, 1]$, $f \in A$, induces an isomorphism $\iota^* : K^0(B) \to K^0(A)$.*

Proof. Let $\pi : A \to M_n$ be evaluation at $1 \in \mathbb{T}$. Let $\alpha : M_n \to M_n \oplus M_n$ denote the map $\alpha(x) = (x, x)$. Let $\beta : B \to M_n \oplus M_n$ be the map $\beta(f) = (f(0), f(1))$. We have a pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & M_n \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ B & \xrightarrow[\beta]{} & M_n \oplus M_n. \end{array}$$

By [2, Theorem 21.5.1] this induces a six-term exact sequence

$$\begin{array}{ccccc} K^0(M_n \oplus M_n) & \xrightarrow{(-\alpha^*,\beta^*)} & K^0(M_n) \oplus K^0(B) & \xrightarrow{\pi^*+\iota^*} & K^0(A) \\ \big\uparrow & & & & \big\downarrow \\ K^1(A) & \xleftarrow[\pi^*+\iota^*]{} & K^1(M_n) \oplus K^1(B) & \xleftarrow[(-\alpha^*,\beta^*)]{} & K^1(M_n). \end{array}$$

$K^0(M_n \oplus M_n) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[\pi_1]$ and $[\pi_2]$ where $\pi_1, \pi_2 : M_n \oplus M_n \to M_n$ are the coordinate projections. $K^0(M_n) \cong \mathbb{Z}$ is generated by the class of the identity map id on M_n. Note that

$$\pi^*([id]) = \frac{n}{d_1}[\Lambda_1^A],$$
$$\iota^*([\Lambda_i^B]) = [\Lambda_i^A], \qquad i = 1, 2, \ldots, N,$$
$$(-\alpha^*, \beta^*)(a[\pi_1] + b[\pi_2]) = (-(a+b)[id], (a+b)\frac{n}{d_1}[\Lambda_1^B]).$$

As $\pi^* + \iota^*$ maps onto $K^0(A)$ (because $K^1(M_n) = 0$) and as $im(\pi^*) \subseteq im(\iota^*)$, we see that ι^* is surjective. Assume that $\iota^*(x) = 0$. Then $(0, x) \in im(-\alpha^*, \beta^*)$ and hence $x = 0$ by the above. □

Corollary 4.3. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a circle building block. Then $K^0(A)$ is generated by $[\Lambda_1], [\Lambda_2], \ldots, [\Lambda_N]$. For $a_1, a_2, \ldots, a_N \in \mathbb{Z}$ we have that*

$$\begin{aligned} a_1[\Lambda_1] + a_2[\Lambda_2] + \cdots + a_N[\Lambda_N] = 0 \ \Leftrightarrow \ & \exists b_1, \ldots, b_{N-1} \in \mathbb{Z} : \\ & a_1 = -b_1 \frac{n}{d_1}, \\ & \cdots \\ & a_{N-1} = -b_{N-1} \frac{n}{d_{N-1}}, \\ & a_N = (b_1 + \cdots + b_{N-1}) \frac{n}{d_N}. \end{aligned}$$

Finally, $K^0(A) \cong K_1(A)$.

Proof. Combine Lemma 4.1 and Lemma 4.2 to get the first statement. The second statement follows from the universal coefficient theorem, [20, Theorem 1.17]. □

Lemma 4.4. *Let $A = A(n, d_1, d_2, \ldots, d_N)$, $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks. Let $h \in Hom(K^0(B), K^0(A))$. For every $j = 1, 2, \ldots, M$, $i = 1, 2, \ldots, N$, there is a uniquely determined integer h_{ji}, with $0 \leq h_{ji} < \frac{n}{d_i}$ for $i \neq N$, such that*

$$\begin{pmatrix} h([\Lambda_1^B]) \\ h([\Lambda_2^B]) \\ \vdots \\ h([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \ldots & h_{1N} \\ h_{21} & h_{22} & \ldots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \ldots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

This will be called the standard form for h.

The integers determined by h above satisfy the equations

$$\frac{m}{e_j}h_{ji} \equiv \frac{m}{e_M}h_{Mi} \quad mod \quad \frac{n}{d_i}, \quad j = 1, 2, \ldots, M, \; i = 1, 2, \ldots, N,$$

$$\frac{m}{e_j}\sum_{i=1}^{N} h_{ji}d_i = \frac{m}{e_M}\sum_{i=1}^{N} h_{Mi}d_i, \quad j = 1, 2, \ldots, M.$$

Proof. By Corollary 4.3, or simply because homotopic *-homomorphisms $A \to M_n$ define the same elements in $K^0(A)$,

$$\frac{n}{d_N}[\Lambda_N^A] = \frac{n}{d_i}[\Lambda_i^A], \quad i = 1, 2, \ldots, N.$$

From this the existence follows.

To check uniqueness, assume

$$\begin{pmatrix} h_{11} & h_{12} & \ldots & h_{1N} \\ h_{21} & h_{22} & \ldots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \ldots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix} = 0$$

where

$$-\frac{n}{d_i} < h_{ji} < \frac{n}{d_i}, \quad i = 1, 2, \ldots, N-1, \; j = 1, 2, \ldots, M.$$

By Corollary 4.3 there are, for $j = 1, 2, \ldots, M$, integers b_{ji}, $i = 1, 2, \ldots, N-1$, such that

$$\begin{aligned} h_{j1} &= -b_{j1}\frac{n}{d_1}, \\ &\ldots \\ h_{j(N-1)} &= -b_{j(N-1)}\frac{n}{d_{N-1}}, \\ h_{jN} &= (b_{j1} + \cdots + b_{j(N-1)})\frac{n}{d_N}. \end{aligned}$$

Therefore $h_{j1} = h_{j2} = \cdots = h_{j(N-1)} = 0$ and hence $h_{jN} = 0$.

Finally, for each $j = 1, 2, \ldots, M$ we have that

$$0 = h(0) = h(-\frac{m}{e_j}[\Lambda_j^B] + \frac{m}{e_M}[\Lambda_M^B]) = \sum_{i=1}^{N}(-\frac{m}{e_j}h_{ji} + \frac{m}{e_M}h_{Mi})[\Lambda_i^A].$$

Hence there exist integers b_{ji}, $i = 1, 2, \ldots, N-1$, such that

$$-\frac{m}{e_j}h_{ji} + \frac{m}{e_M}h_{Mi} = -b_{ji}\frac{n}{d_i}, \quad i = 1, 2, \ldots, N-1,$$

and

$$-\frac{m}{e_j}h_{jN} + \frac{m}{e_M}h_{MN} = (b_{j1} + \cdots + b_{j(N-1)})\frac{n}{d_N}.$$

The desired conclusion follows easily from these equations. □

From now on, let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks. Define a group homomorphism

$$\Gamma : KK(A, B) \to Hom(K^0(B), K^0(A)) \oplus K_1(B)$$

by

$$\Gamma(\kappa) = (\kappa^*, \kappa_*([V_A]))$$

and let $\Gamma_1 : KK(A,B) \to Hom(K^0(B), K^0(A))$ be the projection onto the first coordinate, i.e $\Gamma_1(\kappa) = \kappa^*$. We want to show that Γ is an isomorphism when $s(B)$ is large.

Proposition 4.5. *Let $h : K^0(B) \to K^0(A)$ be a group homomorphism with standard form*

$$\begin{pmatrix} h([\Lambda_1^B]) \\ h([\Lambda_2^B]) \\ \vdots \\ h([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \dots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}$$

*where $h_{jN} \geq \frac{n}{d_N}$ for $j = 1, 2, \dots, M$, and $\sum_{i=1}^N h_{Mi} d_i = e_M$. Let $\chi \in K_1(B)$. There is a unital *-homomorphism $\varphi : A \to B$ such that $\Gamma([\varphi]) = (h, \chi)$.*

Proof. Let $1 \leq i \leq N$. By Lemma 4.4 there is an integer s_i, $0 \leq s_i < \frac{n}{d_i}$, and integers l_{ji}, $j = 1, 2, \dots, M$, such that

$$\frac{m}{e_j} h_{ji} = l_{ji} \frac{n}{d_i} + s_i. \tag{1}$$

Note that for each j, $l_{ji} \geq 0$ for $i = 1, 2, \dots, N-1$, and $l_{jN} \geq 1$. By Lemma 4.4 we see that for $j = 1, 2, \dots, M$,

$$m = \frac{m}{e_M} \sum_{i=1}^N h_{Mi} d_i = \frac{m}{e_j} \sum_{i=1}^N h_{ji} d_i = \sum_{i=1}^N (l_{ji} n + s_i d_i).$$

By (1) there exists a unitary $V_j \in M_m$ such that the matrix

$$V_j \mathrm{diag}\big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), \underbrace{f(x_1), \dots, f(x_1)}_{l_{j1} \text{ times}}, \dots, \underbrace{f(x_N), \dots, f(x_N)}_{l_{jN} \text{ times}}\big) V_j^*$$

belongs to $M_{e_j} \subseteq M_m$ for all $f \in A$.

Set

$$L = \frac{1}{n}(m - \sum_{i=1}^N s_i d_i) = \sum_{i=1}^N l_{ji}, \quad j = 1, 2, \dots, M.$$

Let $x_1, x_2, \dots, x_N$ denote the exceptional points of A and let $y_1, y_2, \dots, y_M$ be those of B. Choose continuous functions $\lambda_1, \lambda_2, \dots, \lambda_{L-1} : \mathbb{T} \to \mathbb{T}$ such that

$$\big(\lambda_1(y_j), \lambda_2(y_j), \dots, \lambda_{L-1}(y_j)\big) = \big(\underbrace{x_1, \dots, x_1}_{l_{j1} \text{ times}}, \dots, \underbrace{x_{N-1}, \dots, x_{N-1}}_{l_{j(N-1)} \text{ times}}, \underbrace{x_N, \dots, x_N}_{l_{jN}-1 \text{ times}}\big)$$

as ordered tuples. Choose a unitary $U \in C(\mathbb{T}) \otimes M_m$ such that $U(y_j) = V_j$. Define a unital *-homomorphism $\psi : A \to B$ by

$$\psi(f)(z) = U(z) \mathrm{diag}\big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), f(\lambda_1(z)), \dots, f(\lambda_{L-1}(z)), f(x_N)\big) U(z)^*.$$

By Theorem 3.2 we have that $\chi = \sum_{j=1}^M a_j [U_j^B]$ for some $a_1, a_2, \dots, a_M \in \mathbb{Z}$. Let

$$\psi_*[V_A] = \sum_{i=1}^M b_j [U_j^B]$$

in $K_1(B)$. Define $\xi : \mathbb{T} \to \mathbb{T}$ by

$$\xi(z) = \prod_{j=1}^{M} Det(U_j^B(z))^{a_j - b_j},$$

and define $\lambda_L : \mathbb{T} \to \mathbb{T}$ by $\lambda_L(z) = \xi(z)x_N$. Now define $\varphi : A \to B$ by

$$\varphi(f)(z) = U(z)\mathrm{diag}\big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), f(\lambda_1(z)), \dots, f(\lambda_L(z))\big)U(z)^*.$$

Then by Lemma 3.3, we see that in $K_1(B)$,

$$\begin{aligned}\varphi_*[V_A] &= \psi_*[V_A] + [z \mapsto U(z)\mathrm{diag}(1, 1, \dots, 1, V_A(\lambda_L(z))V_A(x_N)^*)U(z)^*] \\ &= \psi_*[V_A] + \sum_{j=1}^{M}(a_j - b_j)[U_j^B] = \sum_{j=1}^{M} a_j[U_j^B].\end{aligned}$$

As $\lambda_L(y_j) = x_N$ we see that $\varphi(f)(y_j) = \psi(f)(y_j)$, $f \in A$, $j = 1, 2, \dots, M$. Thus

$$\varphi^*([\Lambda_j^B]) = [\Lambda_j^B \circ \varphi] = [\Lambda_j^B \circ \psi] = \sum_{i=1}^{N}(s_i + l_{ji}\frac{n}{d_i})\frac{e_j}{m}[\Lambda_i^A] = \sum_{i=1}^{N} h_{ji}[\Lambda_i^A].$$

□

Proposition 4.6. *Assume that there exists a homomorphism $h' : K^0(B) \to K^0(A)$ with standard form*

$$\begin{pmatrix} h'([\Lambda_1^B]) \\ h'([\Lambda_2^B]) \\ \vdots \\ h'([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h'_{11} & h'_{12} & \dots & h'_{1N} \\ h'_{21} & h'_{22} & \dots & h'_{2N} \\ \vdots & \vdots & & \vdots \\ h'_{M1} & h'_{M2} & \dots & h'_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}$$

where $h'_{jN} \geq \frac{n}{d_N}$ for $j = 1, 2, \dots, M$, and $\sum_{i=1}^{N} h'_{Mi}d_i = e_M$. Then the map $\Gamma_1 : KK(A, B) \to Hom(K^0(B), K^0(A))$ is surjective.

Proof. $h' \in im(\Gamma_1)$ by Proposition 4.5. Let $h \in Hom(K^0(B), K^0(A))$ have standard form

$$\begin{pmatrix} h([\Lambda_1^B]) \\ h([\Lambda_2^B]) \\ \vdots \\ h([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \dots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

By adding an integer-multiple of h' we may assume that $h_{jN} \geq 0$ for $j = 1, 2, \dots, M$. Define l_{ji} and s_i, $i = 1, 2, \dots, N$, as in the proof of Proposition 4.5. Let

$$c = \frac{m}{e_M}\sum_{i=1}^{N} h_{Mi}d_i = \frac{m}{e_j}\sum_{i=1}^{N} h_{ji}d_i = \sum_{i=1}^{N}(l_{ji}n + s_i d_i), \quad j = 1, 2, \dots, M.$$

Choose a positive integer d such that $c \leq dm$. Then there is for each $j = 1, 2, \dots, M$, a unitary $V_j \in M_{dm}$ such that the matrix

$$V_j\mathrm{diag}\big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), \underbrace{f(x_1), \dots, f(x_1)}_{l_{j1}\text{ times}}, \dots, \underbrace{f(x_N), \dots, f(x_N)}_{l_{jN}\text{ times}}, \underbrace{0, \dots, 0}_{dm-c}\big)V_j^*$$

belongs to $M_{de_j} \subseteq M_{dm}$ for all $f \in A$.

As in the proof of Proposition 4.5 these matrices can be connected to give a *-homomorphism $\varphi : A \to M_d(B)$. We leave it with the reader to check that $\varphi^* = h$ on $K^0(B)$. $\square$

Theorem 4.7. *Assume that there exists a homomorphism $h' : K^0(B) \to K^0(A)$ with standard form*

$$\begin{pmatrix} h'([\Lambda_1^B]) \\ h'([\Lambda_2^B]) \\ \vdots \\ h'([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h'_{11} & h'_{12} & \dots & h'_{1N} \\ h'_{21} & h'_{22} & \dots & h'_{2N} \\ \vdots & \vdots & & \vdots \\ h'_{M1} & h'_{M2} & \dots & h'_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}$$

where $h'_{jN} \geq \frac{n}{d_N}$ for $j = 1, 2, \dots, M$, and $\sum_{i=1}^N h'_{Mi} d_i = e_M$. Then the map $\Gamma : KK(A, B) \to Hom(K^0(B), K^0(A)) \oplus K_1(B)$ is an isomorphism.

Proof. By Theorem 3.2 there exist finite abelian groups G and H such that $K_1(A) \cong \mathbb{Z} \oplus G$, $K_1(B) \cong \mathbb{Z} \oplus H$. According to the universal coefficient theorem, [20, Theorem 1.17],

$$\begin{aligned} KK(A,B) &\cong Ext(K_0(A), K_1(B)) \oplus Ext(K_1(A), K_0(B)) \oplus \\ &\qquad Hom(K_0(A), K_0(B)) \oplus Hom(K_1(A), K_1(B)) \\ &\cong 0 \oplus G \oplus \mathbb{Z} \oplus Hom(G, H) \oplus K_1(B). \end{aligned}$$

As $K^0(A) \cong K_1(A)$, $K^0(B) \cong K_1(B)$ by Corollary 4.3, we also have that

$$Hom(K^0(B), K^0(A)) \oplus K_1(B) \cong K_1(A) \oplus Hom(H, G) \oplus K_1(B).$$

Hence $Hom(K^0(B), K^0(A)) \oplus K_1(B)$ and $KK(A, B)$ are isomorphic groups. As any surjective endomorphism of a finitely generated abelian group is an isomorphism, it suffices to show that Γ is surjective.

Let therefore $(h, \chi) \in Hom(K^0(B), K^0(A)) \oplus K_1(B)$. By Proposition 4.6 there exists a $\kappa \in KK(A, B)$ such that $\Gamma(\kappa) = (h - h', \eta)$ for some $\eta \in K_1(B)$. Next, by Proposition 4.5 there exists a $\beta \in KK(A, B)$ such that $\Gamma(\beta) = (h', \chi - \eta)$. Thus $\Gamma(\kappa + \beta) = (h, \chi)$. $\square$

Theorem 4.8. *Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks such that $s(B) \geq Nn$. Let $\kappa \in KK(A, B)$ be an element such that $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit. Then the map $\Gamma : KK(A, B) \to Hom(K^0(B), K^0(A)) \oplus K_1(B)$ is an isomorphism and there exists a unital *-homomorphism $\varphi : A \to B$ such that $[\varphi] = \kappa$.*

Proof. Let $\kappa^* : K^0(B) \to K^0(A)$ have standard form

$$\begin{pmatrix} \kappa^*([\Lambda_1^B]) \\ \kappa^*([\Lambda_2^B]) \\ \vdots \\ \kappa^*([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \dots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

Let $\cdot$ denote the Kasparov product. By assumption we have that $[1_A] \cdot \kappa = [1_B] \in KK(\mathbb{C}, B) \cong K_0(B)$. Thus

$$[1_B] \cdot [\Lambda_j^B] = [1_A] \cdot \kappa \cdot [\Lambda_j^B] = [1_A] \cdot (\sum_{i=1}^N h_{ji} [\Lambda_i^A])$$

in $KK(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$. Hence for $j = 1, 2, \ldots, M$,

$$e_j = \sum_{i=1}^{N} h_{ji} d_i.$$

This implies that $h_{jN} > \frac{n}{d_N}$ as

$$Nn \leq \sum_{i=1}^{N} h_{ji} d_i < \sum_{i=1}^{N-1} \frac{n}{d_i} d_i + h_{jN} d_N = (N-1)n + h_{jN} d_N.$$

Thus by Proposition 4.5 there is a unital *-homomorphism $\varphi : A \to B$ such that $\Gamma([\varphi]) = \Gamma(\kappa)$. Hence $[\varphi] = \kappa$ by Theorem 4.7. $\square$

Let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks. Let $\varphi : A \to B$ be a unital *-homomorphism. As in [21, Chapter 1] we define the small remainders of φ, $s^{\varphi}(j,i)$, to be the multiplicity of the representation Λ_i^A in the representation $\Lambda_j^B \circ \varphi$ for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Note that approximately unitarily equivalent *-homomorphisms have the same small remainders. Let t_j^{φ} be the total multiplicity of all representations that are evaluations at non-exceptional points.

Proposition 4.9. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks and let $\varphi : A \to B$ be a unital *-homomorphism. There exist integers r_i with $0 \leq r_i < \frac{n}{d_i}$, $i = 1, 2, \ldots, N$, and an integer $L \geq 0$ such that, if $\psi : A \to B$ is a unital *-homomorphism with $\varphi^* = \psi^*$ in $Hom(K^0(B), K^0(A))$, then there exist continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_L : [0,1] \to \mathbb{T}$ and $\mu_1, \mu_2, \ldots, \mu_L : [0,1] \to \mathbb{T}$ together with unitaries $u, v \in C[0,1] \otimes M_m$ such that φ is approximately unitarily equivalent to a *-homomorphism of the form*

$$\varphi'(f)(e^{2\pi it}) = u(t)\, diag\big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f(\lambda_1(t)), f(\lambda_2(t)), \ldots, f(\lambda_L(t))\big) u(t)^*,$$

*and ψ is approximately unitarily equivalent to a *-homomorphism of the form*

$$\psi'(f)(e^{2\pi it}) = v(t)\, diag\big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f(\mu_1(t)), f(\mu_2(t)), \ldots, f(\mu_L(t))\big) v(t)^*,$$

where $f \in A$, $t \in [0,1]$. Furthermore, for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$,

$$s^{\varphi}(j,i) \equiv s^{\psi}(j,i) \mod \frac{n}{d_i}.$$

Proof. By [21, Corollary 1.5] there exist integers r_i with $0 \leq r_i < \frac{n}{d_i}$, $i = 1, 2, \ldots, N$, an integer $L \geq 0$, and continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_L : [0,1] \to \mathbb{T}$ such that φ is approximately unitarily equivalent to a *-homomorphism of the form

$$\varphi'(f)(e^{2\pi it}) = u(t) \mathrm{diag}\big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f(\lambda_1(t)), f(\lambda_2(t)), \ldots, f(\lambda_L(t))\big) u(t)^*.$$

Assume that $\psi : A \to B$ is a unital *-homomorphism such that $\varphi^* = \psi^*$. Let

$$s^{\varphi}(j,i) = a^{\varphi}(j,i) \frac{n}{d_i} + b^{\varphi}(j,i),$$

where $0 \le b^\varphi(j,i) < \frac{n}{d_i}$. Let $\{h^\varphi_{ji}\}$ and $\{h^\psi_{ji}\}$ be the (unique) $M \times N$ matrices from Lemma 4.4 corresponding to φ^* and ψ^*, respectively. Note that by Corollary 4.3,

$$\begin{aligned}\varphi^*([\Lambda_j^B]) = [\Lambda_j^B \circ \varphi] &= \sum_{i=1}^{N} s^\varphi(j,i)[\Lambda_i^A] + t_j^\varphi \frac{n}{d_N}[\Lambda_N^A] \\ &= \sum_{i=1}^{N-1} b^\varphi(j,i)[\Lambda_i^A] + ((\sum_{i=1}^{N} a^\varphi(j,i) + t_j^\varphi)\frac{n}{d_N} + b^\varphi(j,N))[\Lambda_N^A].\end{aligned}$$

It follows that for $i = 1, 2, \dots, N$,

$$h^\varphi_{ji} \equiv s^\varphi(j,i) \mod \frac{n}{d_i}.$$

Similarly,

$$h^\psi_{ji} \equiv s^\psi(j,i) \mod \frac{n}{d_i}.$$

Hence

$$s^\varphi(j,i) \equiv s^\psi(j,i) \mod \frac{n}{d_i}.$$

By [21, Corollary 1.5] we get that ψ is approximately unitarily equivalent to a *-homomorphism of the form

$$\psi'(f)(e^{2\pi i t}) = v(t)\mathrm{diag}(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), f(\mu_1(t)), f(\mu_2(t)), \dots, f(\mu_K(t)))v(t)^*,$$

for $f \in A$, $t \in [0,1]$, where s_i is an integer such that $0 \le s_i < \frac{n}{d_i}$, $i = 1, 2, \dots, N$, and $\mu_1, \mu_2, \dots, \mu_K : [0,1] \to \mathbb{T}$ are continuous functions. Since for $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$,

$$\begin{aligned}s^\varphi(j,i)\frac{m}{e_j} &= s^{\varphi'}(j,i)\frac{m}{e_j} = r_i + \#\{r = 1, 2, \dots, L : \lambda_r(y_j) = x_i\}\frac{n}{d_i}, \\ s^\psi(j,i)\frac{m}{e_j} &= s^{\psi'}(j,i)\frac{m}{e_j} = s_i + \#\{r = 1, 2, \dots, L : \mu_r(y_j) = x_i\}\frac{n}{d_i},\end{aligned}$$

it follows that $r_i = s_i$, $i = 1, 2, \dots, N$. And as

$$m = Ln + \sum_{i=1}^{N} r_i d_i = Kn + \sum_{i=1}^{N} s_i d_i$$

we see that $K = L$. □

5. The commutator subgroup of the unitary group

In [19] Nielsen and Thomsen introduced a new group into the classification programme: The unitary group modulo the closure of its commutator subgroup. We start this section by repeating some of their notation.

Let A be a unital C^*-algebra. Let $U(A)$ denote the unitary group of A and let $DU(A)$ denote its commutator subgroup (i.e the group generated by all unitaries of the form uvu^*v^*). Let $q'_A : U(A) \to U(A)/\overline{DU(A)}$ be the canonical map. We equip $U(A)/\overline{DU(A)}$ with the quotient metric

$$D_A\big(\, q'_A(u)\,,\, q'_A(v)\,\big) = \inf\{\|uv^* - x\| : x \in \overline{DU(A)}\}.$$

Let $r_A : T(A) \to SK_0(A)$ be the restriction map, where $SK_0(A)$ denotes the state space of $K_0(A)$. Let $\rho_A : K_0(A) \to \mathrm{Aff}\, T(A)$ be the group homomorphism

$$\rho_A(x)(\omega) = r_A(\omega)(x), \quad \omega \in T(A),\ x \in K_0(A).$$

We equip the group $\operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))}$ with the quotint metric d'_A. This space can be equipped with another metric which gives rise to the same topology, namely

$$d_A(f,g) = \begin{cases} 2 & d'_A(f,g) \geq \frac{1}{2}, \\ |e^{2\pi i d'_A(f,g)} - 1| & d'_A(f,g) < \frac{1}{2}, \end{cases}$$

cf. [19, Chapter 3]. If $a \in A$ is self-adjoint we define $\widehat{a} \in \operatorname{Aff} T(A)$ by $\widehat{a}(\omega) = \omega(a)$, $\omega \in T(A)$.

Theorem 5.1. *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks. Then the canonical maps $\pi_0(U(A)) \to K_1(A)$ and $\pi_1(U(A)) \to K_0(A)$ are isomorphisms.*

Proof. Let k_n denote the non-stable K-groups defined in [22], $n = -1, 0, 1, \ldots$. We need to show that the canonical maps $k_{-1}(A) \to k_{-1}(A \otimes \mathcal{K}) \cong K_1(A)$ and $k_0(A) \to k_0(A \otimes \mathcal{K}) \cong K_0(A)$ are isomorphisms, cf. [22, Proposition 2.6], where $\mathcal{K}$ denotes the set of compact operators on a separable infinite dimensional Hilbert-space. As noted in [22] it follows from [12, Proposition 4.4] that k_n is a continuous functor. Since it obviously is additive it suffices to prove the theorem in the case that A is a building block.

As in the proof of Theorem 3.2 we see that there exists finite dimensional C^*-algebras F and G such that we have a short exact sequence of the form

$$0 \longrightarrow SF \longrightarrow A \longrightarrow G \longrightarrow 0.$$

Since $k_{-1}(G) = k_{-1}(G \otimes \mathcal{K}) = 0$ and $k_1(G) = k_1(G \otimes \mathcal{K}) = 0$ we get by [22, Proposition 2.5] commutative diagrams of the form

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & k_0(SF) \arrow[r] \arrow[d] & k_0(A) \arrow[r] \arrow[d] & k_0(G) \arrow[r] \arrow[d] & k_{-1}(SF) \arrow[d] \\
0 \arrow[r] & k_0(SF \otimes \mathcal{K}) \arrow[r] & k_0(A \otimes \mathcal{K}) \arrow[r] & k_0(G \otimes \mathcal{K}) \arrow[r] & k_{-1}(SF \otimes \mathcal{K})
\end{tikzcd}$$

$$\begin{tikzcd}
k_0(G) \arrow[r] \arrow[d] & k_{-1}(SF) \arrow[r] \arrow[d] & k_{-1}(A) \arrow[r] \arrow[d] & 0 \arrow[d] \\
k_0(G \otimes \mathcal{K}) \arrow[r] & k_{-1}(SF \otimes \mathcal{K}) \arrow[r] & k_{-1}(A \otimes \mathcal{K}) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and the vertical maps are canonical. It is well-known (cf. [22, Lemma 2.3]) that the vertical maps, except those involving A, are isomorphisms. The theorem follows. □

Let A be a unital C^*-algebra such that the canonical maps $\pi_0(U(A)) \to K_1(A)$ and $\pi_1(U(A)) \to K_0(A)$ are isomorphisms. Then there exists an injective group homomorphism $\lambda_A : \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \to U(A)/\overline{DU(A)}$ given by $\lambda_A(q_A(\widehat{a})) = q'_A(e^{2\pi i a})$, cf. [19, Chapter 3]. Let $\pi_A : U(A)/\overline{DU(A)} \to K_1(A)$ be the canonical map.

Proposition 5.2. *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks. There exists a split exact sequence*

$$0 \longrightarrow \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \xrightarrow{\lambda_A} U(A)/\overline{DU(A)} \xrightarrow{\pi_A} K_1(A) \longrightarrow 0.$$

λ_A is an isometry when $\operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))}$ is equipped with the metric d_A.

Proof. Combine Theorem 5.1 with [21, Lemma 6.4]. ☐

Lemma 5.3. *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. Let $a \in A$ be a self-adjoint element such that $Tr(a(z)) = 0$ for $z \in \mathbb{T}$. Then $e^{2\pi i a} \in \overline{DU(A)}$.*

Proof. Let $x_1, x_2, \dots, x_N$ denote the exceptional points of A. By [10, Lemma 3.5] there exists for each $z \in \mathbb{T}$ an element $c_z \in M_n$ such that $a(z) = c_z c_z^* - c_z^* c_z$. We may assume that $c_{x_i} \in M_{d_i}$, $i = 1, 2, \dots, N$. Let $\epsilon > 0$. For each $z \in \mathbb{T}$ there exists a neighbourhood V_z of z such that

$$\|a(x) - (c_z(x)c_z(x)^* - c_z(x)^* c_z(x))\| < \epsilon, \quad x \in V_z.$$

Since $\mathbb{T}$ is compact there exist points $z_1, z_2, \dots, z_L \in \mathbb{T}$ such that $\mathbb{T} = \cup_{j=1}^L V_{z_j}$. We may assume that $L \geq N$, that $z_i = x_i$, and that $x_i \notin V_j$ for $j \neq i$, $i = 1, 2, \dots, N$. Let $\{h_j : j = 1, 2, \dots, L\}$ be a continuous partition of unity in $C(\mathbb{T})$ subordinate to the cover $\{V_j : j = 1, 2, \dots, L\}$. Define $b_j \in A$ by $b_j(x) = \sqrt{h_j(x)} c_{z_j}(x)$, $x \in \mathbb{T}$, $j = 1, 2, \dots, L$. Then

$$\|a - \sum_{j=1}^{L} (b_j b_j^* - b_j^* b_j)\| < \epsilon.$$

Note that

$$q_A'(e^{2\pi i \sum_{j=1}^{L} (b_j b_j^* - b_j^* b_j)}) = \prod_{j=1}^{L} q_A'(e^{2\pi i (b_j b_j^* - b_j^* b_j)}) = q_A'(1) \quad \text{in } U(A)/\overline{DU(A)},$$

where the last equality follows from the last part of the proof of [25, Lemma 3.1]. Since $\epsilon > 0$ was arbitrary we conclude that $e^{2\pi i a} \in \overline{DU(A)}$. ☐

Lemma 5.4. *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. Assume that $\lambda^d = 1$ for some $\lambda \in \mathbb{T}$, where $d = d(A)$. Then $\lambda 1 \in DU(A)$.*

Proof. As there exists a unital *-homomorphism $M_d \to A$ it suffices to prove this for the C^*-algebra M_d. And this case follows from the fact that there exists a unitary $u \in M_d$ such that

$$\begin{aligned}
&\operatorname{diag}(\lambda, \lambda, \dots, \lambda) \operatorname{diag}(1, \lambda, \lambda^2, \dots, \lambda^{d-1}) \\
&= \operatorname{diag}(\lambda, \lambda^2, \dots, \lambda^d) \\
&= u \operatorname{diag}(1, \lambda, \lambda^2, \dots, \lambda^{d-1})\, u^*.
\end{aligned}$$

☐

The main result of this section is the following:

Theorem 5.5. *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. Let $u \in A$ be a unitary. Assume that*

$$Det(u(z)) = 1, \quad z \in \mathbb{T},$$
$$Det(\Lambda_i(u)) = 1, \quad i = 1, 2, \dots, N.$$

Then $u \in \overline{DU(A)}$.

Proof. By Lemma 3.3 we see that u is trivial in $K_1(A)$. Hence by Theorem 5.1,

$$u = e^{2\pi i a_1} e^{2\pi i a_2} \dots e^{2\pi i a_L}$$

for self-adjoint elements $a_1, a_2, \dots, a_L \in A$. Set $b = a_1 + a_2 + \dots + a_L$. As $Det(u(z)) = \exp(2\pi i Tr(b(z)))$, $z \in \mathbb{T}$, it follows that $Tr(b(\cdot))$ is constantly equal to some $k \in \mathbb{Z}$. Set $c = b - \frac{k}{n}1$. By Lemma 5.3 we see that $e^{2\pi i c} \in \overline{DU(A)}$. Hence $q'_A(u) = q'_A(e^{2\pi i b}) = q'_A(\lambda 1)$ in $U(A)/\overline{DU(A)}$, where $\lambda = \exp(2\pi i \frac{k}{n})$. By assumption $\lambda^{d_i} = 1$. Hence $\frac{n}{d_i}$ divides k for each $i = 1, 2, \dots, N$. It follows that $\frac{n}{d}$ divides k, where $d = d(A)$. Thus $\lambda^d = 1$. The conclusion follows from Lemma 5.4. □

We conclude this section with some lemmas for later use.

