DISTRIBUTION RESULTS IN AUTOMORPHIC FORMS AND ANALYTIC NUMBER THEORY

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Preface

This dissertation is written as a part of the requirements in the PhD program in The Department of Mathematical Sciences at Aarhus University. It represents the research part of my work during my PhD studies, which has been carried out under the supervision of Morten S. Risager (University of Copenhagen) and Alexei B. Venkov (Aarhus University).

Abstract

In this thesis three different distribution problems are studied. In the first problem we consider the Eisenstein series $E(g, s, \chi)$ on $\mathrm{GL}_2(\mathbf{A})$, where $\mathbf{A}$ is the adele ring of a number field. We prove (quantitatively) that the measure $|E(g, 1/2 + it, \chi)|^2 d\mu$ becomes equidistributed in the limit $t \to \infty$. Here $d\mu$ is the measure derived from the Haar measure on $\mathrm{GL}_2(\mathbf{A})$. This generalizes previous results due to W. Luo and P. Sarnak and S. Koyama.

The second problem concerns angles in hyperbolic lattices. We prove that in a suitable (and natural) setting these angles are equidistributed with an effective error term for the equidistribution rate. We use this to generalize a result due to F. Boca.

The last problem studied in the thesis is about the pair correlation for the fractional parts of $n^2\alpha$. It has been proved by Z. Rudnick and P. Sarnak that the pair correlation is Poissonian for almost all α. However, one does not know of any specific α for which it holds. We show that the problem is closely related to a divisor problem, which gives a better arithmetic understanding of the problem. The divisor problem considered seems to be hard, but we can show that it is true on average in a suitable sense.

About the Dissertation

Due to various circumstances during my PhD studies (as I will describe) I have – in agreement with my advisors – worked on three different problems. While two of the problems concern spectral theory of automorphic forms the last one is a problem in more classical analytic number theory. The dissertation consists of four manuscripts:

Manuscript A: *Quantum Unique Ergodicity of Eisenstein Series on the Hilbert Modular Group over a Totally Real Field*. Submitted.

Manuscript B: *Quantitative Mass Equidistribution of Eisenstein Series on* GL_2.

Manuscript C: *Distribution of Angles in Hyperbolic Lattices*. Joint with M. Risager. Accepted for publication in Quarterly Journal of Mathematics.

Manuscript D: *Divisor Problems and the Pair Correlation for the Fractional Parts of* $n^2\alpha$.

In addition to the four manuscripts there are brief introductions to the problems studied in the manuscripts.

In Chapter 1 we review some very basic facts about spectral theory of automorphic forms. The purpose of the chapter is to set the stage for Chapters 2 and 3 and Manuscripts A, B and C.

In Chapter 2 we introduce the problem studied in Manuscripts A and B. This was the problem originally suggested to me by M. Risager for my PhD. The research carried out during the first two years of my studies resulted in Manuscript A. In this manuscript we generalize the Luo-Sarnak result [**29**] on QUE of Eisenstein series on the modular group to Eisenstein series on the Hilbert modular group over a real field. We also give an expository account for the theory of Hecke operators on non-holomorphic Hilbert modular forms.

In the spring semester 2008 I had the opportunity to visit P. Sarnak and Claus M. Sørensen at Princeton University. During that period the bulk of the work for Manuscript B was made, which generalizes the equidistribution result in Manuscript A to Eisenstein series on GL_2 over a number field.

Chapter 3 is an introduction to Manuscript C which is joint with M. Risager. In this manuscript we prove an effective equidistribution result for angles in hyperbolic lattices. We use this to generalize a result due to F. Boca [**5**]. I worked on this problem just after I obtained my masters degree (as a part of the PhD program), i.e. in the beginning of my third year.

Chapter 4 is an introduction to Manuscript D. We study the pair correlation for the fractional parts of $n^2\alpha$ for specific α's. An unsolved conjecture in the area is to show that the pair correlation for the sequence is Poissonian for a class of α's with a certain Diophantine approximation property. We make a conjecture for a divisor problem and show that this conjecture implies the pair correlation conjecture. Furthermore, we show that the conjecture for the divisor problem holds on average.

The problem was suggested to me by P. Sarnak (since I had read the paper [**41**] on a different occasion). In the fall semester 2008 I was visiting D. R. Heath-Brown in Oxford who was interested in the problem and he has had great influence on Manuscript D.

We note that it is not the purpose of the introductory chapters to be references for researchers in the area. They should merely give a brief introduction to the problems considered in this dissertation (and provide motivation). Therefore the statements can be imprecise and the accounts given are by no means complete.

Acknowledgements

I would like to my thank my advisors Morten S. Risager and Alexei B. Venkov for their encouragement, help and support. In particular it was a joy to work with Morten Risager on Manuscript C and he has spent much more time supervising me than he was obligated to.