Lemma 5.6. *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. Let $u \in A$ be a unitary that is trivial in $K_1(A)$ and let p be a non-zero integer. Then there exists a unitary $v \in A$ that is trivial in $K_1(A)$ such that*

$$q'_A(v^p) = q'_A(u) \quad \textit{in } U(A)/\overline{DU(A)}.$$

Proof. By Theorem 5.1 there exists self-adjoint elements $a_1, a_2, \dots, a_k \in A$ such that

$$u = e^{2\pi i a_1} e^{2\pi i a_2} \dots e^{2\pi i a_k}.$$

Set $a = a_1 + a_2 + \dots + a_k$ and define $\beta : \mathbb{T} \to \mathbb{T}$ by

$$\beta(z) = \exp(\frac{2\pi i Tr(a(z))}{np}).$$

Define $v \in A$ by

$$v(z) = \operatorname{diag}(\underbrace{\beta(z), \beta(z), \dots, \beta(z)}_{n \text{ times}}), \quad z \in \mathbb{T}.$$

Then

$$Det(v(z)^p) = \beta(z)^{np} = e^{2\pi i Tr(a(z))} = Det(e^{2\pi i a(z)}) = Det(u(z)), \quad z \in \mathbb{T},$$

and for $i = 1, 2, \dots, N$,

$$Det(\Lambda_i(v^p)) = e^{2\pi i Tr(a(x_i))\frac{d_i}{n}} = e^{2\pi i Tr(\Lambda_i(a))} = Det(\Lambda_i(e^{2\pi i a})) = Det(\Lambda_i(u)).$$

The conclusion follows by applying Theorem 5.5. □

Lemma 5.7. *Let A be a building block and let $u \in A$ be a unitary of finite order in $K_1(A)$. Then there exists a unitary $w \in A$ such that $[u] = [w]$ in $K_1(A)$ and such that $Det(w(\cdot))$ is constant.*

Proof. Let p denote the order of u in $K_1(A)$. By Lemma 5.6 there exists a unitary $v \in A$ that is trivial in $K_1(A)$ such that

$$q'_A(v^p) = q'_A(u^p) \quad \text{in } U(A)/\overline{DU(A)}.$$

Set $w = uv^*$. Then w^p is trivial in $U(A)/\overline{DU(A)}$. □

If $\varphi : A \to B$ is a unital *-homomorphism we let $\varphi^\# : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be the group homomorphism $\varphi^\#(q'_A(u)) = q'_B(\varphi(u))$.

Lemma 5.8. *Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks and let $\varphi, \psi : A \to B$ be unital *-homomorphisms such that $\varphi^* = \psi^*$ in $Hom(K^0(B), K^0(A))$. If $W \in A$ is a unitary such that $Det(W(\cdot))$ is constant then*

$$\varphi^\#(q'_A(W)) = \psi^\#(q'_B(W)) \quad \textit{in } U(B)/\overline{DU(B)}.$$

Proof. By Proposition 4.9 we see that φ and ψ are approximately unitarily equivalent to *-homomorphisms of the form

$$\varphi'(f)(e^{2\pi it}) = u(t)\mathrm{diag}\big(\Lambda_1^{r_1}(f),\ldots,\Lambda_N^{r_N}(f), f(\lambda_1(t)),\ldots,f(\lambda_L(t))\big)u(t)^*,$$
$$\psi'(f)(e^{2\pi it}) = v(t)\mathrm{diag}\big(\Lambda_1^{r_1}(f),\ldots,\Lambda_N^{r_N}(f), f(\mu_1(t)),\ldots,f(\mu_L(t))\big)v(t)^*,$$

for continuous functions $\lambda_1,\lambda_2,\ldots,\lambda_L,\mu_1,\mu_2,\ldots,\mu_L : [0,1]\to\mathbb{T}$, integers r_i with $0\le r_i<\frac{n}{d_i}$, $i=1,2,\ldots,N$, and unitaries $u,v\in C[0,1]\otimes M_m$. It follows that

$$Det(\varphi(W)(z)) = Det(\psi(W)(z)),\quad z\in\mathbb{T}.$$

Fix some $j=1,2,\ldots,M$. Also by Proposition 4.9 $s^{\varphi}(j,i)\equiv s^{\psi}(j,i) \mod \frac{n}{d_i}$, $i=1,2,\ldots,N$. Choose s_i, $0\le s_i<\frac{n}{d_i}$, such that $s_i\equiv s^{\varphi}(j,i) \mod \frac{n}{d_i}$. By Lemma 2.1 there exist points $a_1,a_2,\ldots,a_D,b_1,b_2,\ldots,b_D\in\mathbb{T}$ and unitaries $w_1,w_2\in M_{e_j}$ such that

$$\Lambda_j\circ\varphi(f) = w_1\mathrm{diag}(\Lambda_1^{s_1}(f),\Lambda_2^{s_2}(f),\ldots,\Lambda_N^{s_N}(f), f(a_1),f(a_2),\ldots,f(a_N))w_1{}^*,$$
$$\Lambda_j\circ\psi(f) = w_2\mathrm{diag}(\Lambda_1^{s_1}(f),\Lambda_2^{s_2}(f),\ldots,\Lambda_N^{s_N}(f), f(b_1),f(b_2),\ldots,f(b_N))w_2{}^*.$$

Hence

$$Det(\Lambda_j\circ\varphi(W)) = Det(\Lambda_j\circ\psi(W)),\quad j=1,2,\ldots,M.$$

The conclusion follows again from Theorem 5.5. ☐

6. *-HOMOMORPHISMS

Lemma 6.1. *Assume that*

$$\big(\exp(2\pi i\theta_1),\ldots,\exp(2\pi i\theta_L)\big) = \big(\exp(2\pi i\omega_1),\ldots,\exp(2\pi i\omega_L)\big)$$

as unordered L-tuples, where $\theta_1,\theta_2,\ldots,\theta_L$ and $\omega_1,\omega_2,\ldots,\omega_L$ are real numbers such that

$$\theta_1\le\theta_2\le\cdots\le\theta_L\le\theta_1+1,$$
$$\omega_1\le\omega_2\le\cdots\le\omega_L\le\omega_1+1.$$

There exists an integer r such that

$$\theta_j=\omega_{r+j},\quad j=1,2,\ldots,L,$$

when we define

$$\omega_{kL+j}=\omega_j+k,\quad j=1,2,\ldots,L,\ k\in\mathbb{Z}.$$

Proof. Define

$$\theta_{kL+j}=\theta_j+k,\quad j=1,2,\ldots,L,\ k\in\mathbb{Z}.$$

Choose $m\in\mathbb{Z}$ such that $\theta_m<\theta_{m+1}$ and choose $n\in\mathbb{Z}$ such that

$$\theta_{m+1}=\omega_{n+1}>\omega_n.$$

Assume that

$$\theta_{m+p}=\omega_{n+q}+k.$$

for some $p,q\in\mathbb{Z}$, $1\le p\le L$, $1\le q\le L$, and $k\in\mathbb{Z}$. Then

$$-1<\theta_{m+1}-\theta_{m+L}\le\theta_{m+1}-\theta_{m+p}=\omega_{n+1}-\omega_{n+q}-k\le -k,$$
$$0\le\theta_{m+p}-\theta_{m+1}=\omega_{n+q}+k-\omega_{n+1}<\omega_{n+q}+k-\omega_n\le 1+k.$$

Hence $k=0$. By assumption it follows that for every $x\in\mathbb{R}$,

$$\#\{j=1,2,\ldots,L:\theta_{m+j}=x\}=\#\{j=1,2,\ldots,L:\omega_{n+j}=x\}.$$

Thus

$$\big(\theta_{m+1}, \theta_{m+2}, \dots, \theta_{m+L}\big) = \big(\omega_{n+1}, \omega_{n+2}, \dots, \omega_{n+L}\big)$$

as unordered L-tuples. Therefore

$$\theta_{m+j} = \omega_{n+j}, \quad j = 1, 2, \dots, L.$$

We conclude that

$$\theta_j = \omega_{n-m+j}, \quad j = 1, 2, \dots, L.$$

□

Lemma 6.2. *Let $\lambda_1, \lambda_2, \dots, \lambda_L : [0,1] \to \mathbb{T}$ be continuous functions. There exist continuous functions $F_1, F_2, \dots, F_L : [0,1] \to \mathbb{R}$ such that*

$$F_1(t) \leq F_2(t) \leq \dots \leq F_L(t) \leq F_1(t) + 1$$

and such that for each $t \in [0,1]$,

$$\big(\lambda_1(t), \lambda_2(t), \dots, \lambda_L(t)\big) = \big(\exp(2\pi i F_1(t)), \exp(2\pi i F_2(t)), \dots, \exp(2\pi i F_L(t))\big)$$

as unordered L-tuples.

Proof. Choose a positive integer k such that

$$|s - t| \leq \frac{1}{k} \;\Rightarrow\; \rho(\lambda_j(s), \lambda_j(t)) < \frac{1}{2L}, \quad s, t \in [0,1],\ j = 1, 2, \dots, L.$$

We will prove by induction in m that there exist functions $F_1, \dots, F_L$ that satisfy the above for $t \in [0, \frac{m}{k}]$.

Choose $z_0 \in \mathbb{T}$ such that $\rho(z_0, \lambda_j(0)) \geq \frac{1}{2L}$, $j = 1, 2, \dots, L$. Choose $\alpha_0 \in \mathbb{R}$ such that $\exp(2\pi i \alpha_0) = z_0$. There exist for $j = 1, 2, \dots, L$, continuous functions $F_j : [0, \frac{1}{k}] \to]\alpha_0, \alpha_0 + 1[$ such that

$$F_1(t) \leq F_2(t) \leq \dots \leq F_L(t)$$

for each $t \in [0, \frac{1}{k}]$, and

$$\big(\lambda_1(t), \lambda_2(t), \dots, \lambda_L(t)\big) = \big(\exp(2\pi i F_1(t)), \exp(2\pi i F_2(t)), \dots, \exp(2\pi i F_L(t))\big)$$

as unordered L-tuples.

Now assume that we have constructed functions $F_1, F_2, \dots, F_L : [0, \frac{m}{k}] \to \mathbb{R}$ such that for each $t \in [0, \frac{m}{k}]$, $F_1(t) \leq F_2(t) \leq \dots \leq F_L(t)$ and

$$\big(\lambda_1(t), \lambda_2(t), \dots, \lambda_L(t)\big) = \big(\exp(2\pi i F_1(t)), \exp(2\pi i F_2(t)), \dots, \exp(2\pi i F_L(t))\big)$$

as unordered L-tuples. Choose $z_m \in \mathbb{T}$ such that $\rho(z_m, \lambda_j(\frac{m}{k})) \geq \frac{1}{2L}$ for $j = 1, 2, \dots, L$. Choose $\alpha_m \in \mathbb{R}$ such that $\exp(2\pi i \alpha_m) = z_m$. Choose continuous functions $G_j : [\frac{m}{k}, \frac{m+1}{k}] \to]\alpha_m, \alpha_m + 1[$ such that for each $t \in [\frac{m}{k}, \frac{m+1}{k}]$,

$$G_1(t) \leq G_2(t) \leq \dots \leq G_L(t)$$

and

$$\big(\lambda_1(t), \lambda_2(t), \dots, \lambda_L(t)\big) = \big(\exp(2\pi i G_1(t)), \exp(2\pi i G_2(t)), \dots, \exp(2\pi i G_L(t))\big)$$

as unordered L-tuples.

Set for $j = 1, 2, \dots, L$, $p \in \mathbb{Z}$,

$$G_{pL+j}(t) = G_j(t) + L, \quad t \in [\frac{m}{k}, \frac{m+1}{k}].$$

By Lemma 6.1 there exists an integer r such that for $j = 1, 2, \dots, L$,

$$F_j(\frac{m}{k}) = G_{r+j}(\frac{m}{k}).$$

Define for $j = 1, 2, \dots, L$, $F'_j : [0, \frac{m+1}{k}] \to \mathbb{R}$ by

$$F'_j(t) = \begin{cases} F_j(t) & t \in [0, \frac{m}{k}], \\ G_{r+j}(t) & t \in [\frac{m}{k}, \frac{m+1}{k}]. \end{cases}$$

$F'_1, F'_2, \dots, F'_L$ satisfy the conclusion of the lemma for $t \in [0, \frac{m+1}{k}]$. □

Proposition 6.3. *Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks, and let $\varphi : A \to B$ be a unital *-homomorphism. There exist for $i = 1, 2, \dots, N$, integers r_i with $0 \le r_i < \frac{n}{d_i}$, an integer $L \ge 0$ and a unitary w in M_m such that if $\psi : A \to B$ is a unital *-homomorphism such that $[\varphi] = [\psi]$ in $KK(A, B)$, then ψ is approximately unitarily equivalent to a *-homomorphism of the form*

$$\psi'(f)(e^{2\pi it}) = u(t)\, diag\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(t)}), \dots, f(e^{2\pi i F_L(t)})\big) u(t)^*,$$

for $t \in [0,1]$, $f \in A$, where $u \in C[0,1] \otimes M_m$ is a unitary with $u(0) = 1$, $u(1) = w$, and where $F_1, F_2, \dots, F_L : [0,1] \to \mathbb{T}$ are continuous functions for that for every $t \in [0,1]$,

$$F_1(t) \le F_2(t) \le \dots \le F_L(t) \le F_1(t) + 1.$$

*If k is an integer then ψ is also approximately unitarily equivalent to a unital *-homomorphism of the form*

$$\psi''(f)(e^{2\pi it}) = v(t)\, diag\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i G_1(t)}), \dots, f(e^{2\pi i G_L(t)})\big) v(t)^*,$$

for $t \in [0,1]$, $f \in A$, where $v \in C[0,1] \otimes M_m$ is a unitary with $v(0) = 1$, $v(1) = w$, and where $G_1, G_2, \dots, G_L : [0,1] \to \mathbb{T}$ are continuous functions for that for every $t \in [0,1]$,

$$G_1(t) \le G_2(t) \le \dots \le G_L(t) \le G_1(t) + 1,$$

and such that

$$\sum_{r=1}^{L} (G_r(t) - F_r(t)) = k, \quad t \in [0,1].$$

Proof. Let l denote the winding number of the loop $Det(\varphi(V_A)(\cdot))$. Let y be a unitary $(Ln) \times (Ln)$ matrix such that

$$y\, \mathrm{diag}(a_1, a_2, \dots, a_L)\, y^* = \mathrm{diag}(a_L, a_1, a_2, \dots, a_{L-1})$$

for all $a_1, a_2, \dots, a_L \in M_m$. Set

$$w = \mathrm{diag}(\underbrace{1, 1, \dots, 1}_{m-Ln \text{ times}}, y^l).$$

Choose L and $r_1, r_2, \dots, r_N$ according to Proposition 4.9.

Now assume that $\psi : A \to B$ is a unital *-homomorphism such that $[\varphi] = [\psi]$ in $KK(A, B)$. By Proposition 4.9 there exist continuous functions $\mu_1, \mu_2, \dots, \mu_L$ and a unitary $v \in C[0,1] \otimes M_m$ such that ψ is approximately unitarily equivalent to a *-homomorphism $\alpha : A \to B$ of the form

$$\alpha(f)(e^{2\pi it}) = v(t) \mathrm{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(\mu_1(t)), f(\mu_2(t)), \dots, f(\mu_L(t))\big) v(t)^*.$$

By Proposition 6.2 there exist continuous functions $F_1, \dots, F_L : [0,1] \to \mathbb{R}$ such that for each $t \in [0,1]$,

$$F_1(t) \le F_2(t) \le \dots \le F_L(t) \le F_1(t) + 1$$

and

$$\big(\mu_1(t), \mu_2(t), \dots, \mu_L(t)\big) = \big(\exp(2\pi i F_1(t)), \exp(2\pi i F_2(t)), \dots, \exp(2\pi i F_L(t))\big) \quad (2)$$

as unordered L-tuples. Since $[\varphi] = [\psi]$ in $KK(A, B)$ we have that $Det(\varphi(V_A)(\cdot))$ and $Det(\psi(V_A)(\cdot))$ have the same winding number. Thus

$$l = \sum_{r=1}^{L} (F_r(1) - F_r(0)).$$

As

$$\big(\exp(2\pi i F_1(0)), \dots, \exp(2\pi i F_L(0))\big) = \big(\exp(2\pi i F_1(1)), \dots, \exp(2\pi i F_L(1))\big)$$

as unordered L-tuples, we conclude by Lemma 6.1 that $F_r(1) = F_{r+l}(0)$ for each $r = 1, 2, \dots, L$. Therefore, for every $f \in A$,

$$\begin{aligned} &\operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(0)}), \dots, f(e^{2\pi i F_L(0)})\big) \\ = \;& w \operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(1)}), \dots, f(e^{2\pi i F_L(1)})\big)\, w^*. \end{aligned}$$

Let $t_1, t_2, \dots, t_M \in]0, 1[$ be numbers such that $e^{2\pi i t_j}$ are the exceptional points of B. By (2) there exist a unitary $u_j \in M_m$ such that

$$u_j \operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(t_j)}), \dots, f(e^{2\pi i F_L(t_j)})\big) u_j^* \in M_{e_j} \subseteq M_m$$

for every $f \in A$. Choose a unitary $u \in C[0, 1] \otimes M_m$ such that $u(0) = 1$, $u(1) = w$, and $u(t_j) = u_j$, $j = 1, 2, \dots, M$. Note that we can define a unital *-homomorphism $\psi' : A \to B$ by

$$\psi'(f)(e^{2\pi i t}) = u(t) \operatorname{diag}\big(\Lambda_1^{r_1}(f), \dots, \Lambda_N^{r_N}(f), f(e^{2\pi i F_1(t)}), \dots, f(e^{2\pi i F_L(t)})\big) u(t)^*,$$

for $f \in A$, $t \in [0, 1]$. Then for every $f \in A$, $z \in \mathbb{T}$,

$$Tr(\psi(f)(z)) = Tr(\alpha(f)(z)) = Tr(\psi'(f)(z)).$$

Hence ψ and ψ' are approximately unitarily equivalent by [21, Theorem 1.4].

To see the last part of the proposition, set $G_r = F_{r+k}$ and substitute G_r for F_r in the proof above. □

7. Uniqueness

The purpose of this section is to prove a uniqueness theorem, i.e a theorem saying that two unital *-homomorphisms between (finite direct sums of) building blocks are close in a suitable sense if they approximately agree on the invariant. Many of the arguments here are inspired by similar arguments in [7], [8], [19], [21], and [13].

We start out with some definitions. Let k be a positive integer. A k-arc is an arc-segment of the form

$$I = \{e^{2\pi i t} : t \in [\frac{m}{k}, \frac{n}{k}]\}$$

where m and n are integers, $m < n$. We set

$$I \pm \epsilon = \{e^{2\pi i t} : t \in [\frac{m}{k} - \epsilon, \frac{n}{k} + \epsilon]\}.$$

A permutation σ of the set $\{1, 2, \dots, n\}$ will be called cyclic if there exists an integer l such that $\sigma(i) \equiv i + l \mod n$ for $i = 1, 2, \dots, n$. Finally we define a metric on the set of unordered L-tuples consisting of elements from $\mathbb{T}$ by

$$R_L\big((a_1, a_2, \dots, a_L), (b_1, b_2, \dots, b_L)\big) = \min_{\sigma \in \Sigma_n} \big(\max_{1 \le i \le n} \rho(a_i, b_{\sigma(i)}) \big),$$

where Σ_n denotes the group of permutations of the set $\{1, 2, \ldots, n\}$.

Lemma 7.1. *Let $a_1, a_2, \ldots, a_L, b_1, b_2, \ldots, b_L \in \mathbb{T}$. Assume that there is a positive integer k such that $\epsilon < \frac{1}{2k}$ and*

$$\#\{r : \lambda_r \in I\} \leq \#\{r : \mu_r \in I \pm \epsilon\}$$

for all k-arcs I. Then

$$R_L\big((a_1, a_2, \ldots, a_L), (b_1, b_2, \ldots, b_L)\big) < \epsilon + \frac{1}{k}.$$

Proof. Let $S \subseteq \{1, 2, \ldots, n\}$ be a subset. Set

$$P(S) = \bigcup_{j \in S} \{r : \rho(\mu_r, \lambda_j) < \epsilon + \frac{1}{k}\}.$$

We will show that $\#S \leq \#P(S)$.

Let $J \subseteq \mathbb{T}$ be the union of those k-arcs of length $\frac{1}{k}$ that intersect non-trivially with $\{\lambda_j : j \in S\}$. Let $I_1, I_2, \ldots, I_q$ be the connected components of J. By assumption we have that for $j = 1, 2, \ldots, q$,

$$\#\{r : \lambda_r \in I_j\} \leq \#\{r : \mu_r \in I_j \pm \epsilon\}.$$

As the sets $\{r : \mu_r \in I_j \pm \epsilon\}$, $j \in S$, are disjoint and contained in $P(S)$ we see that $\#S \leq \#P(S)$. By Hall's marriage lemma, see e.g [4, Theorem 2.2], we see that there exists a permutation σ of $\{1, 2, \ldots, L\}$ such that $\rho(\mu_j, \lambda_{\sigma(j)}) < \epsilon + \frac{1}{k}$ for $j = 1, 2, \ldots, L$. $\square$

Lemma 7.2. *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. Assume that there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $|a_j - b_{\sigma(j)}| < \epsilon$ for some $\epsilon > 0$. Then $|a_j - b_j| < \epsilon$.*

Proof. If e.g $a_j \leq b_j - \epsilon$ for some j then σ must map the set $\{1, 2, \ldots, j\}$ into $\{1, 2, \ldots, j-1\}$. Contradiction. $\square$

If the statement of the next lemma is clear to the reader, we urge him or her to skip its awkward proof.

Lemma 7.3. *Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be naturally numbered tuples in $\mathbb{T}$. Assume that there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that*

$$\rho(a_j, b_{\sigma(j)}) < \epsilon, \quad j = 1, 2, \ldots, n,$$

for some $\epsilon < \frac{1}{8}$. Then there exists a cyclic permutation ω such that

$$\rho(a_j, b_{\omega(j)}) < \epsilon, \quad j = 1, 2, \ldots, n.$$

Proof. We may assume that $b_1, b_2, \ldots, b_n$ are mutually different. Let j be an integer. If $j = kn + r$ for some integers k and r, where $1 \leq r \leq n$, then we set $a_j = a_r$, $b_j = b_r$.

Let $a_j = \exp(2\pi i x_j)$, $x_j \in [0, 1[$, $j = 1, 2, \ldots, n$. By assumption there exist numbers $y_j \in]-\frac{1}{8}, \frac{9}{8}[$, $j = 1, 2, \ldots, n$, such that

$$\big(b_1, b_2, \ldots, b_n\big) = \big(\exp(2\pi i y_1), \exp(2\pi i y_2), \ldots, \exp(2\pi i y_n)\big)$$

as unordered n-tuples, $y_1 < y_2 < \cdots < y_n$, and such that $|x_j - y_{\tau(j)}| < \epsilon$ for a permutation τ of $\{1, 2, \ldots, n\}$. Thus by Lemma 7.2,

$$|x_j - y_j| < \epsilon, \ j = 1, 2, \ldots, n. \tag{3}$$

There are integers s and t, $1 \leq s \leq t-1 \leq n-1$, such that

$$\{j = 1, 2, \ldots, n : \frac{1}{4} \leq x_j \leq \frac{3}{4}\} = \{s, s+1, \ldots, t\}.$$

Then $y_j \in]\frac{1}{8}, \frac{7}{8}[$, $j = s, s+1, \ldots, t$. Hence there exists an integer m such that

$$b_{m+j} = \exp(2\pi i y_j), \quad j = s, s+1, \ldots, t.$$

Thus

$$\rho(b_{m+j}, a_j) < \epsilon, \quad j = s, s+1, \ldots, t. \tag{4}$$

By (3) there exists a permutation χ of $\{t+1, t+2, \ldots, n+s-1\}$ such that

$$\rho(b_{m+j}, a_{\chi(j)}) < \epsilon, \quad j = t+1, t+2, \ldots, n+s-1.$$

Choose $z_j, w_j \in]\frac{5}{8}, \frac{11}{8}[$ such that

$$a_j = \exp(2\pi i z_j), \quad j = t, t+1, \ldots, n+s-1,$$
$$b_{j+m} = \exp(2\pi i w_j), \quad j = t, t+1, \ldots, n+s-1.$$

As $|w_j - z_{\chi(j)}| < \epsilon$, we see by Lemma 7.2 that $|z_j - w_j| < \epsilon$. Hence

$$\rho(a_j, b_{j+m}) < \epsilon, \quad j = t+1, t+2, \ldots, n+s-1.$$

The lemma follows from this and (4). □

Lemma 7.4. *Let $\theta_1, \theta_2, \ldots, \theta_L$ and $\omega_1, \omega_2, \ldots, \omega_L$ be real numbers such that*

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_L \leq \theta_1 + 1,$$
$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_L \leq \omega_1 + 1.$$

Assume that $\epsilon > 0$ and $\delta > 0$ satisfy that $L\epsilon \leq \delta$, $2\delta \leq L$ and $\epsilon < \frac{1}{8}$. Assume furthermore that

$$R_L\big((e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \ldots, e^{2\pi i\theta_L}), (e^{2\pi i\omega_1}, e^{2\pi i\omega_2}, \ldots, e^{2\pi i\omega_L})\big) < \epsilon,$$

$$|\sum_{j=1}^{L}(\theta_j - \omega_j)| < \delta, \tag{5}$$

and that $s > 8$ is an integer such that for every s-arc I,

$$\#\{j : e^{2\pi i\omega_j} \in I\} \geq \delta.$$

Then

$$|\theta_j - \omega_j| < \epsilon + \frac{3}{s}, \quad j = 1, 2, \ldots, L.$$

Proof. Set

$$\omega_{Ln+j} = \omega_j + n, \quad j = 1, 2, \ldots, L, \ n \in \mathbb{Z}.$$

By Lemma 7.3 there exists an integer p, $0 \leq p \leq L-1$, such that

$$\rho(e^{2\pi i\theta_j}, e^{2\pi i\omega_{j+p}}) < \epsilon, \quad j = 1, 2, \ldots, L.$$

Hence there exist integers q_j, $j = 1, 2, \ldots, L$, such that

$$|\theta_j - \omega_{j+p} + q_j| < \epsilon, \quad j = 1, 2, \ldots, L.$$

By assumption $\{e^{2\pi i\omega_j} : j = 1, 2, \dots, L\}$ intersects non-trivially with every arc-segment of length $\frac{2}{s}$. Hence $\{e^{2\pi i\theta_j} : j = 1, 2, \dots, L\}$ intersects non-trivially with every arc-segment of length $\frac{2}{s} + 2\epsilon$. Thus

$$|\theta_{j+1} - \theta_j| \leq 2\epsilon + \frac{2}{s}.$$

Therefore, for $j = 1, 2, \dots, L-1$,

$$\begin{aligned} &|q_{j+1} - q_j| \\ &\leq |\theta_{j+1} - \omega_{j+1+p} + q_{j+1}| + |\theta_j - \omega_{j+p} + q_j| + |\theta_{j+1} - \theta_j| + |\omega_{j+1+p} - \omega_{j+p}| \\ &< \epsilon + \epsilon + (2\epsilon + \frac{2}{s}) + \frac{2}{s} < 1. \end{aligned}$$

Let $q = q_1 = q_2 = \cdots = q_L$. Then

$$|\sum_{j=1}^{L}(\theta_j - \omega_j) + Lq - p| = |\sum_{j=1}^{L}(\theta_j - \omega_{j+p} + q)| < L\epsilon.$$

By (5) it follows that

$$-L\epsilon - Lq + p < \delta \;\Rightarrow\; q > -\epsilon + \frac{p}{L} - \frac{\delta}{L} \geq -(\epsilon + \frac{\delta}{L}) \geq -1, \tag{6}$$

and

$$L\epsilon - Lq + p > -\delta \;\Rightarrow\; q < \epsilon + \frac{p}{L} + \frac{\delta}{L} \leq \epsilon + \frac{\delta}{L} + 1 \leq 2. \tag{7}$$

Therefore $q = 0$ or $q = 1$.

If $q = 0$ then by (6)

$$-L\epsilon + p < \delta \;\Rightarrow\; p < \delta + L\epsilon \leq 2\delta.$$

Hence if we set $J = \{e^{2\pi it} : \omega_j < t < \omega_{j+p}\}$ then

$$\#\{j = 1, 2, \dots, L : e^{2\pi i\omega_j} \in J\} < 2\delta.$$

Thus J intersects non-trivially with at most 3 s-arcs. Therefore

$$|\omega_j - \omega_{j+p}| \leq \frac{3}{s}.$$

It follows that for $j = 1, 2, \dots, L$,

$$|\theta_j - \omega_j| \leq |\theta_j - \omega_{j+p}| + |\omega_{j+p} - \omega_j| < \epsilon + \frac{3}{s}.$$

Similarly, if $q = 1$ then by (7)

$$L\epsilon - L + p > -\delta \;\Rightarrow\; L - p < L\epsilon + \delta \leq 2\delta.$$

Set $J = \{e^{2\pi it} : \omega_{j-(L-p)} < t < \omega_j\}$. Then J intersects non-trivially with at most 3 s-arcs. Hence

$$|\omega_j - \omega_{j-(L-p)}| \leq \frac{3}{s}.$$

It follows that for $j = 1, 2, \dots, L$,

$$|\theta_j - \omega_j| \leq |\theta_j - \omega_{j+p} + 1| + |\omega_{j-(L-p)} - \omega_j| < \epsilon + \frac{3}{s}.$$

$\square$

Let $\varphi : A \to B$ be a unital *-homomorphism. We define an affine continuous map $\varphi^* : T(B) \to T(A)$ by $\varphi^*(\omega) = \omega \circ \varphi$. We define a positive linear order unit preserving map $\widehat{\varphi} : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ by $\widehat{\varphi}(f)(\omega) = f(\varphi^*(\omega))$, $f \in \operatorname{Aff} T(A)$, $\omega \in T(B)$.

Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block and p a positive integer. Let I be a p-arc. Choose a continuous function $f_A^I : \mathbb{T} \to [0, \frac{1}{n}]$ such that $\emptyset \neq \operatorname{supp} f_A^I \subseteq I$ and such that f_A^I equals 0 at all the exceptional points of A. Choose a continuous function $g_A^I : \mathbb{T} \to [0,1]$ such that g_A^I equals 1 on I, such that $\operatorname{supp} g_A^I \subseteq I \pm \frac{1}{5p}$, and such that $\operatorname{supp} g_A^I \setminus I$ contains no exceptional points of A. Set

$$H(A,p) = \{f_A^I \otimes 1 : I \ p\text{-arc}\},$$
$$\widetilde{H}(A,p) = \{g_A^I \otimes 1 : I \ p\text{-arc}\}.$$

Theorem 7.5. *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block. Let $\epsilon > 0$ and let $F \subseteq A$ be a finite set. There exists a positive integer l_0 such that if l, p, k and q are positive integers with $l_0 \le l \le p \le k \le q$, if $B = A(m, e_1, e_2, \dots, e_M)$ is a building block with exceptional points $y_1, y_2, \dots, y_M$, if $\varphi, \psi : A \to B$ are unital *-homomorphisms, if $\alpha : \mathbb{T} \to]-\frac{m}{k}, \frac{m}{k}[$ is a continuous function, and if $\delta > 0$, such that*

(i) $\widehat{\psi}(\widehat{h}) > \frac{8}{p}$, $\quad h \in H(A,l)$;

(ii) $\widehat{\psi}(\widehat{h}) > \frac{6}{k}$, $\quad h \in H(A,p)$;

(iii) $\widehat{\psi}(\widehat{h}) > \frac{1}{q}$, $\quad h \in H(A,k)$;

(iv) $\|\widehat{\varphi}(\widehat{h}) - \widehat{\psi}(\widehat{h})\| < \delta$, $\quad h \in \widetilde{H}(A, 2q)$;

(v) $\widehat{\psi}(\widehat{h}) > \delta$, $\quad h \in H(A, 10q)$;

(vi) $[\varphi] = [\psi]$ *in* $KK(A,B)$;

(vii) $Det(\varphi(V_A)(z)) = Det(\psi(V_A)(z)) \exp(2\pi i \alpha(z))$, $\quad z \in \mathbb{T}$;

and such that at least one of the two statements

(a) $d(B) \ge k$,

(b) $Det(\Lambda_j \circ \varphi(V_A)) = Det(\Lambda_j \circ \psi(V_A))$ *and* $\alpha(y_j) = 0$, $\quad j = 1, 2, \dots, M$,

is true; then there exists a unitary $W \in B$ such that

$$\|\varphi(f) - W\psi(f)W^*\| < \epsilon, \quad f \in F.$$

Proof. We may assume that $\epsilon < 12$. Choose l_0 such that $l_0 \ge 32n$, such that $l_0 \ge 2Nn$, and such that for $x, y \in \mathbb{T}$,

$$\rho(x,y) < \frac{10}{l_0} \Rightarrow \|f(x) - f(y)\| < \frac{\epsilon}{6}, \quad f \in F \cup \{V_A\}.$$

Let integers $q \ge k \ge p \ge l \ge l_0$, a building block $B = A(m, e_1, e_2, \dots, e_M)$, and unital *-homomorphisms $\varphi, \psi : A \to B$ be given such that (i)-(vii) and either (a) or (b) are satisfied. Choose $c > 0$ such that for $x, y \in \mathbb{T}$,

$$\rho(x,y) < c \Rightarrow \|\varphi(f)(x) - \varphi(f)(y)\| < \frac{\epsilon}{6}, \quad f \in F,$$
$$\rho(x,y) < c \Rightarrow \|\psi(f)(x) - \psi(f)(y)\| < \frac{\epsilon}{6}, \quad f \in F.$$

Let for each $j = 1, 2, \dots, M$, $t_j \in]0,1[$ be the number such that $e^{2\pi i t_j} = y_j$. Let $\tau : \mathbb{T} \to \mathbb{T}$ be a continuous function such that $|\tau(z) - z| < c$ for every $z \in \mathbb{T}$, and such that for each $j = 1, 2, \ldots, M$, τ is constantly equal to y_j on some arc

$$I_j = \{e^{2\pi it} : t \in [a_j, b_j]\},$$

where $0 < a_j < t_j < b_j < 1$. Define a unital *-homomorphism $\chi : B \to B$ by $\chi(f) = f \circ \tau$. Set $\varphi_1 = \chi \circ \varphi$ and $\psi_1 = \chi \circ \psi$. Then

$$\|\varphi(f) - \varphi_1(f)\| < \frac{\epsilon}{6}, \quad f \in F,$$
$$\|\psi(f) - \psi_1(f)\| < \frac{\epsilon}{6}, \quad f \in F.$$

φ_1 and ψ_1 satisfies (i)-(vi), whereas (vii) can be replaced by

$$Det(\varphi_1(V_A)(z)) = Det(\psi_1(V_A)(z)) \exp(2\pi i \alpha_1(z)), \quad z \in \mathbb{T},$$

where $\alpha_1 = \alpha \circ \tau$. Note that $\|\alpha_1\|_\infty < \frac{m}{k}$, and that $\alpha_1(y_j) = \alpha(y_j)$, $j = 1, 2, \ldots, M$.

Fix some $j = 1, 2, \ldots, M$. Let $\iota_j : M_{e_j} \to M_m$ denote the (unital) inclusion. By Proposition 4.9 and (vi) we have that $s^\varphi(j, i) \equiv s^\psi(j, i) \bmod \frac{n}{d_i}$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Choose s_i^j, $0 \le s_i^j < \frac{n}{d_i}$, such that $s_i^j = s^\varphi(j, i) \bmod \frac{n}{d_i}$. By Lemma 2.1 we see that for each $z \in I_j$,

$$\varphi_1(f)(z) = \iota_j(y_1^j \operatorname{diag}\big(\Lambda_1^{s_1^j}(f), \ldots, \Lambda_N^{s_N^j}(f), f(e^{2\pi i\theta_1^j}), \ldots, f(e^{2\pi i\theta_{D_j}^j})\big) y_1^{j*}),$$
$$\psi_1(f)(z) = \iota_j(y_2^j \operatorname{diag}\big(\Lambda_1^{s_1^j}(f), \ldots, \Lambda_N^{s_N^j}(f), f(e^{2\pi i\omega_1^j}), \ldots, f(e^{2\pi i\omega_{D_j}^j})\big) y_2^{j*}),$$

for some unitaries $y_1^j, y_2^j \in M_{e_j}$ and numbers $\theta_1^j, \ldots, \theta_{D_j}^j, \omega_1^j, \ldots, \omega_{D_j}^j \in \mathbb{R}$. By changing y_1^j and y_2^j we may assume that

$$\theta_1^j \le \theta_2^j \le \cdots \le \theta_{D_j}^j \le \theta_1^j + 1,$$
$$\omega_1^j \le \omega_2^j \le \cdots \le \omega_{D_j}^j \le \omega_1^j + 1,$$

and

$$|\sum_{r=1}^{D_j} (\omega_r^j - \theta_r^j)| < 1. \tag{8}$$

Let I be a $2q$-arc. By (iv) and (v),

$$\begin{aligned}
&\#\{r : e^{2\pi i\theta_r^j} \in I\}n + \sum_{\{i : x_i \in I\}} s_i^j d_i \\
&\le Tr(\Lambda_j \circ \varphi_1(g_I^A \otimes 1)) \\
&< e_j\delta + Tr(\Lambda_j \circ \psi_1(g_I^A \otimes 1)) \\
&\le e_j\delta + \#\{r : e^{2\pi i\omega_r^j} \in I \pm \frac{1}{10q}\}n + \sum_{\{i : x_i \in \operatorname{supp} g_I^A\}} s_i^j d_i \\
&\le \#\{r : e^{2\pi i\omega_r^j} \in I \pm \frac{1}{5q}\}n + \sum_{\{i : x_i \in \operatorname{supp} g_I^A\}} s_i^j d_i.
\end{aligned}$$

Hence

$$\#\{r : e^{2\pi i\theta_r^j} \in I\} \le \#\{r : e^{2\pi i\omega_r^j} \in I \pm \frac{1}{5q}\}.$$

Therefore by Lemma 7.1,

$$R_{D_j}\big((e^{2\pi i\theta_1^j}, e^{2\pi i\theta_2^j}, \ldots, e^{2\pi i\theta_{D_j}^j}), (e^{2\pi i\omega_1^j}, e^{2\pi i\omega_2^j}, \ldots, e^{2\pi i\omega_{D_j}^j})\big) < \frac{1}{2q} + \frac{1}{5q} < \frac{1}{q}.$$

By (iii), if J is a k-arc then

$$\#\{r : e^{2\pi i \omega_r^j} \in J\} \geq \frac{e_j}{q}.$$

Clearly $\frac{D_j}{q} \leq \frac{e_j}{q}$. From (v) it follows that $e_j \geq 10q > q$. Thus

$$|\sum_{r=1}^{D_j} (\omega_r^j - \theta_r^j)| < 1 < \frac{e_j}{q},$$

and

$$e_j = D_j n + \sum_{i=1}^{N} s_i^j d_i < D_j n + Nn \leq D_j n + \frac{q}{2} < D_j n + \frac{e_j}{2} \;\Rightarrow\; 2\frac{e_j}{q} < D_j \frac{4n}{q} < D_j.$$

By Lemma 7.4 it follows that

$$|\theta_r^j - \omega_r^j| < \frac{1}{q} + \frac{3}{k} \leq \frac{4}{k}, \quad r = 1, 2, \ldots, D_j.$$

For each $r = 1, 2, \ldots, D_j$, choose a_j', b_j' such that $a_j < a_j' < t_j < b_j' < b_j$. Let $g_r^j : [a_j, b_j] \to \mathbb{R}$ be the continuous function such that $g_r^j(a_j) = g_r^j(b_j) = \theta_r^j$, $g_r^j(a_j') = g_r^j(b_j') = \omega_r^j$, and such that g_r^j is linear when restricted to each of the intervals $[a_j, a_j']$, $[a_j', b_j']$, and $[b_j', b_j]$. Note that

$$|g_r^j(t) - \theta_r^j| < \frac{4}{k}, \quad r = 1, 2, \ldots, D_j. \tag{9}$$

Define a *-homomorphism $\xi_j : A \to C(I_j) \otimes M_m$ by

$$\xi_j(f)(e^{2\pi i t}) = \iota_j(y_1^j \operatorname{diag}\big(\Lambda_1^{s_1^j}(f), \ldots, \Lambda_N^{s_N^j}(f), f(e^{2\pi i g_1^j(t)}), \ldots, f(e^{2\pi i g_{D_j}^j(t)})\big) y_1^{j*}),$$

for $t \in [a_j, b_j]$, $f \in A$.

Define $\xi : A \to B$ by

$$\xi(f)(z) = \begin{cases} \xi_j(f)(z), & z \in I_j,\ j = 1, 2, \ldots, M, \\ \varphi_1(f)(z), & z \in \mathbb{T} \setminus \cup_{j=1}^{M} I_j. \end{cases}$$

Then

$$\|\varphi_1(f) - \xi(f)\| < \frac{\epsilon}{6}, \quad f \in F \cup \{V_A\}.$$

Hence $[\varphi_1(V_A)] = [\xi(V_A)]$ in $K_1(B)$. Note that

$$Det(\xi(V_A)(z)) = Det(\varphi_1(V_A)(z)) \exp(2\pi i \beta(z)), \quad z \in \mathbb{T},$$

where $\beta : \mathbb{T} \to \mathbb{R}$ is a continuous function defined by, for $t \in [0, 1]$,

$$\beta(e^{2\pi i t}) = \begin{cases} \sum_{r=1}^{D_j} (g_r^j(t) - \theta_r^j) \frac{m}{e_j} & t \in [a_j, b_j],\ j = 1, 2, \ldots, M, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$Det(\xi(V_A)(z)) = Det(\psi_1(V_A)(z)) \exp(2\pi i \gamma_1(z))$$

where $\gamma_1 = \alpha_1 + \beta$. Note that

$$|\gamma_1(z)| \leq |\alpha_1(z)| + |\beta(z)| < \frac{m}{k} + 4\frac{m}{k} = 5\frac{m}{k}, \quad z \in \mathbb{T},$$

and that $f \mapsto \Lambda_j \circ \xi(f)$ and $f \mapsto \Lambda_j \circ \psi_1(f)$ are equivalent representations of A on M_{e_j}, $j = 1, 2, \ldots, M$. Hence $\xi^* = \psi_1^* = \varphi_1^*$ in $Hom(K^0(B), K^0(A))$. By (v) it follows that $e_j \geq 10q > q > Nn$, $j = 1, 2, \ldots, M$. Thus by Theorem 4.8 we see that $[\xi] = [\varphi_1]$ in $KK(A, B)$.

Next we will construct a *-homomorphism $\lambda : A \to B$ such that $[\lambda] = [\varphi_1]$ in $KK(A, B)$, such that

$$\|\varphi_1(f) - \lambda(f)\| < \frac{\epsilon}{6}, \quad f \in F,$$

and such that $f \mapsto \Lambda_j \circ \lambda(f)$ and $f \mapsto \Lambda_j \circ \psi_1(f)$ are equivalent representations of A on M_{e_j}, $j = 1, 2, \ldots, M$, together with a continuous function $\gamma : \mathbb{T} \to \mathbb{R}$ such that $\gamma(y_j) = 0$, $j = 1, 2, \ldots, M$, such that $|\gamma(z)| < 8\frac{m}{p}$, $z \in \mathbb{T}$, and such that

$$Det(\lambda(V_A)(z)) = Det(\psi_1(V_A)(z)) \exp(2\pi i \gamma(z)), \quad z \in \mathbb{T}.$$

First assume that (a) is true. Since $[\xi(V_A)] = [\varphi_1(V_A)] = [\psi_1(V_A)]$ in $K_1(B)$ and since for $j = 1, 2, \ldots, M$, $Det(\Lambda_j \circ \xi(V_A)) = Det(\Lambda_j \circ \psi_1(V_A))$, we get by Theorem 3.4 integers $l_1, l_2, \ldots, l_M$ and C, such that

$$\gamma_1(y_j) = l_j \frac{m}{e_j} + C.$$

Set $e = d(B)$. As $\frac{m}{e_j}$ divides $\frac{m}{e}$ for every $j = 1, 2, \ldots, M$, (actually, we have that $\frac{m}{e} = lcm(\frac{m}{e_1}, \frac{m}{e_2}, \ldots, \frac{m}{e_M})$) we may assume that $0 \le C < \frac{m}{e} \le \frac{m}{k}$. Thus

$$|l_j \frac{m}{e_j}| \le 6\frac{m}{k} \;\Rightarrow\; |l_j| \le 6\frac{e_j}{k}.$$

Fix some $j = 1, 2, \ldots, M$. Let I be a p-arc and let I° denote its interior. By (ii)

$$\#\{r = 1, 2, \ldots, D_j : e^{2\pi i \omega_r^j} \in I^\circ\} \ge Tr(\Lambda_j \circ \psi_1(f_A^I \otimes 1)) > 6\frac{e_j}{k}. \tag{10}$$

Assume that $|\omega_r^j - \omega_{r+|l_j|}^j| \ge \frac{2}{p}$ for some $r \in \mathbb{Z}$. Then there exists a p-arc I such that $I^\circ \subseteq J$, where

$$J = \{e^{2\pi i t} : \omega_r^j < t < \omega_{r+|l_j|}^j\}.$$

Thus J contains at least $6\frac{e_j}{k}$ elements from the set $\{e^{2\pi i \omega_r^j} : r = 1, 2, \ldots, D_j\}$. On the other hand, by (10)

$$D_j > 6\frac{e_j}{k} \ge |l_j|,$$

and therefore

$$\begin{aligned} &\#\{s = 1, 2, \ldots, D_j : e^{2\pi i \omega_s^j} \in J\} \\ &= \#\{s = r, r+1, \ldots, r + D_j - 1 : \omega_r^j < \omega_s^j < \omega_{r+|l_j|}\} \le |l_j| - 1 < 6\frac{e_j}{k}. \end{aligned}$$

Contradiction. Hence $|\omega_r^j - \omega_{r-l_j}^j| < \frac{2}{p}$, $r = 1, 2, \ldots, D_j$.

For $r = 1, 2, \ldots, D_j$, let $h_r^j : [a_j, b_j] \to \mathbb{R}$ be the continuous function such that $h_r^j(t) = g_r^j(t)$ for $t \in [a_j, a_j'] \cup [b_j', b_j]$, such that $h_r^j(t_j) = \omega_{r-l_j}^j$ and such that h_r^j is linear when restricted to each of the intervals $[a_j', t_j]$ and $[t_j, b_j']$. Note that

$$|h_r^j(t) - \theta_r^j| \le |h_r^j(t) - g_r^j(t)| + |g_r^j(t) - \theta_r^j| < \frac{2}{p} + \frac{4}{k} \le \frac{6}{p}, \quad t \in [a_j, b_j]. \tag{11}$$

Define a *-homomorphism $\lambda_j : A \to C(I_j) \otimes M_m$ by, for $t \in [a_j, b_j]$, $f \in A$,

$$\lambda_j(f)(e^{2\pi i t}) = \iota_j(y_1^j \operatorname{diag}(\Lambda_1^{s_1^j}(f), \ldots, \Lambda_N^{s_N^j}(f), f(e^{2\pi i h_1^j(t)}), \ldots, f(e^{2\pi i h_{D_j}^j(t)})) y_1^{j*}).$$

Define $\lambda : A \to B$ by

$$\lambda(f)(z) = \begin{cases} \lambda_j(f)(z), & z \in I_j, \; j = 1, 2, \ldots, M, \\ \varphi_1(f)(z), & z \in \mathbb{T} \setminus \cup_{j=1}^M I_j. \end{cases}$$

Note that

$$\|\lambda(f) - \varphi_1(f)\| < \frac{\epsilon}{6}, \quad f \in F \cup \{V_A\}.$$

By Theorem 4.8 we see that $[\lambda] = [\varphi_1]$ in $KK(A, B)$. Also

$$Det(\lambda(V_A)(z)) = Det(\xi(V_A)(z)) \exp(2\pi i \gamma_2(z)), \quad z \in \mathbb{T},$$

where $\gamma_2 : \mathbb{T} \to \mathbb{R}$ is a continuous function defined by, for $t \in [0, 1]$,

$$\gamma_2(e^{2\pi it}) = \begin{cases} \sum_{r=1}^{D_j} (h_r^j(t) - \omega_r^j) \frac{m}{e_j} & t \in [a_j', b_j'],\ j = 1, 2, \ldots, M, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $|\gamma_2(z)| \leq 2\frac{m}{p}$, $z \in \mathbb{T}$, and

$$Det(\lambda(V_A)(z)) = Det(\psi_1(V_A)(z)) \exp(2\pi i \gamma(z)), \quad z \in \mathbb{T},$$

where $\gamma = \gamma_1 + \gamma_2 - C$. For each $j = 1, 2, \ldots, M$,

$$\gamma(y_j) = \gamma_1(y_j) + \gamma_2(y_j) - C = l_j \frac{m}{e_j} + C + \sum_{r=1}^{D_j} (\omega_{r-l_j}^j - \omega_r^j) \frac{m}{e_j} - C = 0,$$

and

$$|\gamma(z)| \leq |\gamma_1(z)| + |\gamma_2(z)| + |C| < 5\frac{m}{k} + 2\frac{m}{p} + \frac{m}{k} \leq 8\frac{m}{p}, \quad z \in \mathbb{T}.$$

Now assume that (b) is true. Set $\lambda = \xi$ and $\gamma = \gamma_1$. Since

$$Det(\Lambda_j \circ \varphi_1(V_A)) = Det(\Lambda_j \circ \psi_1(V_A)), \quad j = 1, 2, \ldots, M,$$

we get that the left-hand side of (8) is 0. Hence $\beta(y_j) = 0$ and thus we see that

$$\gamma(y_j) = \gamma_1(y_j) = \alpha_1(y_j) + \beta(y_j) = 0 + 0 = 0, \quad j = 1, 2, \ldots, M.$$

Also

$$|\gamma(z)| = |\gamma_1(z)| \leq 5\frac{m}{k} < 8\frac{m}{p}, \quad z \in \mathbb{T}.$$

This completes the construction in the case that (b) is true.

By Proposition 6.3, φ_1, ψ_1, and λ are approximately unitarily equivalent to φ_1', ψ_1', and λ', respectively, where $\varphi_1', \psi_1', \lambda' : A \to B$ are *-homomorphisms of the form

$$\varphi_1'(f)(e^{2\pi it}) = u(t)\mathrm{diag}\big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f(e^{2\pi iF_1(t)}), \ldots, f(e^{2\pi iF_L(t)})\big)u(t)^*,$$
$$\psi_1'(f)(e^{2\pi it}) = v(t)\mathrm{diag}\big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f(e^{2\pi iG_1(t)}), \ldots, f(e^{2\pi iG_L(t)})\big)v(t)^*,$$
$$\lambda'(f)(e^{2\pi it}) = w(t)\mathrm{diag}\big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f(e^{2\pi iH_1(t)}), \ldots, f(e^{2\pi iH_L(t)})\big)w(t)^*,$$

for integers $r_1, r_2, \ldots, r_N$ with $0 \leq r_i < \frac{n}{d_i}$, $i = 1, 2, \ldots, N$, unitaries u, v, w in $C[0, 1] \otimes M_m$ with $u(0) = v(0) = w(0)$, $u(1) = v(1) = w(1)$, and continuous functions $F_r, G_r, H_r : [0, 1] \to \mathbb{R}$, $r = 1, 2, \ldots, L$, such that for $t \in [0, 1]$,

$$F_1(t) \leq F_2(t) \leq \cdots \leq F_L(t) \leq F_1(t) + 1,$$
$$G_1(t) \leq G_2(t) \leq \cdots \leq G_L(t) \leq G_1(t) + 1,$$
$$H_1(t) \leq H_2(t) \leq \cdots \leq H_L(t) \leq H_1(t) + 1.$$

Note that for $t \in [0, 1]$,

$$Det(\lambda'(V_A)(e^{2\pi it})) = Det(\psi_1'(V_A)(e^{2\pi it})) \exp(2\pi i \sum_{r=1}^{L} (H_r(t) - G_r(t))).$$

On the other hand, by the above

$$Det(\lambda'(V_A)(e^{2\pi it})) = Det(\psi_1'(V_A)(e^{2\pi it}))\exp(2\pi i\gamma(e^{2\pi it})).$$

By the last part of Proposition 6.3 we may thus assume that for each $t \in [0,1]$,

$$\gamma(e^{2\pi it}) = \sum_{r=1}^{L}(H_r(t) - G_r(t)). \tag{12}$$

Hence

$$|\sum_{r=1}^{L}(H_r(t) - G_r(t))| < 8\frac{m}{p}.$$

From (11) if (a) is true, or (9) if (b) is true, we get that for each $t \in [0,1]$,

$$R_L\big((e^{2\pi iF_1(t)},\dots,e^{2\pi iF_L(t)}),(e^{2\pi iH_1(t)},\dots,e^{2\pi iH_L(t)})\big) < \frac{6}{p}.$$

Let $t \in [0,1]$ and let I be a $2q$-arc. Then by (iv) and (v)

$$\begin{aligned}
&\#\{r : e^{2\pi iF_r(t)} \in I\}n + \sum_{\{i:x_i\in I\}} r_i d_i \\
&\le Tr(\varphi_1'(g_I^A \otimes 1)(e^{2\pi it})) \\
&< m\delta + Tr(\psi_1'(g_I^A \otimes 1)(e^{2\pi it})) \\
&\le m\delta + \#\{r : e^{2\pi iG_r(t)} \in I \pm \frac{1}{10q}\}n + \sum_{\{i:x_i\in \mathrm{supp}\, g_I^A\}} r_i d_i \\
&\le \#\{r : e^{2\pi iG_r(t)} \in I \pm \frac{1}{5q}\}n + \sum_{\{i:x_i\in \mathrm{supp}\, g_I^A\}} r_i d_i.
\end{aligned}$$

Hence

$$\#\{r : e^{2\pi iF_r(t)} \in I\} \le \#\{r : e^{2\pi iG_r(t)} \in I \pm \frac{1}{5q}\}.$$

It follows from Lemma 7.1 that for each $t \in [0,1]$,

$$R_L\big((e^{2\pi iF_1(t)},\dots,e^{2\pi iF_L(t)}),(e^{2\pi iG_1(t)},\dots,e^{2\pi iG_L(t)})\big) < \frac{1}{2q} + \frac{1}{5q} < \frac{1}{q}.$$

We conclude that

$$R_L\big((e^{2\pi iG_1(t)},\dots,e^{2\pi iG_L(t)}),(e^{2\pi iH_1(t)},\dots,e^{2\pi iH_L(t)})\big) < \frac{1}{q} + \frac{6}{p} \le \frac{7}{p}.$$

Since $f \mapsto \psi_1'(f)(y_j)$ and $f \mapsto \lambda'(f)(y_j)$ are equivalent representations of A on M_m for $j = 1,2,\dots,M$, we get that

$$(e^{2\pi iG_1(t_j)},\dots,e^{2\pi iG_L(t_j)}) = (e^{2\pi iH_1(t_j)},\dots,e^{2\pi iH_L(t_j)})$$

as unordered L-tuples. Since $\gamma(y_j) = 0$, $j = 1,2,\dots,M$, it follows from Lemma 6.1 and (12) that

$$G_r(t_j) = H_r(t_j), \quad r = 1,2,\dots,L,\ j = 1,2,\dots,M.$$

As $v(0) = w(0)$, $v(1) = w(1)$, we may thus define a *-homomorphism $\mu : A \to B$ by

$$\mu(f)(e^{2\pi it}) = v(t)\mathrm{diag}\big(\Lambda_1^{r_1}(f),\dots,\Lambda_N^{r_N}(f), f(e^{2\pi iH_1(t)}),\dots,f(e^{2\pi iH_L(t)})\big)v(t)^*,$$

for $f \in A$, $t \in [0,1]$. Since

$$Tr(\mu(f)(z)) = Tr(\lambda'(f)(z)) = Tr(\lambda(f)(z)), \quad z \in \mathbb{T},\ f \in A,$$

we get from [21, Theorem 1.4] that μ and λ are approximately unitarily equivalent.

By (i) we have that for every l-arc J,

$$\#\{r : e^{2\pi i G_r(t)} \in J\} > 8\frac{m}{p}.$$

As $L\frac{7}{p} \leq 8\frac{m}{p}$, and since $m \geq q$, we see that

$$m = Ln + \sum_{i=1}^{N} r_i d_i < Ln + Nn \leq Ln + \frac{q}{2} \leq Ln + \frac{m}{2} \;\Rightarrow\; 16\frac{m}{p} \leq \frac{32Ln}{p} \leq L.$$

We conclude from Lemma 7.4 that

$$|G_r(t) - H_r(t)| < \frac{7}{p} + \frac{3}{l} \leq \frac{10}{l}.$$

Hence

$$\|\mu(f) - \psi_1'(f)\| < \frac{\epsilon}{6}, \quad f \in F.$$

Choose unitaries $U, V \in B$ such that

$$\|\lambda(f) - U\mu(f)U^*\| < \frac{\epsilon}{6}, \quad f \in F,$$
$$\|\psi_1'(f) - V\psi_1(f)V^*\| < \frac{\epsilon}{6}, \quad f \in F.$$

Set $W = UV$. Then for $f \in F$,

$$\begin{aligned}
&\|\varphi(f) - W\psi(f)W^*\| \\
\leq\ &\|\varphi(f) - \varphi_1(f)\| + \|\varphi_1(f) - \lambda(f)\| + \|\lambda(f) - U\mu(f)U^*\| + \\
&\|U\mu(f)U^* - U\psi_1'(f)U^*\| + \|U\psi_1'(f)U^* - UV\psi_1(f)V^*U^*\| + \\
&\|W\psi_1(f)W^* - W\psi(f)W^*\| \\
<\ &\frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.
\end{aligned}$$

□

Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_R$ be a finite direct sum of building blocks. For each $i = 1, 2, \ldots, R$, we define a unitary V_A^i in A by

$$V_A^i = (1, \ldots, 1, V_{A_i}, 1, \ldots, 1).$$

If p is a positive integer, we set

$$H(A, p) = \cup_{i=1}^{R} \iota_i(H(A_i, p)),$$
$$\widetilde{H}(A, p) = \cup_{i=1}^{R} \iota_i(\widetilde{H}(A_i, p)),$$

where $\iota_i : A_i \to A$ denotes the inclusion, $i = 1, 2, \ldots, R$.

Theorem 7.6. *Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_R$ be a finite direct sum of building blocks. Let $p_1, p_2, \ldots, p_R$ be the minimal non-zero central projections in A. Let $\epsilon > 0$ and let $F \subseteq A$ be a finite set. There exists a positive integer l such that if p, k and q are positive integers with $l \leq p \leq k \leq q$, if B is a finite direct sum of building blocks, if $\varphi, \psi : A \to B$ are unital *-homomorphisms, if $\delta > 0$, and if*

(i) $\widehat{\psi}(\widehat{h}) > \frac{8}{p}, \quad h \in H(A, l),$

(ii) $\widehat{\psi}(\widehat{h}) > \frac{6}{k}, \quad h \in H(A, p),$

(iii) $\widehat{\psi}(\widehat{h}) > \frac{1}{q}, \quad h \in H(A, k),$

(iv) $\|\widehat{\varphi}(\widehat{h}) - \widehat{\psi}(\widehat{h})\| < \delta, \quad h \in \widetilde{H}(A, 2q),$

(v) $\widehat{\psi}(\widehat{h}) > \delta, \quad h \in H(A, 10q),$

(vi) $[\varphi] = [\psi]$ *in* $KK(A, B)$,

(vii) $D_B\big(\varphi^{\#}(q'_A(V_A^i)), \psi^{\#}(q'_A(V_A^i))\big) < \frac{1}{k^2}, \quad i = 1, 2, \ldots, R,$

(viii) $\widehat{\psi}(\widehat{p_i}) > \frac{1}{k}, \quad i = 1, 2, \ldots, R,$

(ix) $d(B) \geq k^2$;

then there exists a unitary $W \in B$ *such that*

$$\|\varphi(f) - W\psi(f)W^*\| < \epsilon, \quad f \in F.$$

Proof. For each $i = 1, 2, \ldots, R$, let $\iota_i : A_i \to A$ be the inclusion and let $\pi_i : A \to A_i$ be the projection. Choose by Theorem 7.5 a positive integer l_0^i with respect to the finite set $\pi_i(F) \subseteq A_i$ and $\epsilon > 0$. Set $l = \max_i l_0^i$.

Let integers $q \geq k \geq p \geq l$, a finite direct sum of building blocks B, and unital *-homomorphisms $\varphi, \psi : A \to B$ be given such that (i)-(ix) are satisfied. It is easy to reduce to the case where $B = A(m, e_1, e_2, \ldots, e_M)$ is a single building block.

As $\varphi_*[p_i] = \psi_*[p_i]$ in $K_0(B)$ for $i = 1, 2, \ldots, R$, there is by Lemma 3.5 a unitary $U \in B$ such that $U\varphi(p_i)U^* = \psi(p_i)$ for every $i = 1, 2, \ldots, R$. Hence we may assume that $\varphi(p_i) = \psi(p_i)$, $i = 1, 2, \ldots, R$. Set $q_i = \psi(p_i)$. It follows from (viii) that $q_i \neq 0$, $i = 1, 2, \ldots, R$.

Let $\varphi_i, \psi_i : A_i \to q_iBq_i$ be the induced maps. Let $\epsilon_i : q_iBq_i \to B$ be the inclusion. Then $[\epsilon_i] \in KK(q_iBq_i, B)$ is a KK-equivalence. Thus

$$[\varphi_i] = [\epsilon_i]^{-1} \cdot [\varphi] \cdot [\iota_i] = [\epsilon_i]^{-1} \cdot [\psi] \cdot [\iota_i] = [\psi_i]$$

in $KK(A_i, q_iBq_i)$. By Lemma 3.5 we have that

$$q_iBq_i \cong A(m_i, \frac{m_i}{m}e_1, \frac{m_i}{m}e_2, \ldots, \frac{m_i}{m}e_M) \tag{13}$$

where $m_i \in \mathbb{Z}$ denotes the rank of q_i.

Fix some $i = 1, 2, \ldots, M$. There exist some $c_i \in \overline{DU(B)}$ and a selfadjoint element $b_i \in B$ with $\|b_i\| < \frac{1}{k^2}$ such that

$$\varphi(V_A^i) = c_i e^{2\pi i b_i} \psi(V_A^i).$$

Thus

$$Det(\varphi(V_A^i)(z)) = e^{2\pi i Tr(b_i(z))} Det(\psi(V_A^i)(z)), \quad z \in \mathbb{T}.$$

Note that

$$Det(\varphi(V_A^i)(z)) = Det(\varphi_i(V_{A_i})(z)), \quad z \in \mathbb{T},$$

where the latter determinant is calculated in M_{m_i}, and similarly for ψ. Thus

$$Det(\varphi_i(V_{A_i})(z)) = e^{2\pi i Tr(b_i(z))} Det(\psi_i(V_{A_i})(z)), \quad z \in \mathbb{T}.$$

As

$$m_i = Tr(\psi(p_i)) > \frac{m}{k}$$

we see that

$$|Tr(b_i(z))| < \frac{m}{k^2} = \frac{m_i}{k}\frac{m}{m_i k} < \frac{m_i}{k}.$$

and that

$$d(q_iBq_i) = gcd(\frac{m_i}{m}e_1, \frac{m_i}{m}e_2, \dots, \frac{m_i}{m}e_M)$$
$$= \frac{m_i}{m}gcd(e_1, e_2, \dots, e_M) = \frac{m_i}{m}d(B) \geq \frac{m_i}{m}k^2 > k.$$

Finally, as every tracial state on q_iBq_i is of the form $\frac{1}{\omega(q_i)}\omega|_{q_iBq_i}$ for some $\omega \in T(B)$, we see that φ_i and ψ_i also satisfy (i)-(v) in Theorem 7.5. Hence we get a unitary $W_i \in q_iBq_i$ such that

$$\|\varphi_i(f) - W_i\psi_i(f)W_i^*\| < \epsilon, \quad f \in \pi_i(F).$$

Set $W = \sum_{i=1}^R W_i$. Then $W \in B$ is a unitary and

$$\|\varphi(f) - W\psi(f)W^*\| < \epsilon, \quad f \in F.$$

□

8. Existence

In this section we prove some existence results for *-homomorphisms between finite direct sums of building blocks. Together with the uniqueness results of the previous section, these results are the cornerstones in the proof of the classification theorem.

Lemma 8.1. *Let $A = A(n, d_1, d_2, \dots, d_N)$ and B be building blocks where $s(B) \geq Nn$. Let $\varphi : A \to B$ be a unital *-homomorphism and let $\kappa \in KK(A, B)$ be an element such that $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit. Assume that $\varphi^* = \kappa^*$ in $Hom(K^0(B), K^0(A))$ and that $\varphi_*[u] = \kappa_*[u]$ in $K_1(B)$, where $u \in A$ is a unitary such that $Det(u(\cdot))$ has winding number 1. Then $[\varphi] = \kappa$ in $KK(A, B)$.*

Proof. By Theorem 4.8 there exists a unital *-homomorphism $\psi : A \to B$ such that $[\psi] = \kappa$ in $KK(A, B)$. V_Au^* is a unitary of finite order in $K_1(A)$ and hence by Lemma 5.7 $[V_Au^*] = [w]$ in $K_1(A)$ where $w \in A$ is a unitary such that $Det(w(\cdot))$ is constant. By Lemma 5.8 we see that $\varphi_*[w] = \psi_*[w]$ in $K_1(B)$. It follows that $\varphi_*[V_A] = \psi_*[V_A]$. Hence $[\varphi] = [\psi] = \kappa$ in $KK(A, B)$ by Theorem 4.8. □

Let A and B be building blocks and let $\varphi : A \to B$ be a *-homomorphism. We define a continuous function $\lambda : \mathbb{T} \to \mathbb{T}$ to be an eigenvalue function for φ if $\lambda(z)$ is an eigenvalue for the matrix $\varphi(\iota \otimes 1)(z)$ for every $z \in \mathbb{T}$. Here $\iota : \mathbb{T} \to \mathbb{C}$ denotes the inclusion.

Theorem 8.2. *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block, let $\epsilon > 0$, and let C be a positive integer. There exists a positive integer K such that if*

- (i) *$u \in A$ is a unitary such that $Det(u(z)) = z\exp(2\pi i\beta(z))$, $z \in \mathbb{T}$, where $\beta : \mathbb{T} \to \mathbb{R}$ is a continuous function;*
- (ii) *$B = A(m, e_1, e_2, \dots, e_M)$ is a building block and $\kappa \in KK(A, B)$ is an element such that $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit;*
- (iii) *$s(B) \geq K$;*
- (iv) *$\lambda_1, \lambda_2, \dots, \lambda_C : \mathbb{T} \to \mathbb{T}$ are continuous functions;*
- (v) *$v \in B$ is a unitary such that $\kappa_*[u] = [v]$ in $K_1(B)$;*

*then there exists a unital *-homomorphism $\varphi : A \to B$ such that $[\varphi] = \kappa$ in $KK(A,B)$, such that $\lambda_1, \lambda_2 \ldots, \lambda_C$ are eigenvalue functions for φ and such that*

$$\|\widehat{\varphi}(f) - \frac{1}{C}\sum_{k=1}^{C} f \circ \lambda_k\| < \epsilon \|f\|, \quad f \in \mathit{AffT}(A),$$

when we identify $\mathit{AffT}(\cdot)$ and $C_{\mathbb{R}}(\mathbb{T})$ as order unit spaces, together with a continuous function $\alpha : \mathbb{T} \to \mathbb{R}$ with $\|\alpha\|_\infty \leq 2\|\beta\|_\infty$ that equals 0 at all the exceptional points of B such that

$$\mathit{Det}(\varphi(u)(z)) = \mathit{Det}(v(z)) \exp(2\pi i \alpha(z)), \quad z \in \mathbb{T},$$

and

$$\mathit{Det}(\Lambda_j \circ \varphi(u)) = \mathit{Det}(\Lambda_j(v)), \quad j = 1, 2, \ldots, M.$$

Proof. We may assume that $C > \frac{8}{\epsilon}$. Let K be a positive integer such that

$$K \geq \max(\frac{4(N+C+2)n}{\epsilon}, Nn).$$

Let B, κ, u, β, and v be as above. Let $x_1, x_2, \ldots, x_N \in \mathbb{T}$ denote the exceptional points of A and let $y_1, y_2, \ldots, y_M \in \mathbb{T}$ be those of B.