During my PhD studies I have been privileged and had the opportunity to visit Oxford and Princeton and I thank these institutions for hosting me. In particular I thank Roger Heath-Brown for arranging my visit in Oxford and Peter Sarnak and Claus Sørensen for arranging my visit in Princeton.

It was very inspiring to discuss analytic number theory with Roger Heath-Brown. His influence on Manuscript D was invaluable, and I am very grateful for all the time he has spent with me. The idea of studying the pair correlation for the fractional parts of $n^2\alpha$ as a divisor problem came from Peter Sarnak, and his suggestion is much appreciated.

During my stay in Princeton I also had the pleasure of meeting with Claus Sørensen every week in his office. This was a great help in generalizing Manuscript A and resulted in Manuscript B. He has always shown interest in my research, and I greatly appreciate his encouragements.

I thank my office mate Niels M. Møller for many interesting discussions on mathematics (not least during three months in Princeton where we shared an apartment). Finally I would like to thank Bent Ørsted and Simon Kristensen for answering questions from time to time.

Introduction

CHAPTER 1

A Brief Introduction to Spectral Theory of Automorphic Forms

In this chapter we give a (very) concise introduction to spectral theory of automorphic forms. More details can be found in [**21**]. It is the most basic prerequisites – certainly not sufficient to understand the problems in detail – needed in order to read Chapters 2 and 3 and Manuscripts A, B and C.

1. Fuchsian Groups

We let $\mathbf{H}$ denote the upper half-plane of $\mathbf{C}$,

$$\mathbf{H} = \{x + iy \in \mathbf{C} \mid y > 0\}$$

and we equip $\mathbf{H}$ with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The hyperbolic distance derived from the metric is usually denoted ρ. This metric induces a measure $d\mu$ on $\mathbf{H}$ which is given by

$$d\mu = \frac{dxdy}{y^2}.$$

It is well known that $\mathrm{SL}_2(\mathbf{R})$ (the invertible 2×2 matrices with real entries and determinant 1) acts on $\mathbf{H}$ as Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d},$$

and these transformations are orientation preserving isometries on $\mathbf{H}$. In fact $\mathrm{SL}_2(\mathbf{R})$ acts on the one-point compactification of $\mathbf{C}$.

We let $\mathrm{PSL}_2(\mathbf{R})$ denote the orientation preserving isometries on $\mathbf{H}$ and it is well known that $\mathrm{PSL}_2(\mathbf{R}) \cong \mathrm{SL}_2(\mathbf{R})/\{\pm I\}$. A Fuchsian group of the first kind (also called a cofinite group) Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ such that $\Gamma\backslash\mathbf{H}$ has finite volume – the metric (and measure) on $\mathbf{H}$ transfers to the quotient $\Gamma\backslash\mathbf{H}$. It is well known that any compact hyperbolic Riemann surface is of the form $\Gamma\backslash\mathbf{H}$ for suitable Γ.

A function $f : \mathbf{H} \to \mathbf{C}$ is automorphic (with respect to Γ) if $f(\gamma z) = f(z)$ for all $z \in \mathbf{H}$ and $\gamma \in \Gamma$. Thus an automorphic function is a function on $\Gamma\backslash\mathbf{H}$.

A point $w \in \mathbf{R} \cup \{\infty\}$ is called a cusp of Γ if w is fixed by some non-identity element in Γ. Two cusps w and w' of Γ are said to be equivalent if there exists $\gamma \in \Gamma$ such that $\gamma w = w'$. A Fuchsian group of the first kind has a finite number of inequivalent cusps. Our main example of a non-cocompact Fuchsian group of the first kind is the modular group $\mathrm{PSL}_2(\mathbf{Z})$, which has the classical fundamental domain

$$\{z \in \mathbf{H} \mid |z| > 1,\ |\mathrm{Re}(z)| < 1/2\}.$$

Using this fundamental domain one can check that the volume of $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$ is $\pi/3$. One easily checks that the cusps of $\mathrm{PSL}_2(\mathbf{Z})$ are $\mathbf{Q}\cup\{\infty\}$ and they are all equivalent. More generally the principal congruence subgroups of the modular group

$$\Gamma(N)=\left\{\pm\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{PSL}_2(\mathbf{Z})\mid a\equiv d\equiv 1(\mathrm{mod}N), b\equiv c\equiv 0(\mathrm{mod}N)\right\}$$

are all cofinite but non-cocompact.