Let $\kappa^* : K^0(B) \to K^0(A)$ have standard form (cf. Lemma 4.4)

$$\begin{pmatrix} \kappa^*([\Lambda_1^B]) \\ \kappa^*([\Lambda_2^B]) \\ \vdots \\ \kappa^*([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \ldots & h_{1N} \\ h_{21} & h_{22} & \ldots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \ldots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

As in the proof of Theorem 4.8 we see that

$$e_j = \sum_{i=1}^{N} h_{ji} d_i \tag{14}$$

and hence $h_{jN} \geq 0$ for $j = 1, 2, \ldots, M$. Define l_{ji} and s_i for $i = 1, 2, \ldots, N$, and $j = 1, 2, \ldots, M$, as in the proof of Proposition 4.5, i.e such that

$$\frac{m}{e_j} h_{ji} = l_{ji} \frac{n}{d_i} + s_i.$$

For $j = 1, 2, \ldots, M$, choose integers h_{jN}^o, $0 \leq h_{jN}^o < \frac{n}{d_N}$, and $r_j \geq 0$ such that

$$h_{jN} = r_j \frac{n}{d_N} + h_{jN}^o,$$

and note that

$$\frac{m}{e_j} h_{jN}^o = l_{jN}^o \frac{n}{d_N} + s_N$$

for some integers $l_{jN}^o \geq 0$, $j = 1, 2, \ldots, M$. Then

$$l_{jN} - l_{jN}^o = \frac{m}{e_j} r_j.$$

Let for each j

$$r_j = k_j(C+2) + u_j$$

for some integers $k_j \geq 0$ and $0 \leq u_j < C+2$ and set

$$b = \min_{1 \leq j \leq M} k_j \frac{m}{e_j}.$$

Note that for $j = 1, 2, \dots, M$,

$$e_j = \sum_{i=1}^{N} h_{ji} d_i < (N-1)n + r_j n + h^o_{jN} d_N < Nn + r_j n$$
$$= (N + C + 2)n + (r_j - (C+2))n \leq \frac{\epsilon}{4} e_j + (r_j - (C+2))n.$$

Hence

$$(1 - \frac{\epsilon}{4}) e_j < (r_j - (C+2))n.$$

Therefore

$$nk_j(C+2)\frac{m}{e_j} = n(r_j - u_j)\frac{m}{e_j} > n(r_j - (C+2))\,\frac{m}{e_j} > (1 - \frac{\epsilon}{4})m. \tag{15}$$

Since by (14)

$$nk_j(C+2)\frac{m}{e_j} \leq nr_j \frac{m}{e_j} \leq h_{jN} d_N \frac{m}{e_j} \leq m, \tag{16}$$

we see that

$$nb\frac{8}{\epsilon} \leq nbC \leq nb(C+2) \leq m.$$

By this and (15),

$$m(1 - \frac{\epsilon}{4}) < nb(C+2) \leq nbC + \frac{\epsilon}{4} m.$$

Hence from (16) we conclude that

$$0 \leq 1 - \frac{nbC}{m} < \frac{\epsilon}{2}.$$

Define a continuous map $\gamma : \mathbb{T} \to \mathbb{T}$ by

$$\gamma(z) = Det(v(z)).$$

Define $\eta : \mathbb{T} \to \mathbb{T}$ by

$$\eta(z) = Det(u(z)) = z \exp(2\pi i \beta(z)).$$

Set for $j = 1, 2, \dots, M$,

$$a_j = (\prod_{i=1}^{N-1} Det(\Lambda_i(u))^{h_{ji}}) Det(\Lambda_N(u))^{h^o_{jN}}$$

and note that

$$a_j^{\frac{m}{e_j}} = (\prod_{i=1}^{N-1} \eta(x_i)^{l_{ji}}) \eta(x_N)^{l^o_{jN}} \prod_{i=1}^{N} Det(\Lambda_i(u))^{s_i}. \tag{17}$$

Set for $j = 1, 2, \dots, M$,

$$c_j = Det(\Lambda_j(v))$$

and note that

$$c_j^{\frac{m}{e_j}} = \gamma(y_j). \tag{18}$$

As η is surjective, we can choose a continuous function $\lambda_{C+1} : \mathbb{T} \to \mathbb{T}$ such that

$$\eta(\lambda_{C+1}(y_j))^{-k_j} = a_j c_j^{-1} \eta(1)^{(k_j + u_j)} \prod_{k=1}^{C} \eta(\lambda_k(y_j))^{k_j}, \quad j = 1, 2, \dots, M. \tag{19}$$

There exists a unitary $W_j \in M_m$ such that the matrix

$$W_j \operatorname{diag}\Big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), \underbrace{f(x_1), \dots, f(x_1)}_{l_{j1} \text{ times}}, \dots, \underbrace{f(x_{N-1}), \dots, f(x_{N-1})}_{l_{j(N-1)} \text{ times}},$$
$$\underbrace{f(x_N), \dots, f(x_N)}_{l^o_{jN} \text{ times}}, \underbrace{f(\lambda_1(y_j)), \dots, f(\lambda_1(y_j))}_{k_j \frac{m}{e_j} - b \text{ times}}, \dots,$$
$$\underbrace{f(\lambda_{C+1}(y_j)), \dots, f(\lambda_{C+1}(y_j))}_{k_j \frac{m}{e_j} - b \text{ times}}, \underbrace{f(1), \dots, f(1)}_{(k_j+u_j)\frac{m}{e_j} - b \text{ times}},$$
$$\underbrace{f(\lambda_1(y_j)), \dots, f(\lambda_1(y_j))}_{b \text{ times}}, \dots, \underbrace{f(\lambda_{C+1}(y_j)), \dots, f(\lambda_{C+1}(y_j))}_{b \text{ times}},$$
$$\underbrace{f(1), \dots, f(1)}_{b \text{ times}}\Big) W_j^*$$

belongs to $M_{e_j} \subseteq M_m$ for every $f \in A$. Set

$$L = \frac{1}{n}(m - \sum_{i=1}^{N} s_i d_i) - (C+2)b.$$

For each $j = 1, 2, \dots, M$, we have that by (14),

$$L = \sum_{i=1}^{N} l_{ji} - (C+2)b = \sum_{i=1}^{N-1} l_{ji} + l^o_{jN} + \frac{m}{e_j}(k_j(C+2) + u_j) - (C+2)b.$$

Choose for $k = 1, 2, \dots, L$ continuous functions $\mu_k : \mathbb{T} \to \mathbb{T}$ such that for each $j = 1, 2, \dots, M$,

$$(\mu_1(y_j), \mu_2(y_j), \dots, \mu_L(y_j)) = \big(\underbrace{x_1, \dots, x_1}_{l_{j1} \text{ times}}, \dots, \underbrace{x_{N-1}, \dots, x_{N-1}}_{l_{j(N-1)} \text{ times}}, \underbrace{x_N, \dots, x_N}_{l^o_{jN} \text{ times}},$$
$$\underbrace{\lambda_1(y_j), \dots, \lambda_1(y_j)}_{k_j \frac{m}{e_j} - b \text{ times}}, \dots, \underbrace{\lambda_{C+1}(y_j), \dots, \lambda_{C+1}(y_j)}_{k_j \frac{m}{e_j} - b \text{ times}}, \underbrace{1, 1, \dots, 1}_{(k_j+u_j)\frac{m}{e_j} - b \text{ times}}\big)$$

as ordered tuples.

Choose a unitary $W \in C(\mathbb{T}) \otimes M_m$ such that $W(y_j) = W_j$ for $j = 1, 2, \dots, M$. Define a continuous function $g : \mathbb{T} \to \mathbb{T}$ such that

$$g(z) \prod_{k=1}^{L} \eta(\mu_k(z)) \prod_{k=1}^{C+1} \eta(\lambda_k(z))^b \prod_{i=1}^{N} Det(\Lambda_i(u))^{s_i} = \gamma(z)\eta(1)^{-b}, \quad z \in \mathbb{T}.$$

Then by (17), (18), and (19) we have that $g(y_j) = 1$ for $j = 1, 2, \dots, M$. Define $\varphi : A \to B$ by

$$\varphi(f)(z) = W(z) \operatorname{diag}\Big(\Lambda_1^{s_1}(f), \dots, \Lambda_N^{s_N}(f), f(\mu_1(z)), \dots, f(\mu_L(z)),$$
$$\underbrace{f(\lambda_1(z)), \dots, f(\lambda_1(z))}_{b \text{ times}}, \dots, \underbrace{f(\lambda_{C+1}(z)), \dots, f(\lambda_{C+1}(z))}_{b \text{ times}},$$
$$f(g(z)), \underbrace{f(1), \dots, f(1)}_{b-1 \text{ times}}\Big) W(z)^*.$$

φ is a unital *-homomorphism and

$$\begin{aligned}
&\varphi^*([\Lambda_j^B]) = [\Lambda_j^B \circ \varphi] \\
&= \sum_{i=1}^{N-1} \frac{e_j}{m}(s_i + \frac{n}{d_i} l_{ji})[\Lambda_i^A] + \frac{e_j}{m}(s_N + \frac{n}{d_N}(l_{jN}^o + (C+2)k_j \frac{m}{e_j} + u_j \frac{m}{e_j}))[\Lambda_N^A] \\
&= \sum_{i=1}^{N-1} \frac{e_j}{m}(s_i + \frac{n}{d_i} l_{ji})[\Lambda_i^A] + (h_{jN}^o + \frac{n}{d_N} r_j)[\Lambda_N^A] = \sum_{i=1}^{N} h_{ji}[\Lambda_i^A].
\end{aligned}$$

Thus $\varphi^* = \kappa^*$ in $Hom(K^0(B), K^0(A))$. For $\omega \in T(B)$, $f \in \mathrm{Aff}\, T(A)$,

$$\begin{aligned}
&|\widehat{\varphi}(f)(\omega) - \frac{1}{C}\sum_{i=1}^{C} f \circ \lambda_k(\omega)| = |\omega(\varphi(f \otimes 1)) - \frac{1}{C}\sum_{k=1}^{C} \omega((f \circ \lambda_k) \otimes 1)| \\
&\leq |\frac{1}{m}(m - Cbn)|\, \|f\| + \|\frac{1}{m} bn \sum_{k=1}^{C} f \circ \lambda_k - \frac{1}{C}\sum_{k=1}^{C} f \circ \lambda_k\| \\
&\leq |\frac{1}{m}(m - Cbn)|\, \|f\| + |\frac{1}{m} bn - \frac{1}{C}| C\, \|f\| = 2|1 - \frac{Cbn}{m}|\, \|f\| < \epsilon \|f\|.
\end{aligned}$$

Hence

$$\|\widehat{\varphi}(f) - \frac{1}{C}\sum_{k=1}^{C} f \circ \lambda_k\| < \epsilon \|f\|.$$

Furthermore, for $z \in \mathbb{T}$,

$$\begin{aligned}
&Det(\varphi(u)(z)) \\
&= \prod_{i=1}^{N} Det(\Lambda_i(u))^{s_i} \prod_{k=1}^{L} \eta(\mu_k(z)) \big(\prod_{k=1}^{C+1} \eta(\lambda_k(z))^b\big) \eta(g(z)) \eta(1)^{b-1} \\
&= \prod_{i=1}^{N} Det(\Lambda_i(u))^{s_i} \prod_{k=1}^{L} \eta(\mu_k(z)) \big(\prod_{k=1}^{C+1} \eta(\lambda_k(z))^b\big) g(z) \exp\big(2\pi i \beta(g(z))\big)\, \eta(1)^{b-1} \\
&= \gamma(z)\eta(1)^{-b} \exp(2\pi i \beta(g(z)))\, \eta(1)^{b-1} \\
&= Det(v(z)) \exp(2\pi i(\beta(g(z)) - \beta(1))).
\end{aligned}$$

Thus, if we define $\alpha : \mathbb{T} \to \mathbb{R}$ by $\alpha(z) = \beta(g(z)) - \beta(1)$ we conclude that

$$Det(\varphi(u)(z)) = Det(v(z)) \exp(2\pi i \alpha(z)), \quad z \in \mathbb{T},$$

that $\alpha(y_j) = 0$ for $j = 1, 2, \ldots, M$, and that

$$\|\alpha\|_\infty \leq 2\|\beta\|_\infty.$$

Finally by (19), for $j = 1, 2, \ldots, M$,

$$\begin{aligned}
Det(\Lambda_j \circ \varphi(u)) &= \big(\prod_{i=1}^{N-1} Det(\Lambda_i(u))^{h_{ji}}\big) Det(\Lambda_N(u))^{h_{jN}^o} \big(\prod_{k=1}^{C+1} \eta(\lambda_k(y_j))^{k_j}\big) \eta(1)^{(k_j+u_j)} \\
&= \big(\prod_{i=1}^{N-1} Det(\Lambda_i(u))^{h_{ji}}\big) Det(\Lambda_N(u))^{h_{jN}^o} a_j{}^{-1} c_j = Det(\Lambda_j(v)).
\end{aligned}$$

Hence $[\varphi(u)] = [v]$ in $K_1(B)$ by Lemma 3.3. It follows from Lemma 8.1 that $[\varphi] = \kappa$ in $KK(A, B)$. $\square$

The following result is due to Li [15, Theorem 2.1]. It generalises a theorem of Thomsen [24, Theorem 2.1] and it is perhaps the most important step towards Theorem 8.8 below.

Theorem 8.3. *Let X be a path-connected compact Hausdorff space, let $F \subseteq C_{\mathbb{R}}(X)$ be a finite subset and let $\epsilon > 0$. There exists a positive integer L such that for all $N \geq L$, for all compact Hausdorff spaces Y and for all positive linear order unit preserving maps $\Theta : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$, there exist continuous functions $\lambda_k : Y \to X$, $k = 1, 2, \ldots, N$, such that*

$$\|\Theta(f) - \frac{1}{N}\sum_{k=1}^{N} f \circ \lambda_k\| < \epsilon, \quad f \in F.$$

Theorem 8.4. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block, let $\epsilon > 0$, let $F \subseteq \mathrm{Aff}\,T(A)$ be a finite set, and let C be a non-negative integer. There exists a positive integer K such that if*

(i) *$u \in A$ is a unitary such that $Det(u(z)) = z\exp(2\pi i\beta(z))$, $z \in \mathbb{T}$, where $\beta : \mathbb{T} \to \mathbb{R}$ is a continuous function;*
(ii) *$B = A(m, e_1, e_2, \ldots, e_M)$ is a building block and $\kappa \in KK(A, B)$ is an element such that $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit;*
(iii) *$s(B) \geq K$;*
(iv) *$\Theta : \mathrm{Aff}\,T(A) \to \mathrm{Aff}\,T(B)$ is a positive linear order unit preserving map;*
(v) *$\lambda_1, \lambda_2, \ldots, \lambda_C : \mathbb{T} \to \mathbb{T}$ are continuous functions;*
(vi) *$v \in B$ is a unitary such that $\kappa_*[u] = [v]$ in $K_1(B)$;*

*then there exists a unital *-homomorphism $\varphi : A \to B$ such that $[\varphi] = \kappa$ in $KK(A, B)$, such that*

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F,$$

and such that $\lambda_1, \lambda_2, \ldots, \lambda_C$ are eigenvalue functions for φ, together with a continuous function $\alpha : \mathbb{T} \to \mathbb{R}$ with $\|\alpha\|_\infty \leq 2\|\beta\|_\infty$ that equals 0 at all the exceptional points of B such that

$$Det(\varphi(u)(z)) = Det(v(z))\exp(2\pi i\alpha(z)), \quad z \in \mathbb{T},$$

and

$$Det(\Lambda_j \circ \varphi(u)) = Det(\Lambda_j(v)), \quad j = 1, 2, \ldots, M.$$

Proof. We may assume that $\|f\| \leq 1$, $f \in F$. Identify $\mathrm{Aff}\,T(A)$ and $C_{\mathbb{R}}(\mathbb{T})$ as order unit spaces. Choose by Theorem 8.3 an integer L with respect to $F \subseteq \mathrm{Aff}\,T(A)$ and $\frac{\epsilon}{3}$. We may assume that $L > C$ and that $1 - \frac{L-C}{C+L} < \frac{\epsilon}{3}$. Then choose by Theorem 8.2 an integer K with respect to $C + L$ and $\frac{\epsilon}{3}$.

Now let B, κ, Θ, u, β, $\lambda_1, \lambda_2, \ldots, \lambda_C$, and v be given as above. Choose continuous functions $\lambda_{C+1}, \lambda_{C+2}, \ldots, \lambda_{C+L} : \mathbb{T} \to \mathbb{T}$ such that

$$\|\Theta(f) - \frac{1}{L}\sum_{k=C+1}^{C+L} f \circ \lambda_k\| < \frac{\epsilon}{3}, \quad f \in F.$$

By Theorem 8.2 there exists a unital *-homomorphism $\varphi : A \to B$ and a continuous function $\alpha : \mathbb{T} \to \mathbb{R}$ with $\|\alpha\|_\infty \leq 2\|\beta\|_\infty$ that equals zero at the exceptional points of B, such that $[\varphi] = \kappa$ in $KK(A, B)$, such that $\lambda_1, \lambda_2, \ldots, \lambda_{C+L}$ are eigenvalue functions for φ and such that

$$\|\widehat{\varphi}(f) - \frac{1}{C+L}\sum_{k=1}^{C+L} f \circ \lambda_k\| < \epsilon, \quad f \in \mathrm{Aff}\, T(A),$$
$$Det(\varphi(u)(z)) = Det(v(z)) \exp(2\pi i \alpha(z)), \quad z \in \mathbb{T},$$
$$Det(\Lambda_j \circ \varphi(u)) = Det(\Lambda_j(v)), \quad j = 1, 2, \ldots, M.$$

Since for $f \in \mathrm{Aff}\, T(A)$,

$$\begin{aligned} &\|\frac{1}{C+L}\sum_{k=1}^{C+L} f \circ \lambda_k - \frac{1}{L}\sum_{k=C+1}^{C+L} f \circ \lambda_k\| \\ &\leq \|\frac{1}{C+L}\sum_{k=C+1}^{C+L} f \circ \lambda_k - \frac{1}{L}\sum_{k=C+1}^{C+L} f \circ \lambda_k\| + \|\frac{1}{C+L}\sum_{k=1}^{C} f \circ \lambda_k\| \\ &\leq |\frac{1}{C+L} - \frac{1}{L}|\, L\, \|f\| + \frac{1}{C+L} C\, \|f\| = (1 - \frac{L-C}{C+L})\, \|f\| < \frac{\epsilon}{3}\, \|f\|, \end{aligned}$$

we get that

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F.$$

□

Theorem 8.5. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block, let $\epsilon > 0$, let $F \subseteq \mathit{Aff}\, T(A)$ be a finite set, and let $u \in A$ be a unitary of infinite order in $K_1(A)$. There exists a positive integer K such that if*

(i) *$B = A(m, e_1, e_2, \ldots, e_M)$ is a building block and $\kappa \in KK(A, B)$ is an element such that $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit;*
(ii) *$s(B) \geq K$;*
(iii) *$\Theta : \mathit{Aff}\, T(A) \to \mathit{Aff}\, T(B)$ is a positive linear order unit preserving map;*
(iv) *$v \in B$ is a unitary such that $\kappa_*[u] = [v]$ in $K_1(B)$;*

*then there exists a unital *-homomorphism $\varphi : A \to B$ such that $[\varphi] = \kappa$ in $KK(A, B)$ and such that*

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F,$$

together with a continuous function $\gamma : \mathbb{T} \to \mathbb{R}$ with $\|\gamma\|_\infty < \epsilon m$ that equals 0 at all the exceptional points of B such that

$$Det(\varphi(u)(z)) = Det(v(z)) \exp(2\pi i \gamma(z)), \quad z \in \mathbb{T},$$

and

$$Det(\Lambda_j \circ \varphi(u)) = Det(\Lambda_j(v)), \quad j = 1, 2, \ldots, M.$$

Proof. Let

$$[u] = [u_1]^p [u_2]$$

in $K_1(A)$, where $u_1 \in A$ is a unitary such that $Det(u_1(\cdot)) : \mathbb{T} \to \mathbb{T}$ has winding number 1, $u_2 \in A$ is a unitary of finite order in $K_1(A)$, and p is a non-zero integer. By Lemma 5.7 we may assume that $Det(u_2(\cdot))$ is constant.

Let $c \in A$ be the unitary such that

$$u = c {u_1}^p u_2.$$

As c is trivial in $K_1(A)$ there exists by Lemma 5.6 a unitary $c' \in A$ that is trivial in $K_1(A)$ such that

$$q'_A(c')^p = q'_A(c) \quad \text{in } U(A)/\overline{DU(A)}. \tag{20}$$

Set $u'_1 = c'u_1$. Then

$$q'_A(u) = q'_A({u'_1}^p u_2) \quad \text{in } U(A)/\overline{DU(A)}.$$

Choose K by Theorem 8.4 with respect to ϵ, $F \subseteq \operatorname{Aff} T(A)$, and $C = 0$. We may assume that $K > Nn$ and $K > \frac{2}{\epsilon}|p|\,\|\beta\|_\infty$, where $\beta : \mathbb{T} \to \mathbb{R}$ is a continuous function such that $Det(u'_1(z)) = z\exp(2\pi i\beta(z))$, $z \in \mathbb{T}$.

Now let B, κ, Θ, and v be as above. By Theorem 4.8 there exists a unital *-homomorphism $\psi : A \to B$ such that $[\psi] = \kappa$ in $KK(A,B)$. Set $v_2 = \psi(u_2)$ and $w = \psi(u'_1)$. As $[u] = [u_1]^p[u_2] = [u'_1]^p[u_2]$ in $K_1(A)$, it follows that $[v] = [w]^p[v_2]$ in $K_1(B)$. Hence $v = yw^pv_2$ for a unitary $y \in B$ that is trivial in $K_1(B)$. Choose by Lemma 5.6 a unitary $y' \in B$ that is trivial in $K_1(B)$ such that

$$q'_B(y')^p = q'_B(y) \quad \text{in } U(B)/\overline{DU(B)}.$$

Set $v_1 = y'w$, and note that

$$q'_B(v) = q'_B(v_1^p v_2) \quad \text{in } U(B)/\overline{DU(B)}.$$

By Theorem 8.4 there exists a unital *-homomorphism $\varphi : A \to B$ such that $[\varphi] = \kappa$ in $KK(A,B)$, such that

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F,$$

and such that

$$Det(\varphi(u'_1)(z)) = Det(v_1(z))\exp(2\pi i\alpha(z)), \quad z \in \mathbb{T}, \tag{21}$$

$$Det(\Lambda_j \circ \varphi(u'_1)) = Det(\Lambda_j(v_1)), \quad j = 1,2,\ldots,M, \tag{22}$$

where $\alpha : \mathbb{T} \to \mathbb{R}$ is a continuous function with $\|\alpha\|_\infty \leq 2\|\beta\|_\infty$ that equals 0 at all the exceptional points of B. Note that by (20),

$$\varphi^\#(q'_A(u)) = \varphi^\#(q'_A({u'_1}^p u_2)) \quad \text{in } U(B)/\overline{DU(B)}.$$

As $\varphi^* = \kappa^* = \psi^*$ in $Hom(K^0(B), K^0(A))$ we get from Lemma 5.8 that

$$\varphi^\#(q'_A(u)) = \varphi^\#(q'_A({u'_1}^p v_2)) \quad \text{in } U(B)/\overline{DU(B)}.$$

Hence by (21) and (22),

$$Det(\varphi(u)(z)) = Det(v_1(z))^p e^{2\pi i p\alpha(z)} Det(v_2(z)) = Det(v(z))e^{2\pi i\gamma(z)}, \quad z \in \mathbb{T},$$

$$Det(\Lambda_j \circ \varphi(u)) = Det(\Lambda_j(v_1))^p Det(\Lambda_j(v_2)) = Det(\Lambda_j(v)), \quad j = 1,2,\ldots,M,$$

where $\gamma : \mathbb{T} \to \mathbb{R}$ is defined by $\gamma(z) = p\alpha(z)$. Note that

$$\|\gamma\|_\infty = |p|\,\|\alpha\|_\infty \leq 2|p|\,\|\beta\|_\infty < \epsilon K \leq \epsilon m$$

and that γ equals 0 at all the exceptional points of B. □

Lemma 8.6. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block with exceptional points $x_1, x_2, \ldots, x_N$. Let $g : \mathbb{T} \to \mathbb{T}$ be a continuous function and let $s_i \in \mathbb{T}$ be such that $s_i^{\frac{n}{d_i}} = g(x_i)$, $i = 1,2,\ldots,N$. There exists a unitary $u \in A$ such that*

$$Det(u(z)) = g(z), \quad z \in \mathbb{T},$$

$$Det(\Lambda_i(u)) = s_i, \quad i = 1,2,\ldots,N.$$

Proof. Choose a unitary $v \in A$ such that $Det(\Lambda_i(v)) = s_i$, $i = 1, 2, \ldots, N$. Let $Det(v(z)) = h(z)$, $z \in \mathbb{T}$. Then $h(x_i) = s_i^{\frac{n}{d_i}} = g(x_i)$. Thus we can define a unitary $w \in A$ by

$$w(z) = \operatorname{diag}(g(z)h(z)^{-1}, 1, 1, \ldots, 1), \quad z \in \mathbb{T}.$$

Set $u = wv$. □

Lemma 8.7. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block, let $p \in A$ be a non-zero projection, and let $u \in A$ be a unitary. Then there exists a unitary $w \in pAp$ such that*

$$q'_A(u) = q'_A(w + (1 - p)) \quad \textit{in } U(A)/\overline{DU(A)}.$$

Proof. Let r denote the rank of p. Then $pAp \cong A(r, \frac{r}{n}d_1, \frac{r}{n}d_2, \ldots, \frac{r}{n}d_N)$ by Lemma 3.5. By Lemma 8.6 there exists a unitary $w \in pAp$ such that

$$Det(w(z)) = Det(u(z)), \quad z \in \mathbb{T},$$
$$Det(\Lambda_i(w)) = Det(\Lambda_i(u)), \quad i = 1, 2, \ldots, N.$$

Then $q'_A(u) = q'_A(w + (1 - p))$ in $U(A)/\overline{DU(A)}$ by Theorem 5.5. □

Let $A_1, A_2, \ldots, A_R$, be unital C^*-algebras with $T(A_i) \neq \emptyset$, and let A be their direct sum. It is well-known that the map

$$\operatorname{Aff} T(A) \to \oplus_{i=1}^{R} \operatorname{Aff} T(A_i)$$
$$f \mapsto (\widehat{\pi_1}(f), \widehat{\pi_2}(f), \ldots, \widehat{\pi_R}(f)),$$

where $\pi_i : A \to A_i$ denotes the projection, is an isomorphism in the category of order unit spaces (recall that the direct sum of order unit spaces is the vector space direct sum equipped with the supremum norm and the obvious order and order unit). Let $J_i : \operatorname{Aff} T(A_i) \to \operatorname{Aff} T(A)$ be the inclusion.

Theorem 8.8. *Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_R$ be a finite direct sum of building blocks. Let $F \subseteq \operatorname{Aff} T(A)$ be a finite set and let $\epsilon > 0$. There exists a positive integer K such that if*

(i) *$B = B_1 \oplus B_2 \oplus \cdots \oplus B_S$ is a finite direct sum of building blocks and $\kappa \in KK(A, B)$ is an element such that $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit;*

(ii) *for every minimal non-zero central projection p in A,*

$$s(B)\, \rho_B(\kappa_*[p]) \geq K \quad \textit{in } \operatorname{Aff} T(B);$$

(iii) *there exists a linear positive order unit preserving map $\Theta : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ such that the diagram*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) \\ {\scriptstyle \kappa_*}\downarrow & & \downarrow{\scriptstyle \Theta} \\ K_0(B) & \xrightarrow[\rho_B]{} & \operatorname{Aff} T(B) \end{array}$$

commutes;

(iv) *$u_1, u_2, \ldots, u_N \in B$ are unitaries such that*

$$\kappa_*[V_A^i] = [u_i] \quad \textit{in } K_1(A), \quad i = 1, 2, \ldots, R;$$

*then there exists a unital *-homomorphism* $\varphi : A \to B$ *such that* $[\varphi] = \kappa$ *in* $KK(A, B)$*, and such that*

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F,$$
$$\varphi^{\#}(q'_A(V^i_A)) = q'_B(u_i) \quad \text{in } U(B)/\overline{DU(B)}, \quad i = 1, 2, \ldots, R.$$

Proof. Let $\pi^A_i : A \to A_i$ be the projection and $\iota^A_i : A_i \to A$ be the inclusion, $i = 1, 2, \ldots, R$. Let $p_1, p_2, \ldots, p_R$ denote the minimal non-zero central projections in A. Choose by Theorem 8.4 a K_i with respect to each $\widehat{\pi^A_i}(F) \subseteq \operatorname{Aff} T(A_i)$, $\epsilon > 0$ and $C = 0$. Set $K = \max_{1 \le i \le R} K_i$.

Let B, κ, Θ, and $u_1, u_2, \ldots, u_N$ be as above. We may assume that $S = 1$. To see this, assume that the case $S = 1$ has been settled. Let $\pi^B_l : B \to B_l$ be the projection and $\iota^B_l : B_l \to B$ be the inclusion. As the diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \operatorname{Aff} T(A) \\ {\scriptstyle \pi^B_{l\,*} \circ \kappa_*}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\pi^B_l} \circ \Theta} \\ K_0(B_l) & \xrightarrow[\rho_{B_l}]{} & \operatorname{Aff} T(B_l) \end{array}$$

commutes for $l = 1, 2, \ldots, S$, and since $s(B_l)\, \rho_{B_l}(\pi^B_{l\,*} \circ \kappa_*[p_i]) \ge K$ for $i = 1, 2, \ldots, R$, $l = 1, 2, \ldots, S$, we get unital *-homomorphisms $\varphi_l : A \to B_l$ such that

$$[\varphi_l] = [\pi^B_l] \cdot \kappa \qquad \text{in } KK(A, B_l),$$
$$\|\widehat{\varphi}_l(f) - \widehat{\pi^B_l} \circ \Theta(f)\| < \epsilon, \quad f \in F,$$
$$\varphi^{\#}_l(q'_A(V^i_A)) = q'_{B_l}(\pi^B_l(u_i)) \quad \text{in } U(B_l)/\overline{DU(B_l)}, \quad i = 1, 2, \ldots, R.$$

Define $\varphi : A \to B$ by $\varphi(a) = (\varphi_1(a), \varphi_2(a), \ldots, \varphi_S(a))$. Then

$$[\varphi] = [\sum_{l=1}^{S} \iota^B_l \circ \varphi_l] = \sum_{l=1}^{S} [\iota^B_l] \cdot [\pi^B_l] \cdot \kappa = \kappa \qquad \text{in } KK(A, B),$$
$$\|\widehat{\varphi}(f) - \Theta(f)\| = \max_l \|\widehat{\pi^B_l} \circ \widehat{\varphi}(f) - \widehat{\pi^B_l} \circ \Theta(f)\| < \epsilon, \quad f \in F,$$
$$\varphi^{\#}(q'_A(V^i_A)) = q'_B(u_i) \quad \text{in } U(B)/\overline{DU(B)}, \quad i = 1, 2, \ldots, R.$$

So assume $B = A(m, e_1, e_2, \ldots, e_M)$. Note that by assumption $\kappa_*[p_i] > 0$ in $K_0(B)$ for $i = 1, 2, \ldots, R$. Let $e = d(B)$. Choose by Corollary 3.6 orthogonal non-zero projections $r_i \in M_e \subseteq B$, for $i = 1, 2, \ldots, R$, with sum 1 such that $\kappa_*[p_i] = [r_i]$. Let t_i be the normalised trace of r_i. Define

$$\Theta_i : \operatorname{Aff} T(A_i) \to \operatorname{Aff} T(r_i B r_i)$$
$$\Theta_i(f)(\frac{1}{t_i}\tau \circ \epsilon_i) = \frac{1}{t_i}\Theta(J_i(f))(\tau), \quad \tau \in T(B),$$

where $\epsilon_i : r_i B r_i \to B$ denotes the inclusion.

Θ_i is a linear positive order unit preserving map, since

$$\Theta_i(1)(\frac{1}{t_i}\tau \circ \epsilon_i) = \frac{1}{t_i}\Theta(\widehat{p}_i)(\tau) = \frac{1}{t_i}\rho_B \circ \kappa_*[p_i](\tau) = 1.$$

As r_i is a full projection $[\epsilon_i] \in KK(r_i B r_i, B)$ is a KK-equivalence. Note that

$$[\epsilon_i]^{-1} \cdot \kappa \cdot [\iota^A_i] \in KK(A_i, r_i B r_i)$$

induces a homomorphism $K_0(A_i) \to K_0(r_iBr_i)$ that preserves the order unit. Choose by Lemma 8.7 a unitary $w_i \in r_iBr_i$ such that

$$q'_B(w_i + (1 - r_i)) = q'_B(u_i) \quad \text{in } U(B)/\overline{DU(B)}.$$

As $r_iBr_i \cong A(t_im, t_ie_1, t_ie_2, \dots, t_ie_M)$ and as $t_ie_j \geq K$ for $j = 1, 2, \dots, M$, we get by combining Theorem 8.4 and Theorem 5.5 a unital *-homomorphism $\varphi_i : A_i \to r_iBr_i$ such that

$$[\varphi_i] = [\epsilon_i]^{-1} \cdot \kappa \cdot [\iota_i^A] \quad \text{in } KK(A_i, r_iBr_i),$$
$$\|\widehat{\varphi}_i(f) - \Theta_i(f)\| < \epsilon, \quad f \in \widehat{\pi_i^A}(F),$$
$$\varphi_i(V_{A_i}) = w_i \mod \overline{DU(r_iBr_i)}.$$

Now define $\varphi : A \to B$ by

$$\varphi(a) = \sum_{i=1}^{R} \epsilon_i \circ \varphi_i \circ \pi_i^A(a).$$

φ is a unital *-homomorphism and

$$[\varphi] = \sum_{i=1}^{R} [\epsilon_i] \cdot [\varphi_i] \cdot [\pi_i^A] = \sum_{i=1}^{R} \kappa \cdot [\iota_i^A] \cdot [\pi_i^A] = \kappa \quad \text{in } KK(A, B).$$

For $f \in \operatorname{Aff} T(A),\ \tau \in T(B)$, we have that

$$\Theta(f)(\tau) = \sum_{i=1}^{R} \Theta(J_i(\widehat{\pi_i^A}(f)))(\tau) = \sum_{i=1}^{R} t_i \Theta_i(\widehat{\pi_i^A}(f))(\frac{1}{t_i}\tau \circ \epsilon_i),$$

$$\widehat{\varphi}(f)(\tau) = f(\tau \circ \varphi) = f(\sum_{i=1}^{R} t_i \frac{1}{t_i} \tau \circ \epsilon_i \circ \varphi_i \circ \pi_i^A) = \sum_{i=1}^{R} t_i \widehat{\varphi}_i(\widehat{\pi_i^A}(f))(\frac{1}{t_i}\tau \circ \epsilon_i).$$

It follows that

$$\|\widehat{\varphi}(f) - \Theta(f)\| < \epsilon, \quad f \in F.$$

Finally, for $i = 1, 2, \dots, R$,

$$\begin{aligned} q'_B(\varphi(V_A^i)) &= q'_B(\prod_{j=1}^{R} (\varphi_j(V_{A_i}) + (1 - q_j))) \\ &= q'_B(\varphi_i(V_{A_i}) + (1 - q_i)) = q'_B(\omega_i + (1 - q_i)) = q'_B(u_i). \end{aligned}$$

□

9. Injective connecting maps

In this section we show that a simple unital infinite dimensional inductive limit of a sequence of finite direct sums of building blocks can be realised as an inductive limit of a sequence of finite direct sums of building blocks with unital and injective connecting maps.

Define a continuous function $\kappa : \mathbb{T} \to [0, 1]$ by

$$\kappa(e^{2\pi it}) = \begin{cases} 2t & t \in [0, \frac{1}{2}], \\ 2 - 2t & t \in [\frac{1}{2}, 1]. \end{cases}$$

Define continuous functions $\iota_1, \iota_2 : [0, 1] \to \mathbb{T}$ by $\iota_1(t) = e^{\pi it},\ \iota_2(t) = e^{-\pi it}$. Note that $\kappa \circ \iota_1 = \kappa \circ \iota_2 = id_{[0,1]}$.

Let $A = A(n, d_1, d_2, \ldots, d_N)$ be an interval building block with exceptional points $x_1, x_2, \ldots, x_N$. Define a circle building block by

$$A^{\mathbb{T}} = \{f \in C(\mathbb{T}) \otimes M_n : f(\iota_1(x_i)), f(\iota_2(x_i)) \in M_{d_i},\ i = 1, 2, \ldots, N\}.$$

Define unital *-homomorphisms $\xi_A : A \to A^{\mathbb{T}}$ by $\xi_A(f) = f \circ \kappa$, $f \in A$, and $j_A^1, j_A^2 : A^{\mathbb{T}} \to A$ by $j_A^1(g) = g \circ \iota_1$, $j_A^2(g) = g \circ \iota_2$, $g \in A^{\mathbb{T}}$. Then $j_A^1 \circ \xi_A = j_A^2 \circ \xi_A = id_A$.

Consider a finite direct sum of circle and interval building blocks.

$$A = B_1 \oplus \cdots \oplus B_N \oplus C_1 \oplus \cdots \oplus C_M,$$

where $B_1, \ldots, B_N$ are interval building blocks and $C_1, \ldots, C_M$ are circle building blocks. Define a finite direct sum of circle building blocks $A^{\mathbb{T}}$ by

$$A^{\mathbb{T}} = B_1^{\mathbb{T}} \oplus \cdots \oplus B_N^{\mathbb{T}} \oplus C_1 \oplus \cdots \oplus C_M.$$

Define $\xi_A : A \to A^{\mathbb{T}}$ by

$$\xi_A(x_1, \ldots, x_N, y_1, \ldots, y_M) = (\xi_{B_1}(x_1), \ldots, \xi_{B_N}(x_N), y_1, \ldots, y_M).$$

Define $j_A^1, j_A^2 : A^{\mathbb{T}} \to A$ by

$$j_A^k(x_1, \ldots, x_N, y_1, \ldots, y_M) = (j_{A_1}^k(x_1), \ldots, j_{A_N}^k(x_N), y_1, \ldots, y_M), \quad k = 1, 2.$$

Note that $j_A^1 \circ \xi_A = j_A^2 \circ \xi_A = id_A$. Note also that ξ_A is injective and that

$$j_A^1(f) = j_A^2(f) = 0 \ \Rightarrow\ f = 0, \quad f \in A^{\mathbb{T}}. \tag{23}$$

Lemma 9.1. *Let A be a finite direct sum of circle and interval building blocks. Let $G \subseteq A$ be a finite set and let $\epsilon > 0$. There exists a finite set $H \subseteq A$ of positive non-zero elements such that whenever B is a finite direct sum of circle and interval building blocks, and $\varphi : A \to B$ is a unital *-homomorphism such that $\varphi(h) \neq 0$, $h \in H$, there exists a unital and injective *-homomorphism $\psi : A \to B$ such that*

$$\|\varphi(g) - \psi(g)\| < \epsilon, \quad g \in G.$$

Proof. We may assume that A is a circle or an interval building block rather than a finite direct sum of such algebras. Assume first that $A = A(n, d_1, d_2, \ldots, d_N)$ is a circle building block.

Choose $\delta > 0$ such that for $x, y \in \mathbb{T}$,

$$\rho(x, y) < 2\delta \ \Rightarrow\ \|g(x) - g(y)\| < \epsilon, \quad g \in G.$$

Let $\mathbb{T} = \cup_{i=1}^K V_i$ where each V_i is an open arc-segment of length less than δ. Choose for each $i = 1, 2, \ldots, K$, a non-zero continuous function $\chi_i : \mathbb{T} \to [0, 1]$ with support in V_i and such that χ_i is zero at every exceptional point of A. Set

$$H = \{\chi_1 \otimes 1, \chi_2 \otimes 1, \ldots, \chi_K \otimes 1\}.$$

Let $\varphi : A \to B$ be given such that $\varphi(h) \neq 0$, $h \in H$. Let $B = B_1 \oplus \cdots \oplus B_R$, where $B_1, \ldots, B_S$ are circle building blocks and $B_{S+1}, \ldots, B_R$ are interval building blocks. Let $B_r = A(m_r, e_1^r, e_2^r, \ldots, e_{M_r}^r)$. Let $\pi_r : B \to B_r$ denote the coordinate projections, $r = 1, 2, \ldots, R$. By [21, Chapter 1] we may assume that for $r = 1, 2, \ldots, S$, $t \in [0, 1]$,

$$\pi_r \circ \varphi(f)(e^{2\pi i t}) = u_r(t) \operatorname{diag}\big(\Lambda_1^{s_1^r}(f), \ldots, \Lambda_N^{s_N^r}(f), f(\lambda_1^r(t)), \ldots, f(\lambda_{L_r}^r(t))\big) u_r(t)^*,$$

and that for $r = S+1, S+2, \ldots, R$, $t \in [0, 1]$,

$$\pi_r \circ \varphi(f)(t) = u_r(t) \operatorname{diag}\big(\Lambda_1^{s_1^r}(f), \ldots, \Lambda_N^{s_N^r}(f), f(\lambda_1^r(t)), \ldots, f(\lambda_{L_r}^r(t))\big) u_r(t)^*,$$

where $u_r \in C[0,1] \otimes M_{m_r}$ is a unitary, $\lambda_1^r, \dots, \lambda_{L_r}^r : [0,1] \to \mathbb{T}$ are continuous functions, and $s_1^r, s_2^r, \dots, s_N^r$ are non-negative integers, $r = 1, 2, \dots, R$. Since $\varphi(h) \neq 0$, $h \in H$, it follows that the set

$$\bigcup_{r=1}^{R} \bigcup_{k=1}^{L_r} \lambda_k^r([0,1])$$

is δ-dense in $\mathbb{T}$.

Fix some $r = 1, 2, \dots, S$. Let $t_1, t_2, \dots, t_{M_r} \in]0,1[$ be numbers such that $e^{2\pi i t_j}$, $j = 1, 2, \dots, M_r$, are the exceptional points of B_r. For each $k = 1, 2, \dots, L_r$, choose a continuous function $\mu_k^r : [0,1] \to \mathbb{T}$ such that

$$\{x \in \mathbb{T} : \rho\big(x, \lambda_k^r([0,1])\big) \leq \delta\} \subseteq \mu_k^r([0,1]),$$

such that $\rho(\mu_k^r(t), \lambda_k^r(t)) < 2\delta$, $t \in [0,1]$, and such that $\mu_k^r(t) = \lambda_k^r(t)$ for each $t \in \{t_1, t_2, \dots, t_M, 0, 1\}$. Define $\psi_r : A \to B_r$ by, for $t \in [0,1]$,

$$\psi_r(f)(e^{2\pi i t}) = u_r(t) \operatorname{diag}\big(\Lambda_1^{s_1^r}(f), \dots, \Lambda_N^{s_N^r}(f), f(\mu_1^r(t)), \dots, f(\mu_{L_r}^r(t))\big) u_r(t)^*.$$

Now fix some $j = S+1, S+2, \dots, R$. Let $y_1, y_2, \dots, y_{M_r}$ be the exceptional points of B_r. For each $k = 1, 2, \dots, L_r$, choose a continuous function $\mu_k^r : [0,1] \to \mathbb{T}$ such that

$$\{x \in \mathbb{T} : \rho\big(x, \lambda_k^r([0,1])\big) \leq \delta\} \subseteq \mu_k^r([0,1]),$$

such that $\rho(\mu_k^r(t), \lambda_k^r(t)) < 2\delta$, $t \in [0,1]$, and such that $\mu_k^r(t) = \lambda_k^r(t)$ for each $t \in \{y_1, y_2, \dots, y_{M_r}\}$. Define a unital *-homomorphism $\psi_r : A \to B_r$ by, for $t \in [0,1]$,

$$\psi_r(f)(t) = u_r(t) \operatorname{diag}\big(\Lambda_1^{s_1^r}(f), \dots, \Lambda_N^{s_N^r}(f), f(\mu_1^r(t)), \dots, f(\mu_{L_r}^r(t))\big) u_r(t)^*.$$

Define $\psi : A \to B$ by $\psi(f) = (\psi_1(f), \psi_2(f), \dots, \psi_R(f))$. Since

$$\bigcup_{r=1}^{R} \bigcup_{k=1}^{L_r} \mu_k^r([0,1]) = \mathbb{T},$$

it follows that ψ is injective. Note that $\|\varphi(g) - \psi(g)\| < \epsilon$, $g \in G$.

If A is an interval building block, choose by the above a finite set $H^{\mathbb{T}} \subseteq A^{\mathbb{T}}$ of positive non-zero elements with respect to $\xi_A(G) \subseteq A^{\mathbb{T}}$ and ϵ. Set $H = j_A^1(H^{\mathbb{T}})$. If $\varphi : A \to B$ is a unital *-homomorphism and $\varphi(h) \neq 0$, $h \in H$, then by the above there exists a unital and injective *-homomorphism $\mu : A^{\mathbb{T}} \to B$ such that

$$\|\varphi \circ j_A^1(x) - \mu(x)\| < \epsilon, \quad x \in \xi_A(G).$$

Thus

$$\|\varphi(g) - \mu \circ \xi_A(g)\| < \epsilon, \quad g \in G.$$

□

Theorem 9.2. *Let A be a separable unital C^*-algebra. The following are equivalent:*

(i) *A is isomorphic to the inductive limit of a sequence of finite direct sums of circle and interval building blocks with injective and unital connecting maps.*
(ii) *Given $\epsilon > 0$ and a finite subset $F \subseteq A$ there exists a unital C^*-subalgebra $B \subseteq A$ such that B is isomorphic to a finite direct sum of circle and interval building blocks and such that $F \subseteq_\epsilon B$.*

Proof. (i) $\Rightarrow$ (ii) is trivial so assume (ii). Let $\{x_n\}$ be a dense sequence in A. We will construct a sequence of unital C^*-subalgebras, $A_n \subseteq A$, such that A_n is a finite direct sum of circle and interval building blocks, together with unital injective *-homomorphisms $\gamma_n : A_n \to A_{n+1}$, and finite subsets $F_n \subseteq A_n$ containing the unit, such that

$$G_n \subseteq F_n,$$
$$\gamma_n(F_n) \subseteq F_{n+1},$$
$$\|\gamma_n(x) - x\| \leq 2^{-n}, \quad x \in F_n,$$
$$\{x_1, x_2, \ldots, x_n\} \subseteq_{2^{-n}} F_n,$$

where $A_n = C^*\langle G_n | \mathcal{R}_n \rangle$ for a finite set $\mathcal{R}_n$ of stable relations in a finite number of indeterminates, cf. Theorem 2.4.

This is done inductively. A_{n+1}, γ_n and F_{n+1} are constructed in step n.

Choose a unital C^*-subalgebra $A_1 \subseteq A$ isomorphic to a finite direct sum of circle and interval building blocks such that $\{x_1\} \subseteq_{\frac{1}{2}} A_1$. Choose $y_1 \in A_1$ such that $\|x_1 - y_1\| \leq \frac{1}{2}$ and set

$$F_1 = \{y_1\} \cup \{1\} \cup G_1.$$

Now assume that A_n and F_n have been constructed. Apply Lemma 9.1 to choose a finite set of norm 1 elements $H \subseteq A_n$ with respect to $G = F_n$ and $\epsilon = 2^{-(n+1)}$.

As A_n has stable relations [16, Lemma 15.2.1] gives a $\delta > 0$ such that if $\varphi : A_n \to B$ is a *-homomorphism into any C^*-algebra B and $C \subseteq B$ is a C^*-subalgebra with $\varphi(H \cup F_n) \subseteq_\delta C$ then there exists a *-homomorphism $\psi : A_n \to C$ such that $\|\varphi(x) - \psi(x)\| < 2^{-(n+1)}$, $x \in H \cup F_n$. We may assume that $\delta < 2^{-(n+1)}$.

By assumption there is a unital C^*-subalgebra $A_{n+1} \subseteq A$ isomorphic to a finite direct sum of circle and interval building blocks such that

$$\{x_1, x_2, \ldots, x_{n+1}\} \cup F_n \cup H \subseteq_\delta A_{n+1}.$$

Thus there is a *-homomorphism $\psi_n : A_n \to A_{n+1}$ such that

$$\|x - \psi_n(x)\| < 2^{-(n+1)}, \quad x \in H \cup F_n.$$

Note that ψ_n is unital and $\psi_n(x) \neq 0$, $x \in H$. Hence by Lemma 9.1 there is a unital and injective *-homomorphism $\gamma_n : A_n \to A_{n+1}$ such that

$$\|\psi_n(x) - \gamma_n(x)\| < 2^{-(n+1)}, \quad x \in F_n.$$

Thus $\|x - \gamma_n(x)\| < 2^{-n}$, $x \in F_n$.

Choose $\{y_1, y_2, \ldots, y_{n+1}\} \subseteq A_{n+1}$ such that $\|y_i - x_i\| \leq 2^{-(n+1)}$, $i = 1, \ldots, n+1$. Set

$$F_{n+1} = \gamma_n(F_n) \cup G_{n+1} \cup \{y_1, y_2, \ldots, y_{n+1}\}.$$

This concludes the construction. That $A \cong \varinjlim(A_n, \gamma_n)$ follows as in the proof of [16, Lemma 15.2.2]. $\square$

If

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

is a sequence of C^*-algebras and *-homomorphisms with inductive limit A, we let $\alpha_{n,m} = \alpha_{m-1} \circ \alpha_{m-2} \circ \cdots \circ \alpha_n : A_n \to A_m$ when $m > n$. We set $\alpha_{n,n} = id$ and let $\alpha_{n,\infty} : A_n \to A$ denote the canonical map.

Lemma 9.3. *Let A be a unital C^*-algebra which is the inductive limit of a sequence of finite direct sums of circle building blocks, $A = \varinjlim(A_n, \alpha_n)$. Then $A \cong \varinjlim(B_n, \beta_n)$ where B_n is a finite direct sum of circle building blocks and each β_n is a unital *-homomorphism.*

Proof. Let $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_{m_n}^n$ and let $1_n \in A_n$ denote the unit. Let $e_1^n, e_2^n, \ldots, e_{m_n}^n$ denote the minimal non-zero central projections of A_n. Note that we may assume that $\alpha_{n,\infty}(e_i^n) \neq 0$ for $i = 1, 2, \ldots, m_n$ and all positive integers n. Hence by Lemma 2.2, if $\alpha_{n,\infty}(p) = 0$ for a projection $p \in A_n$ then $p = 0$.

As $\{\alpha_{n,\infty}(1_n)\}_{n=1}^\infty$ is an approximate unit for A there exists a positive integer N such that $\alpha_{k,\infty}(1_k) = 1$ for all $k \geq N$. Thus for $k \geq N$,

$$\alpha_{k+1,\infty}(1_{k+1} - \alpha_k(1_k)) = 0.$$

Hence $\alpha_k(1_k) = 1_{k+1}$, $k \geq N$. □

Lemma 9.4. *Let $X \subseteq \mathbb{T}$ be a closed set and let $G \subseteq X$ be a finite subset. Let $\epsilon > 0$ be given. There exist a set $R \subseteq X$, such that $G \subseteq R$ and such that R is a union of finitely many closed arc-segments and points, and a continuous map $\alpha : X \to R$ such that $\alpha(z) = z$, $z \in R$, and $|\alpha(z) - z| \leq \epsilon$, $z \in X$.*

Proof. An easy exercise. □

Lemma 9.5. *Let A be a C^*-algebra of the form $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$ where*

$$A_i = \{f \in C(X_i) \otimes M_{n_i} : f(x_k^i) \in M_{d_k^i},\ k = 1, 2, \ldots, N_i\}$$

for points $x_1^i, x_2^i, \ldots, x_{N_i}^i \in X_i$ and closed subsets $X_i \subseteq \mathbb{T}$, $i = 1, 2, \ldots, m$.

*Let $F \subseteq A$ be a finite subset and let $\epsilon > 0$. There are a C^*-algebra B that is a finite direct sum of circle building blocks, interval building blocks, and matrix algebras, and a unital injective *-homomorphism $\psi : B \to A$ such that $F \subseteq_\epsilon \psi(B)$.*

If none of the sets X_i contains an arc-segment then B can be assumed to be finite dimensional.

Proof. Let $i \in \{1, 2, \ldots, m\}$. Let $\pi_i : A \to A_i$ denote the projection. Choose by Lemma 9.4 a "nice" subset $R_i \subseteq X_i$ such that $x_1^i, x_2^i, \ldots, x_{N_i}^i \in R_i$ and a continuous function $\alpha_i : X_i \to R_i$ such that $\alpha_i(z) = z$, $z \in R_i$, and such that $\|f \circ \alpha_i(z) - f(z)\| < \epsilon$, $f \in \pi_i(F)$, $z \in X_i$. Set

$$B_i = \{f \in C(R_i) \otimes M_{n_i} : f(x_k^i) \in M_{d_k^i},\ k = 1, 2, \ldots, N_i\}.$$

Set $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$. Define $\psi : B \to A$ by $\psi(f_1, f_2, \ldots, f_m) = (f_1 \circ \alpha_1, f_2 \circ \alpha_2, \ldots, f_m \circ \alpha_m)$. Then $F \subseteq_\epsilon \psi(B)$.

If none of the sets X_i contains an arc-segment then each $R_i \subseteq X_i$ is a finite set and hence B is finite dimensional. □

Lemma 9.6. *Let A be a unital inductive limit of a sequence of finite direct sums of circle building blocks. Assume that A is simple and infinite dimensional. Then A contains a unitary u with full spectrum, i.e with $\sigma(u) = \mathbb{T}$.*

Proof. We may assume by Lemma 9.3 that the connecting maps are unital. Hence $A = \varinjlim(A_i, \alpha_i)$ where each α_i is a unital and injective *-homomorphism and each A_i is a quotient of a finite direct sum of circle building blocks. Thus $A_i = A_1^i \oplus A_2^i \oplus \cdots \oplus A_{m_i}^i$ where for $j = 1, 2, \ldots, m_i$,

$$A_j^i = \{f \in C(X_j^i) \otimes M_{n(i,j)} : f(x_k^{i,j}) \in M_{d(i,j,k)},\ k = 1, 2, \ldots, N(i,j)\},$$

for a closed subset $X_j^i \subseteq \mathbb{T}$ and points $x_1^{i,j}, x_2^{i,j}, \ldots, x_{N(i,j)}^{i,j} \in \mathbb{T}$.

If X_j^i contains an arc-segment for some i, j then A_j^i and hence A contains a unitary with full spectrum.

Otherwise, let a finite set $F \subseteq A$ and $\epsilon > 0$ be given. By Lemma 9.5 there is a finite dimensional unital C^*-subalgebra $B \subseteq A_i$ for some positive integer i such that $F \subseteq_\epsilon \alpha_{i,\infty}(B)$. It follows from [5, Theorem 2.2] that A is an AF-algebra. And it is a well-known result that a simple unital infinite dimensional AF-algebra contains a unitary with full spectrum. □

Proposition 9.7. *Let A be the inductive limit of a sequence of finite direct sums of circle building blocks. Assume that A is simple unital and infinite dimensional. Then A is the inductive limit of a sequence*

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \ldots$$

where each B_n is a finite direct sums of circle and interval building blocks and each β_n is unital and injective.

Proof. Let a finite subset $F \subseteq A$ and $\epsilon > 0$ be given. By Lemma 9.3 write $A = \varinjlim(A_n, \alpha_n)$ where each A_n is a quotient of a finite direct sum of circle building blocks and each α_n is unital and injective.

By Lemma 9.5 there is a unital C^*-algebra $B \subseteq A$ that is a finite direct sum of circle building blocks, interval building blocks, and matrix algebras such that $F \subseteq_\epsilon B$. It is sufficient to show that B is contained in a C^*-subalgebra of A that is a finite direct sum of circle and interval building blocks, cf. Theorem 9.2. Write $B \cong C \oplus F$ where C is a finite direct sum of circle and interval building blocks and F is finite dimensional.

Let $p \in A$ be a minimal non-zero projection in the centre of F. As pAp is a simple unital infinite dimensional inductive limit of a sequence of finite direct sums of circle building blocks we see that it is enough to consider the case where $B \cong M_n$ is a unital C^*-subalgebra of A. Then $A \cong M_n(A \cap B')$ and hence $A \cap B'$ is also a simple unital infinite dimensional inductive limit of a sequence of finite direct sums of circle building blocks. Thus by Lemma 9.6 there is a unitary $u \in A \cap B'$ with full spectrum. Then $C^*(u, B) \cong C(\mathbb{T}) \otimes M_n$. □

Theorem 9.8. *Let A be a simple unital infinite dimensional inductive limit of a sequence of finite direct sums of circle building blocks. Then A is the inductive limit of a sequence of finite direct sums of circle building blocks with unital and injective connecting maps.*

Proof. By Proposition 9.7 we see that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots$$

where each A_n is a finite direct sum of circle and interval building blocks and each α_n is a unital and injective *-homomorphism.

By passing to a subsequence, if necessary, we may assume that, either every A_n is a circle or an interval building block, or every A_n is a finite direct sum of at least two circle and/or interval building blocks.

Let us first assume that the latter is the case.

Let $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_{N_n}^n$ where each A_i^n is a circle or interval building block. For each n let $\pi_i^n : A_n \to A_i^n$ denote the coordinate projections, $i = 1, 2, \ldots, N_n$. First we will show that we may assume that all the maps $\pi_i^{n+1} \circ \alpha_n$ are injective.

By Elliott's approximate intertwining argument it suffices to show that given a finite set $G \subseteq A_n$ and $\epsilon > 0$ there exists an integer $m > n$ and a unital *-homomorphism $\psi : A_n \to A_m$ such that $\|\alpha_{n,m}(g) - \psi(g)\| < \epsilon$, $g \in G$, and such that $\pi_i^m \circ \psi$ is injective, $i = 1, 2, \ldots, N_m$.

Choose by Lemma 9.1 a finite set $H \subseteq A_n$ of positive non-zero elements with respect to G and ϵ. As A is simple and the connecting maps are injective, we have that $\widehat{\alpha_{n,\infty}}(\widehat{h}) > 0$, $h \in H$. Thus there exists an integer $m > n$ such that $\widehat{\alpha_{n,m}}(\widehat{h}) > 0$, $h \in H$. Hence $\pi_i^m \circ \alpha_{n,m}(h) \neq 0$, $i = 1, 2, \ldots, N_m$, and the result follows by N_m applications of Lemma 9.1.

Define a unital *-homomorphism $\psi_n : A_n^{\mathbb{T}} \to A_{n+1}$ by

$$\psi_n(x) = (\pi_1^{n+1} \circ \alpha_n \circ j^1_{A_n}(x), \pi_2^{n+1} \circ \alpha_n \circ j^2_{A_n}(x), \ldots, \pi_{N_{n+1}}^{n+1} \circ \alpha_n \circ j^2_{A_n}(x)).$$

As the maps $\pi_i^{n+1} \circ \alpha_n$ are injective, $i = 1, 2, \ldots, N_{n+1}$, and as $N_{n+1} \geq 2$, it follows from (23) that ψ_n is injective. The theorem therefore follows in this case from the commutativity of the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\xi_{A_1}"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\xi_{A_2}"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\xi_{A_3}"] & \cdots \\
A_1^{\mathbb{T}} \arrow[r, "\xi_{A_2} \circ \psi_1"'] \arrow[ur, "\psi_1"'] & A_2^{\mathbb{T}} \arrow[r, "\xi_{A_3} \circ \psi_2"'] \arrow[ur, "\psi_2"'] & A_3^{\mathbb{T}} \arrow[r, "\xi_{A_4} \circ \psi_3"'] \arrow[ur, "\psi_3"'] & \cdots
\end{tikzcd}$$

It remains to prove the theorem in the first case. By passing to a subsequence we may assume that each A_n is an interval building block. Let $\epsilon > 0$, let k be a positive integer, and let $F \subseteq A_k$ be finite. Again by Elliott's approximative intertwining argument, it suffices to show that there exists an integer $l > k$ and a unital and injective *-homomorphism $\psi : A_k^{\mathbb{T}} \to A_l$ such that

$$\|\alpha_{k,l}(x) - \psi \circ \xi_{A_k}(x)\| < \epsilon, \quad x \in F.$$

To prove this equation we will use the uniqueness theorem of [18].

Choose by [18, Theorem 5.1] a finite set $H \subseteq A_k$ of positive non-zero elements with respect to F and ϵ. We may assume that $\|h\| \leq 1$, $h \in H$. Choose $\delta > 0$ such that

$$\widehat{\alpha_{k,\infty}}(\widehat{h}) > 2\delta, \quad h \in H.$$

Let $A_k = A(n, d_1, d_2, \ldots, d_N)$. By [18, Lemma 7.3] there exists an integer $l > k$ such that $s(A_l) > \max(\frac{2n}{\delta}, Nn)$ and such that $\widehat{\alpha_{k,l}}(\widehat{h}) > 2\delta$, $h \in H$. Let $A_l = A(m, e_1, e_2, \ldots, e_M)$. By [21, Chapter 1] $\alpha_{k,l} \circ j^1_{A_k} : A_k^{\mathbb{T}} \to A_l$ is approximately unitarily equivalent to a unital *-homomorphism $\beta : A_k^{\mathbb{T}} \to A_l$ of the form

$$\beta(f)(t) = u(t)\mathrm{diag}\big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f(\mu_1(t)), \ldots, f(\mu_L(t))\big)u(t)^*, \quad t \in [0,1],$$

where $u \in C[0,1] \otimes M_m$ is a unitary and $\mu_1, \mu_2, \ldots, \mu_L : [0,1] \to \mathbb{T}$ are continuous functions. Choose a continuous function $\mu_1' : [0,1] \to \mathbb{T}$ such that $\mu_1' = \mu_1$ at the exceptional points of A_l and such that μ_1' is surjective. Define $\varphi : A_k^{\mathbb{T}} \to A_l$ by

$$\varphi(f)(t) = u(t)\mathrm{diag}\big(\Lambda_1^{r_1}(f), \ldots, \Lambda_N^{r_N}(f), f(\mu_1'(t)), f(\mu_2(t)), \ldots, f(\mu_L(t))\big)u(t)^*.$$

Note that φ is injective, and that for $h \in H$,

$$\|\widehat{\varphi} \circ \widehat{\xi_{A_k}}(\widehat{h}) - \widehat{\alpha_{k,l}}(\widehat{h})\| = \|\widehat{\varphi}(\widehat{\xi_{A_k}}(\widehat{h})) - \widehat{\alpha_{k,l}} \circ \widehat{j^1_{A_k}}(\widehat{\xi_{A_k}}(\widehat{h}))\| \leq \|\widehat{\varphi} - \widehat{\beta}\| \leq \frac{2n}{m} < \delta.$$

Finally, as $\Lambda_j \circ \varphi = \Lambda_j \circ \beta$, $j = 1, 2, \ldots, M$, we see that

$$(\varphi \circ \xi_{A_k})^* = \xi_{A_k}^* \circ \beta^* = \xi_{A_k}^* \circ (j_{A_k}^1)^* \circ \alpha_{k,l}^* = \alpha_{k,l}^* \quad \text{in } Hom(K^0(A_l), K^0(A_k)).$$

Thus $[\varphi \circ \xi_{A_k}] = [\alpha_{k,l}]$ in $KK(A_k, A_l)$ by [18, Theorem 3.9]. Hence by [18, Theorem 5.1] there exists a unitary $W \in A_l$ such that

$$\|Ad(W) \circ \varphi \circ \xi_{A_k}(x) - \alpha_{k,l}(x)\| < \epsilon, \quad x \in G.$$

Set $\psi = Ad(W) \circ \varphi$. □

10. Classification - K_0 non-cyclic

In this section we prove a classification result for simple unital inductive limits of finite direct sums of building blocks with non-cyclic K_0-group. Our proofs depend heavily on the fact that the functors involved are continuous.

$KK(A, \cdot)$ is continuous by [20, Theorem 1.14] and [20, Theorem 7.13] provided that $K_*(A)$ is finitely generated.

Inductive limits in the category of order unit spaces and linear positive order unit preserving maps were introduced by Thomsen [24]. It follows from [24, Lemma 3.3] that $\mathrm{Aff}T(\cdot)$ is a continuous functor from the category of separable unital C^*-algebras and unital *-homomorphisms to the category of order unit spaces.

Finally, it is an elementary exercise to prove that $U(\cdot)/\overline{DU(\cdot)}$ is a continuous functor from the category of unital C^*-algebras and unital *-homomorphisms to the category of complete metric groups and contractive group homomorphisms (a group G equipped with a metric d is a metric group if $d(fg, fh) = d(g, h)$, $d(gf, hf) = d(g, h)$, for $f, g, h \in G$).

Lemma 10.1. *Let A be a simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots$$

of finite direct sums of building blocks with unital and injective connecting maps. Assume that $K_0(A)$ is not cyclic. Then $d(A_n) \to \infty$.

Proof. If B is a building block then $(K_0(B), [1]) \cong (\mathbb{Z}, d(B))$ as groups with order unit, cf. Corollary 3.6. As the connecting maps are unital it follows that $\{d(A_n)\}$ is an increasing sequence.

Since A is simple and the connecting maps are injective we may by passing to a subsequence assume that $\widehat{\varphi_n}(\widehat{p}) > 0$ for every non-zero projection $p \in A_n$. From this it follows that if each A_n is a finite direct sum of at least two building blocks then $d(A_n) \to \infty$.