Let $\mathfrak{a}$ denote a cusp of Γ. It is well known that the stabilizer subgroup $\Gamma_{\mathfrak{a}}\subset\Gamma$ is cyclic. Let $\gamma_{\mathfrak{a}}$ denote a generator of $\Gamma_{\mathfrak{a}}$. We choose $\sigma_{\mathfrak{a}}\in\mathrm{PSL}_2(\mathbf{R})$ such that

$$\sigma_{\mathfrak{a}}\infty=\mathfrak{a}$$

and

$$\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}}=\pm\begin{pmatrix}1&1\\&1\end{pmatrix}.$$

We will refer to $\sigma_{\mathfrak{a}}$ as a scaling matrix.

2. The Automorphic Laplacian

Let us first consider the Laplace-Beltrami operator Δ on a Riemannian manifold M of dimension n. The manifolds considered are always assumed to be connected and oriented. In local coordinates this operator is given in terms of the metric g_{ij} by

$$\Delta=-\frac{1}{\sqrt{\det(g_{ij})}}\frac{\partial}{\partial x^j}\left(\sqrt{\det(g_{ij})}g^{ij}\frac{\partial}{\partial x^i}\right). \tag{1.1}$$

Assume that the manifold M is compact. In this case Δ has pure point spectrum contained in $[0,\infty)$

$$0=\lambda_0<\lambda_1\leq\lambda_2\leq\dots$$

(listed with multiplicity) and $\lambda_k\to\infty$ as $k\to\infty$. In fact we have the so-called Weyl law

$$\#\{j\in\mathbf{N}_0\mid\lambda_j\leq\Lambda\}\sim\frac{\mathrm{Vol}(M)\mathrm{Vol}(B^n)}{(2\pi)^n}\Lambda^{n/2},$$

where B^n is the unit ball in $\mathbf{R}^n$.

The Laplace-Beltrami operator on $\mathbf{H}$ is given by

$$\Delta=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

It induces an essentially self-adjoint operator on smooth bounded functions $f\in L^2(\Gamma\backslash\mathbf{H})$ with the property that Δf is also bounded. The closure of this operator is called the automorphic Laplacian. By abuse of notation this operator is also denoted Δ. An automorphic form (with respect to Γ) is an eigenfunction of Δ (it does not have to be square integrable). A non-zero, smooth and bounded automorphic form is called a cusp form if

$$\int_0^1 f(\sigma_{\mathfrak{a}}z)dx=0$$

for any cusp $\mathfrak{a}$, i.e. if the zeroth Fourier coefficient is 0 at every cusp. An important property of these function is that together with the so-called incomplete Eisenstein series they span $L^2(\Gamma\backslash\mathbf{H})$ (see (1.3) below).

As mentioned before $\mathrm{PSL}_2(\mathbf{Z})$ has finite volume but is not cocompact. However, it is known that there are infinitely many eigenvalues. They all have finite multiplicity and we still have a Weyl law. More precisely let $\lambda_j = \frac{1}{4} + t_j^2$ denote the eigenvalues of Δ counted with multiplicity. Then

$$\#\{j \in \mathbf{N}_0 \mid |t_j| \leq T\} \sim \frac{\mathrm{Vol}(\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H})}{4\pi} T^2. \tag{1.2}$$

More generally let Γ be a non-cocompact Fuchsian group of the first kind with inequivalent cusps $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ and scaling matrices $\sigma_{\mathfrak{a}_k}$. In this case there is both a continuous spectrum (which covers the segment $[1/4, \infty)$ with multiplicity m) and a discrete spectrum. The eigenvalues fall into two categories – residual eigenvalues (which all lie in the interval $[0, 1/4)$ and are finite in number) and cuspidal eigenvalues (which are eigenfunctions of so-called cusp forms – a certain type of automorphic form) – but for the moment it is not necessary to know the precise definitions. Eigenvalues in (0,1/4) play a special role in the theory and these are called small eigenvalues. The modular group has no small eigenvalues. It has been conjectured by A. Selberg that there are no small eigenvalues for the groups $\Gamma(N)$.

For each cusp we have an Eisenstein series

$$E_{\mathfrak{a}_k}(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}_k}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}_k}^{-1}\gamma z)^s,$$

which is convergent for $\mathrm{Re}(s) > 1$, and it has a meromorphic continuation to the entire complex plane. Note that the Eisenstein series is simply the Poincaré series (elements are taken modulo $\Gamma_{\mathfrak{a}_k}$ since $z = x + iy \mapsto y$ is invariant under translation in x) formed by the formal eigenfunctions $z \mapsto y^s$ of the Laplacian on $\mathbf{H}$.