Therefore we may assume that each A_n is a building block. If $d(A_n) \to d$ then by passing to a subsequence again we may assume that $(K_0(A_n), [1]) \cong (\mathbb{Z}, d)$ for every n. Hence $(K_0(A), [1]) \cong (\mathbb{Z}, d)$. Contradiction. □

Lemma 10.2. *Let A be a finite direct sum of building blocks. Let $u \in A$ be a unitary. Assume that u is trivial in $K_1(A)$ and that u has finite order in $U(A)/\overline{DU(A)}$. Then*

$$D_A\big(\, q'_A(u)\,,\, q'_A(1)\,\big) < \frac{\pi}{d(A)}.$$

Proof. We may assume that $A = A(n, d_1, d_2, \ldots, d_N)$ is a building block. By Theorem 5.1,

$$u = e^{2\pi i a_1} e^{2\pi i a_2} \ldots e^{2\pi i a_L}$$

where $a_1, a_2, \ldots, a_L \in A$ are self-adjoint. Set $b = a_1 + a_2 + \cdots + a_L$. As $Det(u(\cdot))$ is constant, $Tr(b(\cdot))$ is constantly equal to some $\mu \in \mathbb{T}$. Set $a = b - \frac{\mu}{n}1$. By Lemma 5.3 $e^{2\pi i a} \in \overline{DU(A)}$, and hence $q'_A(u) = q'_A(\exp(2\pi i \frac{\mu}{n})1)$ in $U(A)/\overline{DU(A)}$. Set $d = d(A)$. Choose $\lambda \in \mathbb{T}$ such that $\lambda^d = 1$ and $\rho(\lambda, exp(2\pi i \frac{\mu}{n})) \leq \frac{d}{2}$. It follows that $\|\lambda 1 - exp(2\pi i \frac{\mu}{n})1\| < \frac{\pi}{d}$. Finally, $\lambda 1 \in DU(A)$ by Lemma 5.4. □

Lemma 10.3. *Let A be an inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots$$

of finite direct sums of building blocks with unital connecting maps. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \ldots$$

with unital connecting maps and such that $d(B_k) \to \infty$. Assume that there exist group homomorphisms $\lambda : K_1(A) \to K_1(B)$ and $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ such that the diagram

$$\begin{array}{ccc} U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\ \Phi \downarrow & & \downarrow \lambda \\ U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) \end{array}$$

*commutes. Let n and m be positive integers and let $\varphi : A_n \to B_m$ be a unital *-homomorphism. Let $u \in A_n$ be a unitary of finite order in $U(A)/\overline{DU(A)}$ such that*

$$\beta_{m,\infty *} \circ \varphi_*[u] = \lambda \circ \alpha_{n,\infty *}[u] \quad \text{in } K_1(B).$$

Then

$$\beta^{\#}_{m,\infty} \circ \varphi^{\#}(q'_{A_n}(u)) = \Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(u)) \quad \text{in } U(B)/\overline{DU(B)}.$$

Proof. Let u have order p in $U(A)/\overline{DU(A)}$. We may assume that $p \neq 0$. Let $0 < \epsilon < \frac{1}{p}$. There exists a positive integer k and a unitary $v \in B_k$ such that

$$D_B\big(\beta^{\#}_{k,\infty}(q'_{B_k}(v)),\, \Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(u))\big) < \epsilon. \tag{24}$$

Hence $\beta_{k,\infty *}[v] = \lambda \circ \alpha_{n,\infty *}[u]$ in $K_1(B)$ and

$$D_B\big(\beta^{\#}_{k,\infty}(q'_{B_k}(v^p)),\, q'_B(1)\big) < p\epsilon < 1.$$

Thus there exists an integer $l \geq k$ such that

$$D_{B_l}\big(\beta^{\#}_{k,l}(q'_{B_k}(v^p)),\, q'_{B_l}(1)\big) < p\epsilon < 1.$$

It follows that there exists a self-adjoint element $b \in B_l$ such that $\|e^{2\pi i b} - 1\| < p\epsilon$ and such that $\beta^{\#}_{k,l}(q'_{B_k}(v^p)) = q'_{B_l}(e^{2\pi i b})$ in $U(B_l)/\overline{DU(B_l)}$.

Set $w = \beta_{k,l}(v)e^{-2\pi i \frac{b}{p}}$. Then

$$D_{B_l}\big(q'_{B_l}(w),\, \beta^{\#}_{k,l}(q'_{B_k}(v))\big) \leq \|e^{2\pi i \frac{b}{p}} - 1\| \leq \|e^{2\pi i b} - 1\| < p\epsilon, \tag{25}$$

and

$$\beta_{m,\infty *} \circ \varphi_*[u] = \lambda \circ \alpha_{n,\infty *}[u] = \beta_{k,\infty *}[v] = \beta_{l,\infty *}[w] \quad \text{in } K_1(B).$$

Thus there exists an integer $j \geq l, m$ such that $d(B_j) > \frac{\pi}{\epsilon}$ and such that

$$\beta_{m,j *} \circ \varphi_*[u] = \beta_{l,j *}[w] \quad \text{in } K_1(B_j).$$

Note that both $\beta_{m,j} \circ \varphi(u^p)$ and $\beta_{l,j}(w^p)$ are trivial in $U(B_j)/\overline{DU(B_j)}$. Hence by Lemma 10.2,

$$D_{B_j}\big(\, \beta^{\#}_{m,j} \circ \varphi^{\#}(q'_{A_n}(u))\, ,\, \beta^{\#}_{l,j}(q'_{A_l}(w))\,\big) < \frac{\pi}{d(B_j)} < \epsilon. \tag{26}$$

It follows from (24), (25), and (26) that

$$D_B\big(\, \beta^{\#}_{m,\infty} \circ \varphi^{\#}(q'_{A_n}(u))\, ,\, \Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(u))\,\big) < (p+2)\epsilon.$$

As $0 < \epsilon < \frac{1}{p}$ was arbitrary the conclusion follows. □

Lemma 10.4. *Let A be a finite direct sum of building blocks, let $\epsilon > 0$, and let $F \subseteq \mathit{Aff}T(A)$ be a finite set. Let B be the inductive limit of a sequence of finite direct sums of building blocks*

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \dots$$

with unital connecting maps. Let $J : \mathit{Aff}T(A) \to \mathit{Aff}T(B)$ be a linear positive order unit preserving map and let $\kappa \in KK(A,B)$ be an element such that $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit. There exists a positive integer n, a linear positive order unit preserving map $M : \mathit{Aff}T(A) \to \mathit{Aff}T(B_n)$, and an element $\omega \in KK(A, B_n)$ such that $\omega_* : K_0(A) \to K_0(B_n)$ preserves the order unit and such that

$$\|J(f) - \widehat{\beta_{n,\infty}} \circ M(f)\| < \epsilon, \quad f \in F,$$
$$\kappa = [\beta_{n,\infty}] \cdot \omega \quad \text{in } KK(A,B).$$

Proof. We may assume that $\|f\| \leq 1$, $f \in F$. Decompose $A = A_1 \oplus A_2 \oplus \cdots \oplus A_N$ as a finite direct sum of building blocks and let $\pi_i : A \to A_i$ denote the projection, $i = 1, 2, \ldots, N$.

For every $i = 1, 2, \ldots, N$, identify $\mathrm{Aff}\, T(A_i)$ and $C_{\mathbb{R}}(\mathbb{T})$. Choose open sets $V_1, V_2, \ldots, V_{k_i} \subseteq \mathbb{T}$ such that $\cup_{j=1}^{k_i} V_j = \mathbb{T}$ and such that

$$x, y \in V_j \;\Rightarrow\; |f(x) - f(y)| < \frac{\epsilon}{2}, \quad f \in \widehat{\pi}_i(F).$$

Let $\{h_j : j = 1, 2, \ldots, k_i\}$ be a continuous partition of unity in $C_{\mathbb{R}}(\mathbb{T})$ subordinate to the cover $\{V_j : j = 1, 2, \ldots, k_i\}$ and let $x_j \in V_j$ be an arbitrary point, $j = 1, 2, \ldots, k_i$. Define linear positive order unit preserving maps $T_i : \mathrm{Aff}\, T(A_i) \to \mathbb{R}^{k_i}$ and $S_i : \mathbb{R}^{k_i} \to \mathrm{Aff}\, T(A_i)$ by

$$T_i(f) = (f(x_1), f(x_2), \ldots, f(x_{k_i})),$$
$$S_i(t_1, t_2, \ldots, t_{k_i}) = \sum_{j=1}^{k_i} t_j h_j.$$

Note that

$$\|S_i \circ T_i(f) - f\| < \frac{\epsilon}{2}, \quad f \in \widehat{\pi}_i(F).$$

Hence there exist linear positive order unit preserving maps

$$T : \mathrm{Aff}\, T(A) \to \mathbb{R}^k,$$
$$S : \mathbb{R}^k \to \mathrm{Aff}\, T(A),$$

where $k = \sum_{i=1}^N k_i$, such that

$$\|S \circ T(f) - f\| < \frac{\epsilon}{2}, \quad f \in F.$$

Let $\{e_j : j = 1, 2, \ldots, k\}$ be the standard basis in $\mathbb{R}^k$. As $\{J \circ S(e_j) : j = 1, 2, \ldots, k\}$ are positive elements with sum 1 in $\operatorname{Aff} T(B)$, there exist a positive integer l and positive elements $x_1, x_2, \ldots, x_k \in \operatorname{Aff} T(B_l)$ such that $\sum_{j=1}^{k} x_j = 1$ and

$$\|\widehat{\beta_{l,\infty}}(x_j) - J \circ S(e_j)\| < \frac{\epsilon}{2k}, \quad j = 1, 2, \ldots, k.$$

Define linear positive order unit preserving maps $V : \mathbb{R}^k \to \operatorname{Aff} T(B_l)$ by

$$V(\sum_{j=1}^{k} t_j e_j) = \sum_{j=1}^{k} t_j x_j,$$

and $W : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B_l)$ by $W = V \circ T$. Since

$$\|\widehat{\beta_{l,\infty}} \circ V - J \circ S\| < \frac{\epsilon}{2}$$

we see that

$$\|\widehat{\beta_{l,\infty}} \circ W(f) - J(f)\| < \epsilon, \quad f \in F.$$

By continuity of $KK(A, \cdot)$ there exist an integer m and an element $\nu \in KK(A, B_m)$ such that $[\beta_{m,\infty}] \cdot \nu = \kappa$. As

$$\beta_{m,\infty *} \circ \nu_*[1] = \kappa_*[1] = [1] = \beta_{m,\infty *}[1] \quad \text{in } K_0(B)$$

we see that there exists an integer $n \geq m, l$ such that

$$\beta_{m,n_*} \circ \nu_*[1] = [1] \quad \text{in } K_0(B_n).$$

Set $\omega = [\beta_{m,n}] \cdot \nu$ and $M = \widehat{\beta_{l,n}} \circ W$. □

Proposition 10.5. *Let A be a simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks with unital and injective connecting maps. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps. Assume that there exist a $\kappa \in KK(A, B)$ such that $\kappa_*[1] = [1]$ in $K_0(B)$ and an affine continuous map $\varphi_T : T(B) \to T(A)$ such that

$$r_B(\omega)(\kappa_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

Let $\varphi_{T*} : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$ denote the positive linear order unit preserving map induced by φ_T. Let $\epsilon > 0$ and let $F \subseteq \operatorname{Aff} T(A_n)$ be a finite subset for some positive integer n. There exist a positive integer m and a linear positive order unit preserving map $M : \operatorname{Aff} T(A_n) \to \operatorname{Aff} T(B_m)$ such that

$$\|\widehat{\beta_{m,\infty}} \circ M(f) - \varphi_{T_*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \epsilon, \quad f \in F,$$

and an element $\omega \in KK(A_n, B_m)$ such that $\omega_* : K_0(A_n) \to K_0(B_m)$ preserves the order unit and such that

$$[\beta_{m,\infty}] \cdot \omega = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KK(A_n, B),$$
$$M \circ \rho_{A_n} = \rho_{B_m} \circ \omega_* \quad \text{on } K_0(A_n).$$

Proof. We may assume that $\|f\| \leq 1$, $f \in F$. Decompose $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_N^n$ as a finite direct sum of building blocks. Let $r_1, r_2, \ldots, r_N \in A_n$ be projections such that $[r_1], [r_2], \ldots, [r_N]$ generate $K_0(A_n)$. There exist positive integers $d_1, d_2, \ldots, d_N$ such that

$$\sum_{i=1}^{N} d_i[r_i] = [1] \quad \text{in } K_0(A_n).$$

As A is simple and the connecting maps are injective there exists a $\delta_0 > 0$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{r_i}) > \delta_0, \quad i = 1, 2, \ldots, N.$$

Choose $\delta > 0$ such that $\delta < \delta_0$ and $\delta(1 + \sum_{i=1}^{N} d_i) < \epsilon$.

By Lemma 10.4 there exist a positive integer l and a linear positive order unit preserving map $V : \operatorname{Aff} T(A_n) \to \operatorname{Aff} T(B_l)$ such that

$$\|\widehat{\beta_{l,\infty}} \circ V(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \delta, \quad f \in F \cup \{\widehat{r_1}, \widehat{r_2}, \ldots, \widehat{r_N}\},$$

and an element $\nu \in KK(A_n, B_l)$ such that $\nu_* : K_0(A_n) \to K_0(B_m)$ preserves the order unit and such that

$$[\beta_{l,\infty}] \cdot \nu = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KK(A_n, B).$$

Since by assumption $\rho_B \circ \kappa_* = \varphi_{T*} \circ \rho_A$ on $K_0(A)$ we see that for $i = 1, 2, \ldots, N$,

$$\widehat{\beta_{l,\infty}} \circ \rho_{B_l} \circ \nu_*[r_i] = \rho_B \circ \beta_{l,\infty *} \circ \nu_*[r_i] = \varphi_{T*} \circ \rho_A \circ \alpha_{n,\infty *}[r_i] = \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(\widehat{r_i}) > \delta_0.$$

Hence

$$\|\widehat{\beta_{l,\infty}} \circ \rho_{B_l} \circ \nu_*[r_i] - \widehat{\beta_{l,\infty}} \circ V(\widehat{r_i})\| < \delta, \quad i = 1, 2, \ldots, N.$$

Choose $m > l$ such that for $i = 1, 2, \ldots, N$,

$$\widehat{\beta_{l,m}} \circ \rho_{B_l} \circ \nu_*[r_i] > \delta_0,$$
$$\|\widehat{\beta_{l,m}} \circ \rho_{B_l} \circ \nu_*[r_i] - \widehat{\beta_{l,m}} \circ V(\widehat{r_i})\| < \delta.$$

Define $W : \operatorname{Aff} T(A_n) \to \operatorname{Aff} T(B_m)$ by $W = \widehat{\beta_{l,m}} \circ V$. Define $\omega \in KK(A_n, B_m)$ by $\omega = [\beta_{l,m}] \cdot \nu$.

Decompose $B_m = B_1^m \oplus B_2^m \oplus \cdots \oplus B_L^m$ as a finite direct sum of building blocks and let $\pi_j : B_m \to B_j^m$ be the projection, $j = 1, 2, \ldots, L$. Identify $\operatorname{Aff} T(B_m)$ with $\oplus_{j=1}^{L} C_{\mathbb{R}}(\mathbb{T})$. Fix some $j = 1, 2, \ldots, L$. Set $W_j = \widehat{\pi_j} \circ W$. As $\delta < \delta_0$, $W_j(\widehat{r_i})$ is a strictly positive function in $C_{\mathbb{R}}(\mathbb{T})$. Thus for each $i = 1, 2, \ldots, N$, we can define $M_j : \operatorname{Aff} T(A_n) \cong \oplus_{i=1}^{N} \operatorname{Aff} T(A_i^n) \to C_{\mathbb{R}}(\mathbb{T})$ by

$$M_j(f_1, f_2, \ldots, f_N) = \sum_{i=1}^{N} W_j(0, \ldots, 0, f_i, 0, \ldots, 0) \frac{1}{W_j(\widehat{r_i})} \widehat{\pi_j}(\rho_{B_m} \circ \omega_*[r_i]).$$

M_j is positive and linear, and it preserves the order unit since

$$M_j(1) = \sum_{i=1}^{N} W_j(d_i \widehat{r_i}) \frac{1}{W_j(\widehat{r_i})} \widehat{\pi_j}(\rho_{B_m} \circ \omega_*[r_i]) = \sum_{i=1}^{N} \widehat{\pi_j}(\rho_{B_m} \circ \omega_*(d_i[r_i])) = 1.$$

Let now $g \in C_{\mathbb{R}}(\mathbb{T}) \cong \operatorname{Aff} T(A_i^n)$, $\|g\| \leq 1$, for $i = 1, 2, \ldots, N$. Since

$$-d_i \widehat{r_i} \leq (0, \ldots, 0, g, 0, \ldots, 0) \leq d_i \widehat{r_i}$$

in $\operatorname{Aff} T(A_n)$ we have that

$$\begin{aligned}
&\|M_j(0,\ldots,g,\ldots,0) - W_j(0,\ldots,g,\ldots,0)\| \\
&= \|W_j(0,\ldots,g,\ldots,0)\frac{1}{W_j(\widehat{r_i})}\big(\widehat{\pi}_j(\rho_{B_m}\circ\omega_*[r_i]) - W_j(\widehat{r_i})\big)\| \\
&\leq d_i\|\widehat{\pi}_j(\rho_{B_m}\circ\omega_*[r_i]) - W_j(\widehat{r_i})\| < \delta d_i.
\end{aligned}$$

Hence if $f \in \operatorname{Aff} T(A_n)$, $\|f\| \leq 1$, then

$$\|M_j(f) - W_j(f)\| < \sum_{i=1}^{N} \delta d_i.$$

Define a linear positive order unit preserving map $M : \operatorname{Aff} T(A_n) \to \operatorname{Aff} T(B_m)$ by

$$M(f) = (M_1(f), M_2(f), \ldots, M_L(f)).$$

Then

$$\|M(f) - W(f)\| < \sum_{i=1}^{N} \delta d_i, \quad f \in \operatorname{Aff} T(A_n),\ \|f\| \leq 1,$$

and hence

$$\|\widehat{\beta_{m,\infty}} \circ M(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \delta + \sum_{i=1}^{N} \delta d_i < \epsilon, \quad f \in F.$$

Finally, $M(\widehat{r_i}) = \rho_{B_m} \circ \omega_*[r_i]$, $i = 1, 2, \ldots, N$. It follows that $M \circ \rho_{A_n} = \rho_{B_m} \circ \omega_*$ on $K_0(A_n)$. $\square$

Proposition 10.6. *Let A be a simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks with unital and injective connecting maps. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps and such that $s(B_k) \to \infty$. Assume that there exist a $\kappa \in KK(A, B)$ such that $\kappa_*[1] = [1]$ in $K_0(B)$ and an affine continuous map $\varphi_T : T(B) \to T(A)$ such that

$$r_B(\omega)(\kappa_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

Let $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be a homomorphism such that the diagram

$$\begin{array}{ccc}
U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\
\Phi\big\downarrow & & \big\downarrow\kappa_* \\
U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B)
\end{array}$$

commutes. Let n be a positive integer, let $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_N^n$, let $F \subseteq \operatorname{Aff} T(A_n)$ be a finite subset, and let $\epsilon > 0$. There exist a positive integer m and a *unital *-homomorphism* $\varphi : A_n \to B_m$ *such that*

$$[\beta_{m,\infty}] \cdot [\varphi] = \kappa \cdot [\alpha_{n,\infty}] \quad \textit{in } KK(A_n, B),$$
$$\|\widehat{\beta_{m,\infty}} \circ \widehat{\varphi}(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \epsilon, \quad f \in F,$$
$$D_B\big(\,\Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(V^i_{A_n}))\,,\,\beta^{\#}_{m,\infty} \circ \varphi^{\#}(q'_{A_n}(V^i_{A_n}))\,\big) < \epsilon, \quad i = 1, 2, \dots, N.$$

Proof. We may assume that $\epsilon < 2$. Let $p_1, p_2, \dots, p_N$ denote the minimal non-zero central projections in A_n. As A is simple and the connecting maps are injective, there exists a $\delta > 0$ such that $\widehat{\alpha_{n,\infty}}(\widehat{p_i}) > \delta$, $i = 1, 2, \dots, N$. By Proposition 10.5 there exist a positive integer l, a linear positive order unit preserving map $M : \operatorname{Aff} T(A_n) \to \operatorname{Aff} T(B_l)$, and an element $\omega \in KK(A_n, B_l)$ such that $\omega_* : K_0(A_n) \to K_0(B_l)$ preserves the order unit and such that

$$[\beta_{l,\infty}] \cdot \omega = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KK(A_n, B),$$
$$\|\widehat{\beta_{l,\infty}} \circ M(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \frac{\epsilon}{2}, \quad f \in F,$$
$$M \circ \rho_{A_n} = \rho_{B_l} \circ \omega_* \quad \text{on } K_0(A_n).$$

Choose an integer K by Theorem 8.8 with respect to $F \subseteq \operatorname{Aff} T(A_n)$ and $\frac{\epsilon}{2}$. Choose a positive integer k and unitaries $u_1, u_2, \dots, u_N \in B_k$ such that

$$D_B\big(\,\Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(V^i_{A_n}))\,,\,\beta^{\#}_{k,\infty}(q'_{A_n}(u_i))\,\big) < \epsilon, \quad i = 1, 2, \dots, N.$$

From this it follows that $\kappa_* \circ \alpha_{n,\infty *}[V^i_{A_n}] = \beta_{k,\infty *}[u_i]$ in $K_1(B)$. Hence

$$\beta_{l,\infty *} \circ \omega_*[V^i_{A_n}] = \beta_{k,\infty *}[u_i], \quad i = 1, 2, \dots, N.$$

As $\rho_B \circ \kappa_* = \varphi_{T*} \circ \rho_A$ we see that for $i = 1, 2, \dots, N$,

$$\widehat{\beta_{l,\infty}}(\rho_{B_l}(\omega_*[p_i])) = \rho_B(\beta_{l,\infty *} \circ \omega_*[p_i]) = \varphi_{T*} \circ \rho_A \circ \alpha_{n,\infty *}[p_i] = \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(\widehat{p_i}) > \delta.$$

Hence there exists an integer $m \geq k, l$ such that $s(B_m) \geq K\delta^{-1}$ and such that

$$\widehat{\beta_{l,m}}(\rho_{B_l}(\omega_*[p_i])) > \delta, \quad i = 1, 2, \dots, N,$$
$$\beta_{l,m *} \circ \omega_*[V^i_{A_n}] = \beta_{k,m *}[u_i] \quad \text{in } K_1(B_m), \quad i = 1, 2, \dots, N.$$

It follows that $s(B_m)\rho_{B_m}(\beta_{l,m *} \circ \omega_*[p_i]) \geq K$ and that

$$\widehat{\beta_{l,m}} \circ M \circ \rho_{A_n} = \widehat{\beta_{l,m}} \circ \rho_{B_l} \circ \omega_* = \rho_{B_m} \circ \beta_{l,m *} \circ \omega_* \quad \text{on } K_0(A_n).$$

Therefore by Theorem 8.8 there exists a unital *-homomorphism $\varphi : A_n \to B_m$ such that

$$[\varphi] = [\beta_{l,m}] \cdot \omega \quad \text{in } KK(A_n, B_m),$$
$$\varphi^{\#}(q'_{A_n}(V^i_{A_n})) = q'_{B_m}(\beta_{k,m}(u_i)) \quad \text{in } U(B_m)/\overline{DU(B_m)}, \quad i = 1, 2, \dots, N,$$
$$\|\widehat{\varphi}(f) - \widehat{\beta_{l,m}} \circ M(f)\| < \frac{\epsilon}{2}, \quad f \in F.$$

It follows that

$$[\beta_{m,\infty}] \cdot [\varphi] = [\beta_{l,\infty}] \cdot \omega = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KK(A_n, B),$$
$$\|\widehat{\beta_{m,\infty}} \circ \widehat{\varphi}(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \epsilon, \quad f \in F,$$
$$\beta^{\#}_{m,\infty} \circ \varphi^{\#}(q'_{A_n}(V^i_{A_n})) = \beta^{\#}_{k,\infty}(q'_{A_n}(u_i)), \quad \text{in } U(B)/\overline{DU(B)}, \quad i = 1, 2, \dots, N.$$

From the last equation we get that

$$D_B\big(\,\Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(V^i_{A_n}))\,,\,\beta^{\#}_{m,\infty} \circ \varphi^{\#}(q'_{A_n}(V^i_{A_n}))\,\big) < \epsilon, \quad i = 1, 2, \dots, N.$$

□

Proposition 10.7. *Let A be a simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \dots$$

of finite direct sums of building blocks with unital and injective connecting maps. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \dots$$

with unital connecting maps and such that $d(B_k) \to \infty$. Assume that there exist a $\kappa \in KK(A,B)$ such that $\kappa_*[1] = [1]$ in $K_0(B)$ and an affine continuous map $\varphi_T : T(B) \to T(A)$ such that

$$r_B(\omega)(\kappa_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

Let $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be a homomorphism such that the diagram

$$\begin{array}{ccccc}
\mathit{Aff}\,T(A)/\overline{\rho_A(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\
\Big\downarrow{\scriptstyle\widetilde{\varphi}} & & \Big\downarrow{\scriptstyle\Phi} & & \Big\downarrow{\scriptstyle\kappa_*} \\
\mathit{Aff}\,T(B)/\overline{\rho_B(K_0(B))} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B)
\end{array}$$

commutes, where $\widetilde{\varphi} : \mathit{Aff}\,T(A)/\overline{\rho_A(K_0(A))} \to \mathit{Aff}\,T(B)/\overline{\rho_B(K_0(B))}$ is the map induced by $\varphi_{T*} : \mathit{Aff}\,T(A) \to \mathit{Aff}\,T(B)$.

*Let n be a positive integer, let $F_1 \subseteq \mathit{Aff}\,T(A_n)$ and $F_2 \subseteq U(A_n)/\overline{DU(A_n)}$ be finite subsets, and let $\epsilon > 0$. There exist a positive integer m and a unital *-homomorphism $\psi : A_n \to B_m$ such that*

$$[\beta_{m,\infty}] \cdot [\psi] = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KK(A_n, B),$$
$$\|\widehat{\beta_{m,\infty}} \circ \widehat{\psi}(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \epsilon, \quad f \in F_1,$$
$$D_B\big(\,\Phi \circ \alpha^{\#}_{n,\infty}(x)\,,\, \beta^{\#}_{m,\infty} \circ \psi^{\#}(x)\,\big) < \epsilon, \quad x \in F_2.$$

Proof. Decompose $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_N^n$ as a finite direct sum of building blocks. For each $x \in U(A_n)/\overline{DU(A_n)}$ there exist by Proposition 5.2 an element $a_x \in \mathrm{Aff}\,T(A_n)/\overline{\rho_{A_n}(K_0(A_n))}$, integers $k_x^1, k_x^2, \dots, k_x^N$, and a unitary $w_x \in A_n$ such that $q'_{A_n}(w_x)$ has finite order in $U(A_n)/\overline{DU(A_n)}$ and such that

$$x = \lambda_{A_n}(a_x)\, q'_{A_n}\big(\prod_{i=1}^{N} (V^i_{A_n})^{k_x^i}\big)\, q'_{A_n}(w_x) \quad \text{in } U(A_n)/\overline{DU(A_n)}.$$

Choose $b_x \in \mathrm{Aff}\,T(A_n)$ such that $q_{A_n}(b_x) = a_x$. Set $F_1' = F_1 \cup \{b_x : x \in F_2\}$. Choose $0 < \delta < \frac{1}{2}$ such that $\delta < \epsilon$ and such that

$$|e^{2\pi i\delta} - 1| + \delta \sum_{i=1}^{N} k_i^x < \epsilon, \quad x \in F_2.$$

By Proposition 10.6 there exists a positive integer m and a unital *-homomorphism $\psi : A_n \to B_m$ such that

$$[\beta_{m,\infty}] \cdot [\psi] = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KK(A_n, B),$$
$$\|\widehat{\beta_{m,\infty}} \circ \widehat{\psi}(f) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)\| < \delta, \quad f \in F_1',$$
$$D_B\big(\, \beta^{\#}_{m,\infty} \circ \psi^{\#}(q'_{A_n}(V^i_{A_n}))\,,\, \Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(V^i_{A_n}))\big) < \delta, \quad i = 1, 2, \ldots, N.$$

Note that

$$\begin{aligned} d'_B\big(\,\widetilde{\beta_{m,\infty}} \circ \widetilde{\psi}(a_x)\,,\, \widetilde{\varphi} \circ \widetilde{\alpha_{n,\infty}}(a_x)\,\big) &= d'_B\big(\, q_B(\widehat{\beta_{m,\infty}} \circ \widehat{\psi}(b_x))\,,\, q_B(\varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(b_x))\big) \\ &\leq \|\widehat{\beta_{m,\infty}} \circ \widehat{\psi}(b_x) - \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(b_x)\| < \delta < \frac{1}{2}. \end{aligned}$$

Hence

$$d_B\big(\,\widetilde{\beta_{m,\infty}} \circ \widetilde{\psi}(a_x)\,,\, \widetilde{\varphi} \circ \widetilde{\alpha_{n,\infty}}(a_x)\,\big) < |e^{2\pi i\delta} - 1|, \quad x \in F_2.$$

By Proposition 5.2, λ_B is an isometry when $\operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))}$ is equipped with the metric d_B. It follows that

$$D_B\big(\,\lambda_B \circ \widetilde{\beta_{m,\infty}} \circ \widetilde{\psi}(a_x)\,,\, \lambda_B \circ \widetilde{\varphi} \circ \widetilde{\alpha_{n,\infty}}(a_x)\,\big) < |e^{2\pi i\delta} - 1|.$$

Thus

$$D_B\big(\,\beta^{\#}_{m,\infty} \circ \psi^{\#} \circ \lambda_{A_n}(a_x)\,,\, \Phi \circ \alpha^{\#}_{n,\infty} \circ \lambda_{A_n}(a_x)\,\big) < |e^{2\pi i\delta} - 1|, \quad x \in F_2.$$

By Lemma 10.3 we see that for $x \in F_2$,

$$\beta^{\#}_{m,\infty} \circ \psi^{\#}(q'_{A_n}(w_x)) = \Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(w_x)) \quad \text{in } U(B)/\overline{DU(B)}.$$

Hence for $x \in F_2$,

$$\begin{aligned} &D_B\big(\,\beta^{\#}_{m,\infty} \circ \psi^{\#}(x)\,,\, \Phi \circ \alpha^{\#}_{n,\infty}(x)\,\big) \\ &\leq D_B\big(\,\beta^{\#}_{m,\infty} \circ \psi^{\#}(\lambda_{A_n}(a_x))\,,\, \Phi \circ \alpha^{\#}_{n,\infty}(\lambda_{A_n}(a_x))\,\big) + \\ &\quad \sum_{i=1}^{N} k^i_x\, D_B\big(\,\beta^{\#}_{m,\infty} \circ \psi^{\#}(q'_{A_n}(V^i_{A_n}))\,,\, \Phi \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(V^i_{A_n}))\,\big) \\ &< |e^{2\pi i\delta} - 1| + \sum_{i=1}^{N} k^i_x \delta < \epsilon. \end{aligned}$$

□

Lemma 10.8. *Let A be a simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

*of finite direct sums of building blocks with unital and injective connecting maps. Let n be a positive integer, let $G \subseteq A_n$ be a finite set and let $\epsilon > 0$. There exist finite sets $D \subseteq \operatorname{Aff} T(A_n)$ and $E \subseteq U(A_n)/\overline{DU(A_n)}$, a $\delta > 0$ and an integer $r > n$ such that if B is a finite direct sum of building blocks with $d(B) \geq \delta^{-1}$ and if $\varphi : A_n \to B$, $\psi : A_r \to B$ are unital *-homomorphisms such that*

$$\|\widehat{\varphi}(f) - \widehat{\psi} \circ \widehat{\alpha_{n,r}}(f)\| < \delta, \quad f \in D,$$
$$D_B\big(\,\varphi^{\#}(f)\,,\, \psi^{\#} \circ \alpha^{\#}_{n,r}(f)\,\big) < \delta, \quad f \in E,$$
$$[\varphi] = [\psi] \cdot [\alpha_{n,r}] \quad \textit{in } KK(A_n, B),$$

then there exists a unitary $W \in B$ such that

$$\|\psi \circ \alpha_{n,r}(x) - W\varphi(x)W^*\| < \epsilon, \quad x \in G_n.$$

Proof. Let $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_N^n$ be a decomposition of A_n as a finite direct sum of building blocks. Let $p_1, p_2, \ldots, p_N$ denote the minimal non-zero central projections in A_n. Choose by Theorem 7.6 a positive integer l with respect to A_n, $\epsilon > 0$, and $G \subseteq A_n$. As A is simple and the connecting maps are injective, there exists an integer $p \geq l$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{8}{p}, \quad h \in H(A, l).$$

Next, choose an integer $k \geq p$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{6}{k}, \quad h \in H(A_n, p) \cup \{p_1, p_2, \ldots, p_N\}.$$

Choose an integer $q \geq k$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{1}{q}, \quad h \in H(A_n, k).$$

Finally, choose $\delta > 0$ such that $\delta < \frac{1}{k^2}$ and

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \delta, \quad h \in H(A_n, 10q).$$

Set $D = \widetilde{H}(A_n, 2q)$. Choose $r > n$ such that

$$\widehat{\alpha_{n,r}}(\widehat{h}) > \frac{8}{p}, \quad h \in H(A_n, l),$$

$$\widehat{\alpha_{n,r}}(\widehat{h}) > \frac{6}{k}, \quad h \in H(A_n, p) \cup \{p_1, p_2, \ldots, p_N\},$$

$$\widehat{\alpha_{n,r}}(\widehat{h}) > \frac{1}{q}, \quad h \in H(A_n, k),$$

$$\widehat{\alpha_{n,r}}(\widehat{h}) > \delta, \quad h \in H(A_n, 10q).$$

Set $E = \{q'_{A_n}(V_{A_n}^1), q'_{A_n}(V_{A_n}^2), \ldots, q'_{A_n}(V_{A_n}^N)\}$. The conclusion follows from Theorem 7.6. $\square$

Theorem 10.9. *Let A be a unital simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots$$

of finite direct sums of building blocks. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \ldots$$

with unital connecting maps and such that $d(B_k) \to \infty$. Assume that there exist a $\kappa \in KK(A, B)$ such that $\kappa_*[1] = [1]$ in $K_0(B)$ and an affine continuous map $\varphi_T : T(B) \to T(A)$ such that

$$r_B(\omega)(\kappa_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

Let $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be a homomorphism such that the diagram

$$\begin{array}{ccccc}
\mathit{Aff}\,T(A)/\overline{\rho_A(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\
\Big\downarrow{\scriptstyle\tilde{\varphi}} & & \Big\downarrow{\scriptstyle\Phi} & & \Big\downarrow{\scriptstyle\kappa_*} \\
\mathit{Aff}\,T(B)/\overline{\rho_B(K_0(B))} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B)
\end{array}$$

commutes, where $\widetilde{\varphi} : \mathrm{Aff}T(A)/\overline{\rho_A(K_0(A))} \to \mathrm{Aff}T(B)/\overline{\rho_B(K_0(B))}$ *is the map induced by* $\varphi_{T*} : \mathrm{Aff}T(A) \to \mathrm{Aff}T(B)$.

*There exists a unital *-homomorphism* $\psi : A \to B$ *such that* $\psi^* = \varphi_T$ *on* $T(B)$, *such that* $\psi^\# = \Phi$ *on* $U(A)/\overline{DU(A)}$, *and such that* $[\psi] \cdot [\mu] = \kappa \cdot [\mu]$ *in* $KK(D, B)$ *whenever* D *is a finite direct sum of building blocks and* $\mu : D \to A$ *is a unital *-homomorphism.*

Proof. We may assume that A is infinite dimensional. Hence by Theorem 9.8 we may assume that each α_n is unital and injective. For each positive integer n, choose a finite set $G_n \subseteq A_n$ such that G_n generates A_n as a C^*-algebra. We may assume that $\alpha_n(G_n) \subseteq G_{n+1}$. Choose by Lemma 10.8 finite sets $D_n \subseteq \mathrm{Aff}T(A_n)$, $E_n \subseteq U(A_n)/\overline{DU(A_n)}$, a positive integer r_n, and a $\delta_n > 0$ with respect to G_n and 2^{-n}.

Choose finite sets $F_n \subseteq \mathrm{Aff}T(A_n)$ such that $D_n \subseteq F_n$, such that $\widehat{\alpha_n}(F_n) \subseteq F_{n+1}$, and such that $\cup_{n=1}^\infty \widehat{\alpha_{n,\infty}}(F_n)$ is dense in $\mathrm{Aff}T(A)$.

Next, choose finite sets $K_n \subseteq U(A_n)/\overline{DU(A_n)}$ such that $E_n \subseteq K_n$, such that $\alpha_n^\#(K_n) \subseteq K_{n+1}$, and such that $\cup_{n=1}^\infty \alpha_{n,\infty}^\#(K_n)$ is dense in $U(A)/\overline{DU(A)}$.

We will construct by induction strictly increasing sequences $\{n_k\}$ and $\{m_k\}$ and unital *-homomorphisms $\psi_k : A_{n_k} \to B_{m_k}$ such that

(i) $\|\beta_{m_{k-1},m_k} \circ \psi_{k-1}(x) - \psi_k \circ \alpha_{n_{k-1},n_k}(x)\| < 2^{-n_{k-1}}, \quad x \in G_{n_{k-1}},\ k \geq 2,$

(ii) $\|\widehat{\beta_{m_k,\infty}} \circ \widehat{\psi_k}(f) - \varphi_{T*} \circ \widehat{\alpha_{n_k,\infty}}(f)\| < \min\{2^{-n_k}, \frac{\delta_{n_k}}{2}\}, \quad f \in F_{n_k},$

(iii) $D_B\big(\beta_{m_k,\infty}^\# \circ \psi_k^\#(x)\,,\, \Phi \circ \alpha_{n_k,\infty}^\#(x)\big) < \min\{2^{-n_k}, \frac{\delta_{n_k}}{2}\}, \quad x \in K_{n_k},$

(iv) $[\beta_{m_k,\infty}] \cdot [\psi_k] = \kappa \cdot [\alpha_{n_k,\infty}]$ in $KK(A_{n_k}, B)$.

n_k, m_k, and ψ_k are constructed in step k. The case $k = 1$ follows immediately from Proposition 10.7.

Assume that n_k, m_k, and ψ_k have been constructed. Set $n_{k+1} = r_{n_k}$. Choose by Proposition 10.7 a positive integer l and a unital *-homomorphism $\lambda : A_{n_{k+1}} \to B_l$ such that

$$\|\widehat{\beta_{l,\infty}} \circ \widehat{\lambda}(f) - \varphi_{T*} \circ \widehat{\alpha_{n_{k+1},\infty}}(f)\| < \min\{2^{-n_{k+1}}, \frac{\delta_{n_k}}{2}, \frac{\delta_{n_{k+1}}}{2}\}, \quad f \in F_{n_{k+1}},$$
$$D_B\big(\beta_{l,\infty}^\# \circ \lambda^\#(x)\,,\, \Phi \circ \alpha_{n_{k+1},\infty}^\#(x)\big) < \min\{2^{-n_{k+1}}, \frac{\delta_{n_k}}{2}, \frac{\delta_{n_{k+1}}}{2}\}, \quad x \in K_{n_{k+1}},$$
$$[\beta_{l,\infty}] \cdot [\lambda] = \kappa \cdot [\alpha_{n_{k+1},\infty}] \quad \text{in } KK(A_{n_{k+1}}, B).$$

It follows that

$$\|\widehat{\beta_{l,\infty}} \circ \widehat{\lambda} \circ \widehat{\alpha_{n_k,n_{k+1}}}(f) - \widehat{\beta_{m_k,\infty}} \circ \widehat{\psi_k}(f)\| < \delta_{n_k}, \quad f \in F_{n_k},$$
$$D_B\big(\beta_{l,\infty}^\# \circ \lambda^\# \circ \alpha_{n_k,n_{k+1}}^\#(x)\,,\, \beta_{m_k,\infty}^\# \circ \psi_k^\#(x)\big) < \delta_{n_k}, \quad x \in K_{n_k},$$
$$[\beta_{m_k,\infty}] \cdot [\psi_k] = [\beta_{l,\infty}] \cdot [\lambda] \cdot [\alpha_{n_k,n_{k+1}}] \quad \text{in } KK(A_k, B).$$

Hence there exists an integer $m_{k+1} \geq l$ such that $d(B_{m_{k+1}}) \geq {\delta_{n_k}}^{-1}$ and such that

$$\|\widehat{\beta_{l,m_{k+1}}} \circ \widehat{\lambda} \circ \widehat{\alpha_{n_k,n_{k+1}}}(f) - \widehat{\beta_{m_k,m_{k+1}}} \circ \widehat{\psi_k}(f)\| < \delta_{n_k}, \quad f \in F_{n_k},$$
$$D_B\big(\beta_{l,m_{k+1}}^\# \circ \lambda^\# \circ \alpha_{n_k,n_{k+1}}^\#(x)\,,\, \beta_{m_k,m_{k+1}}^\# \circ \psi_k^\#(x)\big) < \delta_{n_k}, \quad x \in K_{n_k},$$
$$[\beta_{m_k,m_{k+1}}] \cdot [\psi_k] = [\beta_{l,m_{k+1}}] \cdot [\lambda] \cdot [\alpha_{n_k,n_{k+1}}] \quad \text{in } KK(A_k, B).$$

By Lemma 10.8 there exists a unitary $W \in B_{m_{k+1}}$ such that

$$\|\beta_{m_k,m_{k+1}} \circ \psi_k(x) - W\beta_{l,m_{k+1}} \circ \lambda \circ \alpha_{n_k,n_{k+1}}(x)W^*\| < 2^{-n_k}, \quad x \in G_{n_k}.$$

Set $\psi_{k+1} = Ad(W) \circ \beta_{l,m_{k+1}} \circ \lambda$. It is easily seen that (i)-(iv) are satisfied. This completes the induction step.

By Elliott's approximate intertwining argument, see e.g [23, Lemma 1], there exists a *-homomorphism $\psi : A \to B$ such that

$$\psi(\alpha_{n,\infty}(x)) = \lim_{k\to\infty} \beta_{m_k,\infty} \circ \psi_k \circ \alpha_{n,n_k}(x), \quad x \in A_n.$$

Clearly, ψ is unital. Let $f \in F_n$, $\omega \in T(B)$. The sequence $\omega \circ \beta_{m_k,\infty} \circ \psi_k \circ \alpha_{n,n_k}$ converges to $\omega \circ \psi \circ \alpha_{n,\infty}$ in $T(A_n)$ as $k \to \infty$. Hence it follows that

$$\widehat{\beta_{m_k,\infty}} \circ \widehat{\psi_k} \circ \widehat{\alpha_{n,n_k}}(f)(\omega) \ \to \ \widehat{\psi} \circ \widehat{\alpha_{n,\infty}}(f)(\omega) \quad \text{as } k \to \infty.$$

On the other hand, from (ii) it follows that

$$\widehat{\beta_{m_k,\infty}} \circ \widehat{\psi_k} \circ \widehat{\alpha_{n,n_k}}(f)(\omega) \ \to \ \varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f)(\omega) \quad \text{as } k \to \infty.$$

Hence $\widehat{\psi} = \varphi_{T*}$ on $\operatorname{Aff} T(A)$ and thus $\psi^* = \varphi_T$ on $T(B)$.

As A_n has stable relations there exists by [16, Theorem 15.1.1] a positive integer $l \geq n$ such that $\psi \circ \alpha_{n,\infty}$ is homotopic to $\beta_{m_l,\infty} \circ \psi_l \circ \alpha_{n,n_l}$. Hence

$$[\psi] \cdot [\alpha_{n,\infty}] = [\beta_{m_l,\infty}] \cdot [\psi_l] \cdot [\alpha_{n,n_l}] = \kappa \cdot [\alpha_{n_l,\infty}] \cdot [\alpha_{n,n_l}] = \kappa \cdot [\alpha_{n,\infty}]$$

in $KK(A_n, B)$. Let D be a finite direct sum of building blocks and $\mu : D \to A$ a *-homomorphism. By [16, Corollary 15.1.3] there exist a positive integer n and a *-homomorphism $\lambda : D \to A_n$ such that μ is homotopic to $\alpha_{n,\infty} \circ \lambda$. Thus

$$[\psi] \cdot [\mu] = [\psi] \cdot [\alpha_{n,\infty}] \cdot [\lambda] = \kappa \cdot [\alpha_{n,\infty}] \cdot [\lambda] = \kappa \cdot [\mu]$$

in $KK(D, B)$. □

Corollary 10.10. *Let A be a unital simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

with unital connecting maps and such that $d(B_k) \to \infty$. Assume that there exist a $\kappa \in KK(A, B)$ such that $\kappa_*[1] = [1]$ in $K_0(B)$ and an affine continuous map $\varphi_T : T(B) \to T(A)$ such that

$$r_B(\omega)(\kappa_*(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A), \ \omega \in T(B).$$

*There exists a unital *-homomorphism $\psi : A \to B$ such that $\psi^* = \varphi_T$ on $T(B)$ and such that $[\psi] \cdot [\mu] = \kappa \cdot [\mu]$ in $KK(D, B)$ whenever D is a finite direct sum of building blocks and $\mu : D \to A$ is a unital *-homomorphism.*

Proof. As $\varphi_{T*} \circ \rho_A = \rho_B \circ \kappa_*$, φ_T induces a contractive group homomorphism $\widetilde{\varphi} : \operatorname{Aff} T(A)/\rho_A(K_0(A)) \to \operatorname{Aff} T(B)/\rho_B(K_0(B))$. By Proposition 5.2 there exists a group homomorphism $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ such that the diagram

$$\begin{array}{ccccc}
\operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\
\downarrow{\scriptstyle \widetilde{\varphi}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \kappa_*} \\
\operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B)
\end{array}$$

commutes. The corollary now follows from Theorem 10.9. □

Corollary 10.11. *Let A be a unital simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \dots$$

of finite direct sums of building blocks. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \dots$$

with unital connecting maps and such that $d(B_k) \to \infty$. Let $\varphi_0 : K_0(A) \to K_0(B)$ be a group homomorphism that preserves the order unit. Let $\varphi_1 : K_1(A) \to K_1(B)$ be a group homomorphism, and let $\varphi_T : T(B) \to T(A)$ be a continuous affine map such that

$$r_B(\varphi_0(x)) = r_A(\varphi_T(\omega))(x), \quad x \in K_0(A),\ \omega \in T(B).$$

*There exists a unital *-homomorphism $\psi : A \to B$ such that $\psi^* = \varphi_T$ on $T(B)$, $\varphi_* = \varphi_0$ on $K_0(A)$, and $\varphi_* = \varphi_1$ on $K_1(A)$.*

Proof. By the universal coefficient theorem [20, Theorem 1.17], there exists an element $\kappa \in KK(A, B)$ such that $\kappa_* = \varphi_0$ on $K_0(A)$ and $\kappa_* = \varphi_1$ on $K_1(A)$. The corollary thus follows from Corollary 10.10. □

Theorem 10.12. *Let A be a simple unital inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \dots$$

of finite direct sums of building blocks. Let B be an inductive limit of a similar sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \dots$$

*with unital connecting maps and such that $d(B_k) \to \infty$. Let $\varphi, \psi : A \to B$ be two unital *-homomorphisms and assume that $\varphi^* = \psi^*$ on $T(B)$, that $\varphi^\# = \psi^\#$ on $U(A)/\overline{DU(A)}$, and that $[\varphi] \cdot [\mu] = [\psi] \cdot [\mu]$ in $KK(D, B)$ whenever $\mu : D \to A$ is a *-homomorphism from a finite direct sum of building blocks D. Then φ and ψ are approximately unitarily equivalent.*

Proof. We may assume that A is infinite dimensional and hence by Theorem 9.8 that the connecting maps are unital and injective.

Let n be a positive integer, let $F \subseteq A_n$ be a finite set and let $\epsilon > 0$. It suffices to find a unitary $U \in B$ such that

$$\|\varphi \circ \alpha_{n,\infty}(x) - Ad(U) \circ \psi \circ \alpha_{n,\infty}(x)\| < \epsilon, \quad x \in F.$$

We may assume that $1 \in F$. Let $A_n = A_1^n \oplus A_2^n \oplus \cdots \oplus A_N^n$. Let $p_1, p_2, \dots, p_N$ denote the minimal non-zero central projections in A_n. Choose by uniqueness, Theorem 7.6, a positive integer l with respect to F and $\frac{\epsilon}{3}$. As A is simple and the connecting maps are injective there exists an integer $p \geq l$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{9}{p}, \quad h \in H(A_n, l).$$

Next, choose $k \geq p$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{7}{k}, \quad h \in H(A_n, p) \cup \{p_1, p_2, \dots, p_N\}.$$

Choose $q \geq k$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{2}{q}, \quad h \in H(A_n, k).$$

Finally, choose $\delta > 0$ such that $\delta < \epsilon$, such that $\delta < \frac{1}{k^2}$, such that $\delta < \frac{3}{q}$, and such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \delta, \quad h \in H(A_n, 10q).$$

Since A_n has stable relations there exists by [16, Corollary 15.1.3] a positive integer r and *-homomorphisms $\varphi_1, \psi_1 : A_n \to B_r$ such that $\beta_{r,\infty} \circ \varphi_1$ is homotopic to $\varphi \circ \alpha_{n,\infty}$, such that $\beta_{r,\infty} \circ \psi_1$ is homotopic to $\psi \circ \alpha_{n,\infty}$, such that for $x \in F \cup \widetilde{H}(A_n, 2q) \cup \{V^1_{A_n}, V^2_{A_n}, \dots, V^N_{A_n}\}$,

$$\|\beta_{r,\infty} \circ \varphi_1(x) - \varphi \circ \alpha_{n,\infty}(x)\| < \frac{\delta}{3},$$

and such that for x in the set

$$F \cup \widetilde{H}(A_n, 2q) \cup H(A_n, 10q) \cup H(A_n, l) \cup H(A_n, p) \cup H(A_n, k) \cup \{V^1_{A_n}, V^2_{A_n}, \dots, V^N_{A_n}\},$$

we have that

$$\|\beta_{r,\infty} \circ \psi_1(x) - \psi \circ \alpha_{n,\infty}(x)\| < \frac{\delta}{3} < \frac{1}{q}.$$

In particular we see that by increasing r we may assume that φ_1 and ψ_1 are unital.

By assumption

$$[\beta_{r,\infty}] \cdot [\psi_1] = [\psi] \cdot [\alpha_{n,\infty}] = [\varphi] \cdot [\alpha_{n,\infty}] = [\beta_{r,\infty}] \cdot [\varphi_1] \quad \text{in } KK(A_n, B).$$

As $\varphi^* = \psi^*$ on $T(B)$ we see that for $h \in \widetilde{H}(A_n, 2q)$,

$$\begin{aligned} &\|\widehat{\beta_{r,\infty}} \circ \widehat{\psi_1}(\widehat{h}) - \widehat{\beta_{r,\infty}} \circ \widehat{\varphi_1}(\widehat{h})\| \\ &\le \|\widehat{\beta_{r,\infty}} \circ \widehat{\psi_1}(\widehat{h}) - \widehat{\psi} \circ \widehat{\alpha_{n,\infty}}(\widehat{h})\| + \|\widehat{\varphi} \circ \widehat{\alpha_{n,\infty}}(\widehat{h}) - \widehat{\beta_{r,\infty}} \circ \widehat{\varphi_1}(\widehat{h})\| < \frac{2\delta}{3}. \end{aligned}$$

Note that

$$\begin{aligned} \widehat{\beta_{r,\infty}} \circ \widehat{\psi_1}(\widehat{h}) &> \frac{2\delta}{3}, \quad h \in H(A_n, 10q), \\ \widehat{\beta_{r,\infty}} \circ \widehat{\psi_1}(\widehat{h}) &> \frac{8}{p}, \quad h \in H(A_n, l), \\ \widehat{\beta_{r,\infty}} \circ \widehat{\psi_1}(\widehat{h}) &> \frac{6}{k}, \quad h \in H(A_n, p) \cup \{p_1, p_2, \dots, p_N\}, \\ \widehat{\beta_{r,\infty}} \circ \widehat{\psi_1}(\widehat{h}) &> \frac{1}{q}, \quad h \in H(A_n, k). \end{aligned}$$

As $\varphi^{\#} = \psi^{\#}$ on $U(A)/\overline{DU(A)}$ we see that for $i = 1, 2, \dots, N$,

$$\begin{aligned} &D_B\big(\beta^{\#}_{r,\infty} \circ \varphi^{\#}_1(q'_{A_n}(V^i_{A_n}))\,,\, \beta^{\#}_{r,\infty} \circ \psi^{\#}_1(q'_{A_n}(V^i_{A_n}))\big) \\ &\le D_B\big(\beta^{\#}_{r,\infty} \circ \varphi^{\#}_1(q'_{A_n}(V^i_{A_n}))\,,\, \varphi^{\#} \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(V^i_{A_n}))\big) + \\ &\quad D_B\big(\psi^{\#} \circ \alpha^{\#}_{n,\infty}(q'_{A_n}(V^i_{A_n}))\,,\, \beta^{\#}_{r,\infty} \circ \psi^{\#}_1(q'_{A_n}(V^i_{A_n}))\big) < \frac{\delta}{3} + \frac{\delta}{3} = \frac{2\delta}{3}. \end{aligned}$$

Choose an integer $m \geq r$ such that $d(B_m) \geq k^2$ and such that

$$[\beta_{r,m}] \cdot [\varphi_1] = [\beta_{r,m}] \cdot [\psi_1] \quad \text{in } KK(A_n, B_m),$$

$$\|\widehat{\beta_{r,m}} \circ \widehat{\varphi_1}(\widehat{h}) - \widehat{\beta_{r,m}} \circ \widehat{\psi_1}(\widehat{h})\| < \frac{2\delta}{3}, \quad h \in \widetilde{H}(A_n, 2q),$$

$$\widehat{\beta_{r,m}} \circ \widehat{\psi_1}(\widehat{h}) > \frac{2\delta}{3}, \quad h \in H(A_n, 10q),$$

$$\widehat{\beta_{r,m}} \circ \widehat{\psi_1}(\widehat{h}) > \frac{8}{p}, \quad h \in H(A_n, l),$$

$$\widehat{\beta_{r,m}} \circ \widehat{\psi_1}(\widehat{h}) > \frac{6}{k}, \quad h \in H(A_n, p) \cup \{p_1, p_2, \ldots, p_N\},$$

$$\widehat{\beta_{r,m}} \circ \widehat{\psi_1}(\widehat{h}) > \frac{1}{q}, \quad h \in H(A_n, k),$$

$$D_B\left(\beta^{\#}_{r,m} \circ \varphi_1^{\#}(q'_{A_n}(V^i_{A_n}))\,,\, \beta^{\#}_{r,m} \circ \psi_1^{\#}(q'_{A_n}(V^i_{A_n}))\right) < \frac{2\delta}{3} < \frac{1}{k^2}, \quad i = 1, 2, \ldots, N.$$

By Theorem 7.6 there exists a unitary $W \in B_m$ such that

$$\|\beta_{r,m} \circ \varphi_1(x) - W\beta_{r,m} \circ \psi_1(x)W^*\| < \frac{\epsilon}{3}, \quad x \in F.$$

If we put $U = \beta_{m,\infty}(W)$ we have that

$$\begin{aligned}
&\|\varphi \circ \alpha_{n,\infty}(x) - Ad(U) \circ \psi \circ \alpha_{n,\infty}(x)\| \\
&\leq \|\varphi \circ \alpha_{n,\infty}(x) - \beta_{r,\infty} \circ \varphi_1(x)\| + \|\beta_{r,\infty} \circ \varphi_1(x) - U\beta_{r,\infty} \circ \psi_1(x)U^*\| \; + \\
&\|\beta_{r,\infty} \circ \psi_1(x) - \psi \circ \alpha_{n,\infty}(x)\| \\
&< \frac{\delta}{3} + \frac{\epsilon}{3} + \frac{\delta}{3} < \epsilon, \quad x \in F.
\end{aligned}$$

□

Theorem 10.13. *Let A and B be simple unital inductive limits of sequences of finite direct sum of building blocks. Assume that $K_0(A)$ is not cyclic. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an isomorphism of groups with order units, let $\varphi_1 : K_1(A) \to K_1(B)$ be an isomorphism of groups, and let $\varphi_T : T(B) \to T(A)$ be an affine homeomorphism such that*

$$r_B(\omega)(\varphi_0(x)) = r_A(\varphi_T(\omega)(x)), \quad x \in K_0(A),\ \omega \in T(B).$$

*There exists a *-isomorphism $\varphi : A \to B$ such that $\varphi_* = \varphi_0$ on $K_0(A)$, such that $\varphi_* = \varphi_1$ on $K_1(A)$, and such that $\varphi_T = \varphi^*$ on $T(B)$.*

Proof. By Theorem 9.8 we may assume that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

of finite direct sums of building blocks with unital and injective connecting maps. Similarly we may assume that B is the inductive limit of a sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \cdots$$

of finite direct sums of building blocks with unital and injective connecting maps. By Lemma 10.1 $d(A_n) \to \infty$ and $d(B_n) \to \infty$ as $n \to \infty$.

By [20, Theorem 7.3] there exists a KK-equivalence $\kappa \in KK(A, B)$ such that $\kappa_* = \varphi_0$ on $K_0(A)$ and $\kappa_* = \varphi_1$ on $K_1(A)$. From Proposition 5.2 it follows that there exists a group isomorphism $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \arrow[r, "\lambda_A"] \arrow[d, "\widetilde{\varphi}"'] & U(A)/\overline{DU(A)} \arrow[r, "\pi_A"] \arrow[d, "\Phi"'] & K_1(A) \arrow[r] \arrow[d, "\kappa_*"] & 0 \\ 0 \arrow[r] & \operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))} \arrow[r, "\lambda_B"'] & U(B)/\overline{DU(B)} \arrow[r, "\pi_B"'] & K_1(B) \arrow[r] & 0 \end{tikzcd}$$

commutes.

By Theorem 10.9 there exists a unital *-homomorphism $\lambda : A \to B$ such that $\lambda^* = \varphi_T$ on $T(B)$, such that $\lambda^\# = \Phi$ on $U(A)/\overline{DU(A)}$, and such that $[\lambda]\cdot[\mu] = \kappa\cdot[\mu]$ in $KK(D,B)$ whenever D is a finite direct sum of building blocks and $\mu : D \to A$ is a *-homomorphism.

Similarly there exists a unital *-homomorphism $\psi : A \to B$ such that $\psi^* = {\varphi_T}^{-1}$ on $T(A)$, such that $\lambda^\# = \Phi^{-1}$ on $U(B)/\overline{DU(B)}$, and such that $[\psi]\cdot[\nu] = \kappa^{-1}\cdot[\nu]$ in $KK(C,A)$ whenever C is a finite direct sum of building blocks and $\nu : C \to B$ is a *-homomorphism.

Let $\mu : D \to A$ be a *-homomorphism from a finite direct sum of building blocks D. As $\lambda \circ \mu$ is a *-homomorphism from a finite direct sum of building blocks into B we have that

$$[\psi\circ\lambda]\cdot[\mu] = [\psi]\cdot[\lambda\circ\mu] = \kappa^{-1}\cdot[\lambda\circ\mu] = \kappa^{-1}\cdot\kappa\cdot[\mu] = [\mu]$$

in $KK(D,A)$. Hence by Theorem 10.12 the *-homomorphisms $\psi\circ\lambda$ and id_A are approximately unitarily equivalent. Similarly $\lambda\circ\psi$ and id_B are approximately unitarily equivalent. Thus there are sequences of unitaries $\{u_n\}$ and $\{v_n\}$, in A and B respectively, such that if we put $\lambda_n = \mathrm{Ad}(v_n)\circ\lambda$ and $\psi_n = \mathrm{Ad}(u_n)\circ\psi$, the diagram

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "\lambda_1"] & A \arrow[r, "id_A"] \arrow[d, "\lambda_2"] & A \arrow[r, "id_A"] \arrow[d, "\lambda_3"] & \cdots \\ B \arrow[r, "id_B"'] \arrow[ur, "\psi_1"'] & B \arrow[r, "id_B"'] \arrow[ur, "\psi_2"'] & B \arrow[r, "id_B"'] \arrow[ur, "\psi_3"'] & \cdots \end{tikzcd}$$

becomes an approximate intertwining. Hence by e.g [23, Theorem 3] there is a *-isomorphism $\varphi : A \to B$ such that

$$\varphi(x) = \lim_{n\to\infty} v_n\lambda(x){v_n}^*, \quad x \in A.$$

It follows that $\varphi^* = \lambda^* = \varphi_T$ on $T(B)$, that $\varphi_* = \lambda_* = \varphi_0$ on $K_0(A)$, and that $\varphi_* = \lambda_* = \varphi_1$ on $K_1(A)$. $\square$

11. A partial classification for K_0 cyclic

Lemma 11.1. *Let A be a simple inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots$$

of finite direct sums of building blocks with unital and injective connecting maps. Then $s(A_n) \to \infty$.

Proof. Let N be a positive integer and let $a_1, a_2, \ldots, a_N$ be mutually orthogonal positive non-zero elements in A_1. As A is simple and the connecting maps are injective there exists a positive integer L such that

$$\widehat{\alpha_{1,l}}(\widehat{a_i}) > 0, \quad i = 1, 2, \ldots, N,$$

for $l \geq L$. Hence by composing the projection from A_l onto a direct summand with any exceptional representation we obtain N positive mutually orthogonal non-zero elements. Thus $s(A_l) \geq N$ for $l \geq L$. □

Lemma 11.2. *Let A be the inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots$$

*of building blocks with unital and injective connecting maps. Assume that A is simple. Let n be a positive integer, let $G \subseteq A_n$ be a finite subset, and let $\epsilon > 0$. There exist a finite subset $E \subseteq \mathit{AffT}(A_n)$, a $\delta > 0$, and an integer $h > n$ such that, if $r \geq h$ is an integer, if $B = A(m, e_1, e_2, \ldots, e_M)$ is a building block, if $\gamma : \mathbb{T} \to]-m\delta, m\delta[$ is a continuous function such that γ equals 0 at all the exceptional points of B, and if $\varphi : A_n \to B$, $\psi : A_r \to B$ are unital *-homomorphisms such that*

$$\|\widehat{\varphi}(f) - \widehat{\psi} \circ \widehat{\alpha_{n,r}}(f)\| < \delta, \quad f \in E,$$
$$[\varphi] = [\psi] \cdot [\alpha_{n,r}] \quad \text{in } KK(A_n, B),$$
$$\mathit{Det}(\varphi(V_{A_n})(z)) = \mathit{Det}(\psi \circ \alpha_{n,r}(V_{A_n})(z)) \exp(2\pi i \gamma(z)), \quad z \in \mathbb{T},$$
$$\mathit{Det}(\Lambda_j \circ \varphi(V_{A_n})) = \mathit{Det}(\Lambda_j \circ \psi \circ \alpha_{n,r}(V_{A_n})), \quad j = 1, 2, \ldots, M,$$

then there exists a unitary $W \in B$ such that

$$\|\psi \circ \alpha_{n,r}(x) - W\varphi(x)W^*\| < \epsilon, \quad x \in G.$$

Proof. Choose by Theorem 7.5 a positive integer l with respect to $G \subseteq A_n$ and ϵ. As A is simple and the connecting maps are injective there exists an integer $p \geq l$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{8}{p}, \quad h \in H(A_n, l).$$

Next, choose an integer $k \geq p$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{6}{k}, \quad h \in H(A_n, p).$$

Choose an integer $q \geq k$ such that

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \frac{1}{q}, \quad h \in H(A_n, k).$$

Finally, choose $\delta > 0$ such that $\delta < \frac{1}{k}$ and

$$\widehat{\alpha_{n,\infty}}(\widehat{h}) > \delta, \quad h \in H(A_n, 10q).$$

Set $E = \widetilde{H}(A_n, 2q)$. Choose $h > n$ such that

$$\widehat{\alpha_{n,h}}(\widehat{h}) > \frac{8}{p}, \quad h \in H(A_n, l),$$
$$\widehat{\alpha_{n,h}}(\widehat{h}) > \frac{6}{k}, \quad h \in H(A_n, p),$$
$$\widehat{\alpha_{n,h}}(\widehat{h}) > \frac{1}{q}, \quad h \in H(A_n, k),$$
$$\widehat{\alpha_{n,h}}(\widehat{h}) > \delta, \quad h \in H(A_n, 10q).$$

The conclusion follows from Theorem 7.5. □

The next lemma is formulated rather generally so that it can be used in the proofs of both Theorem 11.4 and Theorem 11.5.

Lemma 11.3. *Let A be the inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \dots$$

of building blocks with unital and injective connecting maps. Let B be the inductive limit of a sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \dots$$

*of building blocks with unital connecting maps and such that $s(B_k) \to \infty$. Let $\varphi_T : T(B) \to T(A)$ be an affine continuous map and let $\kappa \in KK(A, B)$ be an element such that $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit. Let C be a building block, let n and r be positive integers, let $\epsilon_1 > 0$ and $\epsilon_2 > 0$, let $F_1 \subseteq \mathit{Aff}T(C)$ and $F_2 \subseteq \mathit{Aff}T(A_n)$ be finite sets, and let $\mu : C \to B_r$ and $\lambda : C \to A_n$ be unital *-homomorphisms such that $\lambda(V_C)$ is a unitary of infinite order in $K_1(A_n)$ and such that*

$$\|\varphi_{T*} \circ \widehat{\alpha_{n,\infty}} \circ \widehat{\lambda}(f) - \widehat{\beta_{r,\infty}} \circ \widehat{\mu}(f)\| < \frac{\epsilon_1}{3}, \quad f \in F_1,$$
$$\kappa \cdot [\alpha_{n,\infty}] \cdot [\lambda] = [\beta_{r,\infty}] \cdot [\mu] \quad \text{in } KK(C, B).$$

*There exists an integer $p > r$ such that if $k \geq p$ and if $B_k = A(m, e_1, e_2, \dots, e_M)$ then there exists a unital *-homomorphism $\varphi : A_n \to B_k$ and a continuous function $\gamma : \mathbb{T} \to]-\epsilon_1 m, \epsilon_1 m[$ that equals 0 at all the exceptional points of B_k such that*

$$\|\widehat{\varphi} \circ \widehat{\lambda}(f) - \widehat{\beta_{r,k}} \circ \widehat{\mu}(f)\| < \epsilon_1, \quad f \in F_1,$$
$$\|\varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f) - \widehat{\beta_{k,\infty}} \circ \widehat{\varphi}(f)\| < \epsilon_2, \quad f \in F_2,$$
$$\kappa \cdot [\alpha_{n,\infty}] = [\beta_{k,\infty}] \cdot [\varphi] \quad \text{in } KK(A_n, B),$$
$$[\varphi] \cdot [\lambda] = [\beta_{r,k}] \cdot [\mu] \quad \text{in } KK(C, B_k),$$
$$\mathit{Det}(\varphi \circ \lambda(V_C)(z)) = \mathit{Det}(\beta_{r,k} \circ \mu(V_C)(z)) \exp(2\pi i \gamma(z)), \quad z \in \mathbb{T},$$
$$\mathit{Det}(\Lambda_j \circ \varphi \circ \lambda(V_C)) = \mathit{Det}(\Lambda_j \circ \beta_{r,k} \circ \mu(V_C)), \quad j = 1, 2, \dots, M.$$

Proof. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. By Lemma 10.4 there exists a positive integer l, a positive linear order unit preserving map $M : \mathrm{Aff}\,T(A_n) \to \mathrm{Aff}\,T(B_l)$ such that

$$\|\varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f) - \widehat{\beta_{l,\infty}} \circ M(f)\| < \frac{\epsilon}{3}, \quad f \in \widehat{\lambda}(F_1) \cup F_2,$$

and an element $\omega \in KK(A_n, B_l)$ such that $\omega_* : K_0(A_n) \to K_0(B_l)$ preserves the order unit and such that

$$[\beta_{l,\infty}] \cdot \omega = \kappa \cdot [\alpha_{n,\infty}] \quad \text{in } KK(A_n, B).$$

Note that

$$\|\widehat{\beta_{l,\infty}} \circ M \circ \widehat{\lambda}(f) - \widehat{\beta_{r,\infty}} \circ \widehat{\mu}(f)\| < \frac{2\epsilon_1}{3}, \quad f \in F_1,$$
$$[\beta_{l,\infty}] \cdot \omega \cdot [\lambda] = [\beta_{r,\infty}] \cdot [\mu] \quad \text{in } KK(C, B).$$

Therefore there exists an integer $p > l, r$ such that

$$\|\widehat{\beta_{l,p}} \circ M \circ \widehat{\lambda}(f) - \widehat{\beta_{r,p}} \circ \widehat{\mu}(f)\| < \frac{2\epsilon_1}{3}, \quad f \in F_1,$$
$$[\beta_{l,p}] \cdot \omega \cdot [\lambda] = [\beta_{r,p}] \cdot [\mu] \quad \text{in } KK(C, B_p),$$

and such that $s(B_p) \geq K$ where K is the integer from Theorem 8.5, chosen with respect to $\widehat{\lambda}(F_1) \cup F_2 \subseteq \operatorname{Aff} T(A_n)$, $\frac{\epsilon}{3}$, and the unitary $\lambda(V_C)$. Let $k \geq p$ and let $B_k = A(m, e_1, e_2, \ldots, e_M)$. There exists a unital *-homomorphism $\varphi : A_n \to B_k$ and a continuous function $\gamma : \mathbb{T} \to]-\epsilon_1 m, \epsilon_1 m[$ that equals 0 at all the exceptional points of B_k such that

$$\|\widehat{\varphi}(f) - \widehat{\beta_{l,k}} \circ M(f)\| < \frac{\epsilon}{3}, \quad f \in \widehat{\lambda}(F_1) \cup F_2,$$
$$[\varphi] = [\beta_{l,k}] \cdot \omega \quad \text{in } KK(A_n, B_k),$$
$$Det(\varphi(\lambda(V_C))(z)) = Det(\beta_{r,k} \circ \mu(V_C)(z)) \exp(2\pi i \gamma(z)), \quad z \in \mathbb{T},$$
$$Det(\Lambda_j \circ \varphi(\lambda(V_C))) = Det(\Lambda_j \circ \beta_{r,k} \circ \mu(V_C)), \quad j = 1, 2, \ldots, M.$$

It follows that

$$\|\widehat{\varphi} \circ \widehat{\lambda}(f) - \widehat{\beta_{r,k}} \circ \widehat{\mu}(f)\| < \epsilon_1, \quad f \in F_1,$$
$$\|\varphi_{T*} \circ \widehat{\alpha_{n,\infty}}(f) - \widehat{\beta_{k,\infty}} \circ \widehat{\varphi}(f)\| < \epsilon_2, \quad f \in F_2,$$
$$\kappa \cdot [\alpha_{n,\infty}] = [\beta_{k,\infty}] \cdot [\varphi] \quad \text{in } KK(A_n, B),$$
$$[\varphi] \cdot [\lambda] = [\beta_{r,k}] \cdot [\mu] \quad \text{in } KK(C, B_k).$$

□

Theorem 11.4. *Let A be the inductive limit of a sequence*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots$$

of building blocks with unital and injective connecting maps. Assume that A is simple and that $K_1(A)$ contains an element of infinite order. Let B be an inductive limit of a sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \ldots$$

*of building blocks with unital connecting maps and such that $s(B_k) \to \infty$. Assume that there exist group homomorphisms $\varphi_0 : K_0(A) \to K_0(B)$ and $\varphi_1 : K_1(A) \to K_1(B)$ such that $\varphi_0([1]) = [1]$ in $K_0(B)$, and an affine continuous map $\varphi_T : T(B) \to T(A)$. Then there exists a unital *-homomorphism $\varphi : A \to B$ such that $\varphi^* = \varphi_T$ on $T(B)$, $\varphi = \varphi_0$ on $K_0(A)$, and $\varphi_* = \varphi_1$ on $K_1(A)$.*

Proof. By passing to a subsequence we may assume that each unitary $\alpha_s(V_{A_s})$ has infinite order in $K_1(A_{s+1})$.

Choose finite subsets $G_s \subseteq A_s$ such that G_s generates A_s as a C^*-algebra and such that $\alpha_s(G_s) \subseteq G_{s+1}$. Choose by Lemma 11.2 a finite set $E_s \subseteq \mathrm{Aff}\, T(A_s)$, a number $\delta_s > 0$, and an integer h_s with respect to G_s and 2^{-s}.

Choose finite subsets $F_s \subseteq \mathrm{Aff}\, T(A_s)$ such that $E_s \subseteq F_s$, such that $\widehat{\alpha_s}(F_s) \subseteq F_{s+1}$, and such that $\cup_{s=1}^{\infty} \widehat{\alpha_{s,\infty}}(F_s)$ is dense in $\mathrm{Aff}\, T(A)$. By the universal coefficient theorem, [20, Theorem 1.17], there exists an element $\kappa \in KK(A, B)$ such that $\kappa_* = \varphi_0$ on $K_0(A)$ and $\kappa_* = \varphi_1$ on $K_1(A)$.

For each positive integer s we will construct positive integers n_s and m_s such that $n_1 < n_2 < \cdots < n_s$ and $m_1 < m_2 < \cdots < m_s$, and a unital *-homomorphism $\varphi_s : A_{n_s} \to B_{m_s}$ such that

(i) $\|\beta_{m_{s-1},m_s} \circ \varphi_{s-1}(x) - \varphi_s \circ \alpha_{n_{s-1},n_s}(x)\| < 2^{-n_{s-1}}, \quad x \in G_{n_{s-1}}, \ s \geq 2,$

(ii) $[\beta_{m_s,\infty}] \cdot [\varphi_s] = \kappa \cdot [\alpha_{n_s,\infty}] \quad \text{in } KK(A_{n_s}, B),$

(iii) $\|\widehat{\beta_{m_s,\infty}} \circ \widehat{\varphi_s}(f) - \varphi_{T*} \circ \widehat{\alpha_{n_s,\infty}}(f)\| < \frac{1}{3}\min(\delta_{n_s}, 2^{-s}), \quad f \in F_{n_s}.$

This is done by induction. n_s, m_s and φ_s are constructed in step s. The case $s = 1$ is settled by combining Lemma 10.4 and Theorem 8.5.

Assume that n_{s-1}, m_{s-1} and φ_{s-1} have been constructed. Set $n_s = h_{n_{s-1}}$. By Lemma 11.3 we get an integer $m_s > m_{s-1}$ and a unital *-homomorphism $\varphi_s : A_{n_s} \to B_{m_s}$ such that, if we let $B_{m_s} = A(m, e_1, e_2, \ldots, e_M)$,

$$\|\widehat{\varphi_s} \circ \widehat{\alpha_{n_{s-1},n_s}}(f) - \widehat{\beta_{m_{s-1},m_s}} \circ \widehat{\varphi_{s-1}}(f)\| < \min(\delta_{n_{s-1}}, 2^{-(s-1)}), \quad f \in F_{n_{s-1}},$$

$$\|\varphi_{T*} \circ \widehat{\alpha_{n_s,\infty}}(f) - \widehat{\beta_{m_s,\infty}} \circ \widehat{\varphi_s}(f)\| < \frac{1}{3}\min(\delta_{n_s}, 2^{-s}), \quad f \in F_{n_s},$$

$$\kappa \cdot [\alpha_{n_s,\infty}] = [\beta_{m_s,\infty}] \cdot [\varphi_s] \quad \text{in } KK(A_{n_s}, B),$$

$$[\varphi_s] \cdot [\alpha_{n_{s-1},n_s}] = [\beta_{m_{s-1},m_s}] \cdot [\varphi_{s-1}] \quad \text{in } KK(A_{n_{s-1}}, B_{m_s}),$$

together with a continuous function $\gamma : \mathbb{T} \to]-m\delta_{n_{s-1}}, m\delta_{n_{s-1}}[$ that equals 0 at every point of B_{m_s} such that for $z \in \mathbb{T}$,

$$Det(\varphi_s \circ \alpha_{n_{s-1},n_s}(V_{A_{n_{s-1}}})(z)) = Det(\beta_{m_{s-1},m_s} \circ \varphi_{s-1}(V_{A_{n_{s-1}}})(z)) \exp(2\pi i \gamma(z)),$$

and for $j = 1, 2, \ldots, M$,

$$Det(\Lambda_j \circ \varphi_s \circ \alpha_{n_{s-1},n_s}(V_{A_{n_{s-1}}})) = Det(\Lambda_j \circ \beta_{m_{s-1},m_s} \circ \varphi_{s-1}(V_{A_{n_{s-1}}})).$$

By Lemma 11.2 we get a unitary $W \in B_{m_s}$ such that

$$\|\beta_{m_{s-1},m_s} \circ \varphi_{s-1}(x) - W\varphi_s \circ \alpha_{n_{s-1},n_s}(x)W^*\| < 2^{-n_{s-1}}, \quad x \in G_{n_{s-1}}.$$

Hence by substituting $Ad(W) \circ \varphi_s$ for φ_s we complete the induction step.

By Elliott's approximate intertwining argument, see e.g [23, Lemma 1], there exists a *-homomorphism $\varphi : A \to B$ such that

$$\varphi(\alpha_{n,\infty}(x)) = \lim_{s\to\infty} \beta_{m_s,\infty} \circ \varphi_s \circ \alpha_{n,n_s}(x), \quad x \in A_n.$$

Clearly, φ is unital, $\varphi_* = \varphi_0$ on $K_0(A)$, and $\varphi_* = \varphi_1$ on $K_1(A)$. Finally, $\varphi^* = \varphi_T$ follows as in the proof of 10.9. □

Theorem 11.5. *Let A and B be inductive limits of sequences*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots$$

and

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \ldots$$

*of building blocks with unital and injective connecting maps. Assume that A and B are simple and that $K_1(A)$ contains an element of infinite order. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an isomorphism such that $\varphi_0([1]) = [1]$, let $\varphi_1 : K_1(A) \to K_1(B)$ be an isomorphism and let $\varphi_T : T(B) \to T(A)$ be an affine homeomorphism. Then there exists a *-isomorphism $\varphi : A \to B$ such that $\varphi_* = \varphi_0$ on $K_0(A)$, $\varphi_* = \varphi_1$ on $K_1(A)$, and $\varphi^* = \varphi_T$ on $T(B)$.*

Proof. By Lemma 11.1 we see that $s(A_k) \to \infty$ and $s(B_k) \to \infty$. And by passing to subsequences we may assume that for every positive integer n, $\alpha_n(V_{A_n})$ and $\beta_n(V_{B_n})$ have infinite order in $K_1(A_{n+1})$ and $K_1(B_{n+1})$, respectively. By [20, Theorem 7.3] we see that there exists a KK-equivalence $\kappa \in KK(A, B)$ such that $\kappa_* = \varphi_0$ on $K_0(A)$ and $\kappa_* = \varphi_1$ on $K_1(A)$.

Choose finite sets $G_s \subseteq A_s$ and $G'_s \subseteq B_s$ such that G_s and G'_s generate A_s and B_s, respectively, as C^*-algebras, and such that $\alpha_s(G_s) \subseteq G_{s+1}$, $\beta_s(G'_s) \subseteq G'_{s+1}$. Choose finite sets $F_s \subseteq \operatorname{Aff} T(A_s)$, $F'_s \subseteq \operatorname{Aff} T(B_s)$, such that $\widehat{\alpha_s}(F_s) \subseteq F_{s+1}$, $\widehat{\beta_s}(F'_s) \subseteq F'_{s+1}$, and such that $\cup_{s=1}^{\infty} \widehat{\alpha_{s,\infty}}(F_s)$ and $\cup_{s=1}^{\infty} \widehat{\beta_{s,\infty}}(F'_s)$ are dense in $\operatorname{Aff} T(A)$ and $\operatorname{Aff} T(B)$, respectively.

We will inductively construct two strictly increasing sequences of positive integers $\{n_s\}$ and $\{m_s\}$ and unital *-homomorphisms $\varphi_s : A_{n_s} \to B_{m_s}$, $\psi_s : B_{m_s} \to A_{n_{s+1}}$, such that, if we set

$$H_s = G_{n_s} \cup \bigcup_{t=1}^{s-1} \alpha_{n_{t+1},n_s} \circ \psi_t(G'_{m_t}),$$
$$H'_s = G'_{m_s} \cup \bigcup_{t=1}^{s} \beta_{m_t,m_s} \circ \varphi_t(G_{n_t}),$$

if $E_s \subseteq \operatorname{Aff} T(A_{n_s})$ and δ_s are chosen by Lemma 11.2 with respect to H_s and 2^{-s}, and if $E'_s \subseteq \operatorname{Aff} T(B_{m_s})$ and δ'_s are chosen by Lemma 11.2 with respect to H'_s and 2^{-s}, then

(i) $\|\varphi_s \circ \psi_{s-1}(x) - \beta_{m_{s-1},m_s}(x)\| < 2^{-s}, \quad x \in H'_{s-1},\ s \geq 2,$
(ii) $\|\psi_s \circ \varphi_s(x) - \alpha_{n_s,n_{s+1}}(x)\| < 2^{-s}, \quad x \in H_s,$
(iii) $\|\widehat{\beta_{m_s,\infty}} \circ \widehat{\varphi_s}(f) - \varphi_{T*} \circ \widehat{\alpha_{n_s,\infty}}(f)\| < \frac{1}{3}\min(\delta_s, 2^{-s}), \quad f \in E_s \cup F_{n_s},$
(iv) $\|\widehat{\alpha_{n_{s+1},\infty}} \circ \widehat{\psi_s}(f) - \varphi_{T*}^{-1} \circ \widehat{\beta_{m_s,\infty}}(f)\| < \frac{1}{3}\min(\delta'_s, 2^{-s}), \quad f \in E'_s \cup F'_{n_s},$
(v) $[\beta_{m_s,\infty}] \cdot [\varphi_s] = \kappa \cdot [\alpha_{n_s,\infty}]$ in $KK(A_{n_s}, B)$,
(vi) $[\alpha_{n_{s+1},\infty}] \cdot [\psi_s] = \kappa^{-1} \cdot [\beta_{m_s,\infty}]$ in $KK(B_{m_s}, A)$.

n_1, m_1 and φ_1 are constructed by combining Lemma 10.4 with Theorem 8.5.

Assume that n_s, m_s, and φ_s have been constructed. We will construct n_{s+1}, m_{s+1}, ψ_s, and φ_{s+1}.

Let h_s be the integer from Lemma 11.2 chosen along with E_s and δ_s. As

$$\|\varphi_{T*}^{-1} \circ \widehat{\beta_{m_s,\infty}} \circ \widehat{\varphi_s}(f) - \widehat{\alpha_{n_s,\infty}}(f)\| < \frac{\delta_s}{3}, \quad f \in E_s,$$
$$\kappa^{-1} \cdot [\beta_{m_s,\infty}] \cdot [\varphi_s] = [\alpha_{n_s,\infty}],$$

and since $\varphi_s(V_{A_{n_s}})$ has infinite order in $K_1(B_{m_s})$, we get by Lemma 11.3 an integer $n_{s+1} > h_s$ such that if $A_{n_{s+1}} = A(m, e_1, e_2, \ldots, e_M)$, then there exists a unital *-homomorphism $\psi_s : B_{m_s} \to A_{n_{s+1}}$ and a continuous function $\gamma : \mathbb{T} \to]-m\delta_s, m\delta_s[$

that equals 0 at all the exceptional points of $A_{n_{s+1}}$, such that

$$\|\widehat{\psi_s} \circ \widehat{\varphi_s}(f) - \widehat{\alpha_{n_s,n_{s+1}}}(f)\| < \delta_s, \quad f \in E_s,$$

$$\|\varphi_{T*}^{-1} \circ \widehat{\beta_{m_s,\infty}}(f) - \widehat{\alpha_{n_{s+1},\infty}} \circ \widehat{\psi_s}(f)\| < \frac{1}{3}\min(\delta_s', 2^{-s}), \quad f \in E_s' \cup F_{n_s}',$$

$$\kappa^{-1} \cdot [\beta_{m_s,\infty}] = [\alpha_{n_{s+1},\infty}] \cdot [\psi_s] \quad \text{in } KK(B_{m_s}, A),$$

$$[\psi_s] \cdot [\varphi_s] = [\alpha_{n_s,n_{s+1}}] \quad \text{in } KK(A_{n_s}, B),$$

$$Det(\psi_s \circ \varphi_s(V_{A_{n_s}})(z)) = Det(\alpha_{n_s,n_{s+1}}(V_{A_{n_s}})(z)) \exp(2\pi i \gamma(z)), \quad z \in \mathbb{T},$$

$$Det(\Lambda_j \circ \psi_s \circ \varphi_s(V_{A_{n_s}})) = Det(\Lambda_j \circ \alpha_{n_s,n_{s+1}}(V_{A_{n_s}})), \quad j = 1, 2, \ldots, M.$$

By Lemma 11.2 we get a unitary $W \in A_{n_{s+1}}$ such that

$$\|\alpha_{n_s,n_{s+1}}(x) - W\psi_s \circ \varphi_s(x)W^*\| < 2^{-s}, \quad x \in H_s.$$

If we substitute $Ad(W) \circ \psi_s$ for ψ_s we see that (ii), (iv), and (vi) are satisfied.

The construction of m_{s+1} and $\varphi_{s+1} : A_{n_{s+1}} \to B_{m_{s+1}}$ such that (i), (iii), and (v) are satisfied, is similar.

The proof can now be completed with Elliott's approximate intertwining argument. By e.g [23, Theorem 3] we get a *-isomorphism $\varphi : A \to B$ such that

$$\varphi(\alpha_{n,\infty}(x)) = \lim_{s \to \infty} \beta_{m_s,\infty} \circ \varphi_s \circ \alpha_{n,n_s}(x), \quad x \in A_n.$$

As in the proof of Theorem 11.4 we see that $\varphi_* = \varphi_0$ on $K_0(A)$, $\varphi_* = \varphi_1$ on $K_1(A)$, and $\varphi^* = \varphi_T$ on $T(B)$. □

Corollary 11.6. *Let A and B be simple unital inductive limits of sequences of finite direct sums of building blocks. Let d be a positive integer. Assume that $K_0(A) \cong K_0(B) \cong (\mathbb{Z}, d)$ as groups with order unit, that $K_1(A)$ contains an element of infinite order, and that B is infinite dimensional. Let $\varphi_1 : K_1(A) \to K_1(B)$ be a group homomorphism and let $\varphi_T : T(B) \to T(A)$ be an affine homeomorphism. Then there exists a unital *-homomorphism $\varphi : A \to B$ such that $\varphi_* = \varphi_1$ on $K_1(A)$ and $\varphi^* = \varphi_T$ on $T(B)$. Furthermore, if φ_1 is an isomorphism and φ_T is an affine homeomorphism then φ may be chosen to be an isomorphism.*

Proof. By Theorem 9.8 we may assume that A and B are inductive limits of sequences of finite direct sums of building blocks with unital and injective connecting maps. As $K_0(A)$ and $K_0(B)$ are cyclic groups, it follows that A and B are inductive limits of sequences of building blocks (rather than finite directs sums of such algebras). By Lemma 11.1 we see that $s(B_k) \to \infty$. The conclusions follow from Theorem 11.4 and Theorem 11.5. □

12. Range of the invariant

The purpose of this section is to determine the range of the Elliott invariant, i.e to answer the question which quadruples $(K_0(A), K_1(A), T(A), r_A)$ occur as the Elliott invariant for simple unital infinite dimensional C^*-algebras that are inductive limits of sequences of finite direct sums of building blocks. Villadsen [26] has answered this question in the case where A is an inductive limit of a sequence of finite direct sums of circle algebras. Using this result Thomsen has been able to determine the range of the Elliott invariant for those C^*-algebras that are inductive limits of finite direct sums of building blocks of the form $A(n, d, d, \ldots, d)$, see below.

We start out by examining the restrictions on $(K_0(A), K_1(A), T(A), r_A)$. Let A be a simple unital infinite dimensional inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \dots$$

of finite direct sums of building blocks. We may by Theorem 9.8 assume that each α_n is unital and injective. By Corollary 3.6 each $K_0(A_k)$ is isomorphic (as an ordered group with order unit) to the K_0-group of a finite dimensional C^*-algebra. Thus $K_0(A)$ must be a countable dimension group. This group has to be simple as A is simple.

If $K_0(A) \cong \mathbb{Z}$ then by passing to a subsequence, if necessary, we may assume that A is the inductive limit of a sequence of building blocks, rather than finite direct sums of such algebras. By Lemma 3.8 it follows that $K_1(A)$ an inductive limit of groups of the form $\mathbb{Z} \oplus G$, where G is any finite abelian group.

If $K_0(A)$ is not cyclic our only immediate conclusion is that $K_1(A)$ is a countable abelian group.

$T(A)$ must be a metrisable Choquet simplex. If B is a building block then obviously $r_B : T(B) \to SK_0(B)$ maps extreme points to extreme points. By [26, Corollary 1.6] and [26, Corollary 1.7] the same must be the case for r_A. Finally, r_A is surjective by [3, Theorem 3.3] and [11] (or more elementary, because each $r_{A_k} : T(A_k) \to SK_0(A_k)$ is surjective). As we will see in Theorem 12.1 and Theorem 12.4, these are the only restrictions.

As mentioned above, Thomsen has calculated the range of the invariant for a subclass of the class we are considering. By [21, Theorem 9.2] we have the following:

Theorem 12.1. *Let G be a countable non-cyclic abelian group with order unit, H a countable abelian group, Δ a compact metrisable Choquet simplex, and $\lambda : \Delta \to SG$ an affine continuous extreme point preserving surjection. Then there exists a simple unital inductive limit of a sequence of finite direct sums of building blocks A together with an isomorphism $\varphi_0 : K_0(A) \to G$ of ordered groups with order unit, an isomorphism $\varphi_1 : K_1(A) \to H$, and an affine homeomorphism $\varphi_T : \Delta \to T(A)$ such that*

$$r_A(\varphi_T(\omega))(x) = \lambda(\omega)(\varphi_0(x)), \quad \omega \in \Delta,\ x \in K_0(A).$$

A can be realised as an inductive limit of circle algebras and interval building blocks of the form $A(n, d, d)$.

A different proof of this theorem could be based on Theorem 8.2 and [26, Theorem 4.2].

Combining this theorem with Theorem 10.13 we get the following:

Theorem 12.2. *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks such that $K_0(A)$ is non-cyclic. Then A is the inductive limit of a sequence of finite direct sums of circle algebras and interval building blocks of the form $A(n, d, d)$.*

We are left with the case of cyclic K_0-group. Note that the equation

$$r_A(\varphi_T(\omega))(x) = \lambda(\omega)(\varphi_0(x)), \quad \omega \in \Delta,\ x \in K_0(A)$$

is trivial when A is a unital C^*-algebra such that $K_0(A) \cong \mathbb{Z}$.

Lemma 12.3. *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks. Then $(K_0(A), K_0(A)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, 1)$ if and only if A is unital projectionless.*

Proof. This follows easily from Theorem 9.8 and Lemma 3.7. □

Theorem 12.4. *Let d be a positive integer, let Δ be a metrisable Choquet simplex, and let H be the inductive limit of a sequence of groups of the form $\mathbb{Z}\oplus G$, where G is any finite abelian group. There exists a simple unital infinite dimensional inductive limit A of a sequence of building blocks, with $(K_0(A), K_0(A)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$, $K_1(A) \cong H$, and such that $T(A)$ is affinely homeomorphic to Δ.*

Proof. By [24, Lemma 3.8] $\operatorname{Aff}\Delta$ is isomorphic to an inductive limit in the category of order unit spaces of a sequence

$$C_{\mathbb{R}}[0,1] \longrightarrow C_{\mathbb{R}}[0,1] \longrightarrow C_{\mathbb{R}}[0,1] \longrightarrow \dots$$

It is easy to see that this implies that $\operatorname{Aff}\Delta$ is isomorphic to an inductive limit of a sequence of the form

$$C_{\mathbb{R}}(\mathbb{T}) \xrightarrow{\Theta_1} C_{\mathbb{R}}(\mathbb{T}) \xrightarrow{\Theta_2} C_{\mathbb{R}}(\mathbb{T}) \xrightarrow{\Theta_3} \dots$$

Let $H \cong \varinjlim(H_k, h_k)$ in the category of groups, where each H_k is the direct sum of $\mathbb{Z}$ and a finite abelian group. Choose a dense sequence $\{x_k\}_{k=1}^{\infty}$ in $C_{\mathbb{R}}(\mathbb{T})$ and a dense sequence $\{z_k\}_{k=1}^{\infty}$ in $\mathbb{T}$.

For every positive integer k we will construct a building block A_k such that $(K_0(A_k), K_0(A_k)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$ and $K_1(A_k) \cong H_k$, together with a unital and injective *-homomorphism $\alpha_k : A_k \to A_{k+1}$ such that each of the (constant) functions $z \mapsto z_1, z \mapsto z_2, \dots, z \mapsto z_k$ are among the eigenvalue functions for α_k, such that $\alpha_{k*} = h_k$ on $K_1(A_k)$ (under the identification $K_1(A_k) \cong H_k$) and such that

$$\|\widehat{\alpha_k}(f) - \Theta_k(f)\| < 2^{-k}, \quad f \in F_k,$$

under the identification $\operatorname{Aff} T(A_k) \cong C_{\mathbb{R}}(\mathbb{T})$, where

$$F_k = \{x_1, x_2, \dots, x_k\} \bigcup_{j=1}^{k-1} \Theta_{j,k}(\{x_1, x_2, \dots, x_k\}) \bigcup_{j=1}^{k-1} \alpha_{j,k}(\{x_1, x_2, \dots, x_k\}).$$

First choose by Lemma 3.8 a building block A_1 such that $K_1(A_1) \cong H_1$ and $(K_0(A_1), K_0(A_1)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$.

Assume that A_k has been constructed. We will construct A_{k+1} and α_k. Choose K by Theorem 8.4 with respect to $F_k \subseteq \operatorname{Aff} T(A_k)$, $\epsilon = 2^{-k}$ and the integer $k+1$. By Lemma 3.8 there exists a building block A_{k+1} such that $s(A_{k+1}) \geq K$, $K_1(A_{k+1}) \cong H_{k+1}$, and $(K_0(A_{k+1}), K_0(A_{k+1})^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$. By the universal coefficient theorem, [20, Theorem 1.17], there exists an element $\kappa \in KK(A_k, A_{k+1})$ such that $\kappa_* : K_0(A_k) \to K_0(A_{k+1})$ preserves the order unit and $\kappa_* = h_k$ on $K_1(A_k)$. By Theorem 8.4 there exists a unital *-homomorphism $\alpha_k : A_k \to A_{k+1}$ such that the identity function on $\mathbb{T}$ and each of the functions $z \mapsto z_1, z \mapsto z_2, \dots, z \mapsto z_k$ are among the eigenvalue functions for α_k and such that

$$\|\widehat{\alpha_k}(f) - \Theta_k(f)\| < 2^{-k}, \quad f \in F_k,$$
$$[\alpha_k] = \kappa \quad \text{in } KK(A_k, A_{k+1}).$$

This completes the construction.

Set $A = \varinjlim(A_k, \alpha_k)$. A is infinite dimensional as the connecting maps are injective. Obviously, $(K_0(A), K_0(A)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d)$ and $K_1(A) \cong H$. By [24, Lemma 3.4] $\operatorname{Aff} T(A) \cong \varinjlim(C_{\mathbb{R}}[0,1], \Theta_k) \cong \operatorname{Aff}\Delta$, and hence $T(A)$ and Δ are affinely homeomorphic.

Let $I \subseteq A$ be a closed two-sided ideal in A, $I \neq \{0\}$. By (the proof of) [5, Lemma 3.1],

$$I = \overline{\bigcup_{n=1}^{\infty} \alpha_{n,\infty}({\alpha_{n,\infty}}^{-1}(I))}.$$

Choose a positive integer n such that ${\alpha_{n,\infty}}^{-1}(I) \neq \{0\}$. Choose $f \in {\alpha_{n,\infty}}^{-1}(I)$ such that $f \neq 0$. Choose $k > n$ such that $f(z_k) \neq 0$. Then $\alpha_{n,l}(f)(z) \neq 0$ for every $z \in \mathbb{T}$ and $l > k$. Hence by Lemma 2.2 ${\alpha_{l,\infty}}^{-1}(I) = A_l$ for every $l > k$. It follows that $I = A$. Hence A is simple. $\square$

Corollary 12.5. *Let Δ be a metrisable Choquet simplex and H an inductive limit of a sequence of groups of the form $\mathbb{Z} \oplus G$, where G is any finite abelian group. Then there exists an infinite dimensional simple unital projectionless C^*-algebra A that is an inductive limit of building blocks such that $T(A) \cong \Delta$ and $K_1(A) \cong H$.*

Proof. Combine Theorem 12.4 and Lemma 12.3. $\square$

13. Conclusion

We can now state our main result:

Theorem 13.1. *Let A and B be a simple unital inductive limit of a finite direct sum of building blocks such that either $K_0(A)$ is not cyclic or $K_1(A)$ contains an element of infinite order. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an isomorphism of groups with order units, let $\varphi_1 : K_1(A) \to K_1(B)$ be an isomorphism of groups, and let $\varphi_T : T(B) \to T(A)$ be an affine homeomorphism such that*

$$r_B(\omega)(\varphi_0(x)) = r_A(\varphi_T(\omega)(x)), \quad x \in K_0(A),\ \omega \in T(B).$$

*There exists a *-isomorphism $\varphi : A \to B$ such that $\varphi_* = \varphi_0$ on $K_0(A)$, such that $\varphi_* = \varphi_1$ on $K_1(A)$, and such that $\varphi^* = \varphi_T$ on $T(B)$.*

Proof. Combine Theorem 10.13 and Corollary 11.6. $\square$

It is natural to ask whether the techniques used to prove the above classification result in the case that the K_0-group is non-cyclic can be generalised to give a proof of a classification result in the general case. In order to show that this is not immediate, let us give an example to show that Theorem 10.9 does not hold for every simple unital C^*-algebra in the class of simple unital inductive limits of finite direct sums of building blocks.

By [18, Theorem 8.7] there exists a simple unital C^*-algebra A with $K_0(A) \cong \mathbb{Z}$, $K_1(A) \cong \mathbb{Z}_2$, and a unique tracial state. This C^*-algebra can be realised as an inductive limit of a sequence of interval building blocks and is hence contained in our class. By Proposition 5.2 the group $U(A)/\overline{DU(A)}$ can be identified with the group $(\mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2$. Let $\Phi : U(A)/\overline{DU(A)} \to U(A)/\overline{DU(A)}$ be the group homomorphism given by

$$\Phi(x,k) = (x + \frac{k}{2}, k), \quad x \in \mathbb{R}/\mathbb{Z},\ k \in \mathbb{Z}_2.$$

If Theorem 10.9 were true there would exist a unital *-homomorphism $\psi : A \to B$ such that $\psi^{\#} = \Phi$ on $U(A)/\overline{DU(A)}$ and $[\psi] \cdot [\mu] = [id_A] \cdot [\mu]$ in $KK(D,B)$ whenever D is a finite direct sum of building blocks and $\mu : D \to A$ is a unital *-homomorphism. But then [18, Theorem 7.6] would imply that ψ is approximately inner. This contradicts the fact that the action on $U(A)/\overline{DU(A)}$ is non-trivial.

This example takes advantage of the fact that the group $\operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))}$ contains an element of finite order. Let us show that this is not the case for $K_0(A)$ non-cyclic (where A is a simple unital inductive limit of a sequence of finite direct sums of building blocks). As $K_0(A)$ is a simple non-cyclic countable dimension group we see by [1, Proposition 3.1] that the image of the canonical map $K_0(A) \to \operatorname{Aff} SK_0(A)$ is dense. It follows that $\rho_A(K_0(A))$ is dense in some subspace of $\operatorname{Aff} T(A)$. Hence the quotient $\operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))}$ is in fact a vector space.

References

1. B. Blackadar, *Traces on simple AF C^*-algebras*, J. Funct. Anal. **38** (1980), 156–168.
2. ———, *K-theory for operator algebras*, Springer-Verlag, 1986.
3. B. Blackadar and M. Rørdam, *Extending states on preordered semigroups and the existence of quasitraces on C^*-algebras*, J. Algebra **152** (1992), 240–247.
4. B. Bollobás, *Combinatorics*, Cambridge University Press, 1986.
5. O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.
6. S. Eilers, T. A. Loring, and G. K. Pedersen, *Stability of anticommutation relations. An application of noncommutative CW complexes*, J. Reine Angew. Math. **499** (1998), 101–143.
7. G. A. Elliott, *A classification of certain simple C^*-algebras*, Quantum and non-commutative analysis, H. Araki et al. (eds.), Kluwer, Dordrecht (1993), 373–385.
8. ———, *A classification of certain simple C^*-algebras, II*, preprint (1994).
9. G. A. Elliott, G. Gong, X. Jiang, and H. Su, *A classification of simple inductive limits of dimension drop C^*-algebras*, Fields Institute Communications **13** (1997), 125–143.
10. T. Fack, *Finite sums of commutators in C^*-algebras*, Ann. Inst. Fourier, Grenoble **32-1** (1982), 129–137.
11. U. Haagerup, *Quasitraces on exact C^*-algebras are traces*, manuscript (1991).
12. D. Handelman, *K_0 of von Neumann and AF C^*-algebras*, Quart. J. Math. Oxford Ser. **29** (1978), 427–441.
13. X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, preprint (1996).
14. ———, *A classification of splitting interval algebras*, J. Funct. Anal. **151** (1997), 50–76.
15. L. Li, *Classification of simple C^*-algebras: Inductive limits of matrix algebras over 1-dimensional spaces*, preprint.
16. T. A. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs, 1997.
17. G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, 1990.
18. J. Mygind, *Classification of simple inductive limits of interval algebras with dimension drops*, preprint (1998).
19. K. E. Nielsen and K. Thomsen, *Limits of circle algebras*, Expo. Math. **14** (1996), 17–56.
20. J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431–474.
21. K. Thomsen, *Limits of certain subhomogeneous C^*-algebras*, to appear in Mém. Soc. Math. France.
22. ———, *Nonstable K-theory for operator algebras*, K-Theory **4** (1991), 245–267.
23. ———, *On isomorphisms of inductive limit C^*-algebras*, Proc. Amer. Math. Soc. **113** (1991), 947–953.
24. ———, *Inductive limits of interval algebras: The tracial state space*, Amer. J. Math. **116** (1994), 605–620.
25. ———, *Traces, unitary characters and crossed products by $\mathbb{Z}$*, Publ. RIMS, Kyoto **31** (1995), 1011–1029.
26. J. Villadsen, *The range of the Elliott invariant*, J. Reine u. Angew. Math. **462** (1995), 31–55.

Institut for Matematiske Fag, Ny Munkegade, 8000 Aarhus C, Denmark
E-mail address: mygind@imf.au.dk