One easily checks that $E_{\mathfrak{a}_k}(z,s)$ is an automorphic form with eigenvalue $s(1-s)$, but $E_{\mathfrak{a}_k}(z,s)$ is not square integrable. Using the Eisenstein series the spectral resolution of Δ on $\Gamma\backslash\mathbf{H}$ can be made explicit. Let $\{\varphi_j\}$ denote a complete set of orthonormal eigenfunctions of Δ (ordered according to the eigenvalues). Let C denote the j's for which φ_j is a cusp form. Functions of the form (the first sum is finite)

$$f(z) = \sum_{j \in C} a_j \varphi_j(z) + \sum_{k=1}^{m} \sum_{\gamma \in \Gamma_{\mathfrak{a}_k}\backslash\Gamma} h_k(\mathrm{Im}(\sigma_{\mathfrak{a}_k}^{-1}\gamma z)), \tag{1.3}$$

where $h_k \in C_c^\infty(\mathbf{R}_+)$, are dense in $L^2(\Gamma\backslash\mathbf{H})$ and they have the expansion

$$f(z) = \sum_j \langle f, \varphi_j \rangle \varphi_j(z) + \sum_{k=1}^{m} \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f, E_{\mathfrak{a}_k}(\cdot, 1/2 + ir)\rangle E_{\mathfrak{a}_k}(z, 1/2 + ir) dr. \tag{1.4}$$

Here

$$\langle f, g \rangle = \int_{\Gamma\backslash\mathbf{H}} f(z)\overline{g(z)} d\mu$$

is the inner product on $L^2(\Gamma\backslash\mathbf{H})$. The bracket in the last sum is technically not an inner product since $E_{\mathfrak{a}_k}(z, 1/2 + ir)$ is not square integrable. The corresponding integral is convergent though, since the functions in first term of (1.3) are of rapid decay at every cusp and the last term is compactly supported. The functions appearing in the last term of (1.3) are called incomplete Eisenstein series.

3. The Fourier Expansion of an Automorphic Form and Hecke Operators

Let f be an automorphic form on $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$ with Laplace eigenvalue $s(1-s)$ satisfying the growth condition

$$f(z) = o(e^{2\pi y})$$

as $y \to \infty$. Then f has the Fourier expansion

$$f(z) = \frac{a_0(f)}{2}(y^s + y^{1-s}) + \frac{b_0(f)}{2s-1}(y^s + y^{1-s}) + \sum_{n \neq 0} c_n(f)\sqrt{y}K_{s-1/2}(2\pi y)e(x).$$

Here $e(x) = e^{2\pi i x}$ and K_ν denotes the Macdonald Bessel function

$$K_\nu(y) = \frac{1}{2}\int_0^\infty \exp(-y(t+1/t)/2)t^{\nu-1}dt.$$

As an example the Fourier expansion of the Eisenstein series on the modular group is

$$E(z,s) = y^s + c(s)y^{1-s} + \sum_{n \neq 0} c_n(s)\sqrt{y}K_{s-1/2}(2\pi y)e(x),$$

where

$$c(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(s)}$$

and

$$c_n(s) = \frac{2\pi^s}{\Gamma(s)\zeta(2s)\sqrt{n}} \sum_{ab=|n|} \left(\frac{a}{b}\right)^{s-1/2}.$$

Let f be an automorphic function on $\Gamma\backslash\mathbf{H}$. We then define the Hecke operator T_n (see [**21**] Section 8.5) by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right).$$

The Hecke operators are arithmetic objects. We can also define Hecke operators on other Fuchsian groups such as $\Gamma(N)$, but the groups must be arithmetic in some sense.

The Hecke operators satisfy the multiplication identity

$$T_m T_n = \sum_{d|(m,n)} T_{mnd^{-2}}. \tag{1.5}$$

In particular the Hecke operators commute. Moreover these operators are self-adjoint and they commute with the Laplacian. It turns out that we can choose a basis for the space spanned by cusp forms which are eigenfunctions of both the Laplacian and all the Hecke operators. Such cusp forms are called primitive. A very important fact is that with a suitable normalization the Fourier coefficients of the primitive cusp forms are the Hecke eigenvalues. The Hecke eigenvalues (and hence the Fourier coefficients) satisfy a multiplication identity similar to (1.5). This "extra information" is extremely useful and it is often the reason why one can obtain stronger results for Fuchsian groups that are arithmetic in nature.

It is important to mention that we can attach an L-function $L(s,\varphi)$ to a primitive cusp form φ defined by

$$L(s,\varphi) = \sum_{n=1}^\infty \frac{\lambda(n)}{n^s},$$

where $\lambda(n)$ is the eigenvalue of T_n. This L-function will show up in Chapter 2, Section 3.

4. The Hilbert Modular Group and Adele Groups

The purpose of this section is to introduce the notation for Chapter 2, Section 3. A good amount of algebraic number theory is needed. This can be found in [**35**] and [**39**].

We want to consider a generalization of $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$. Rather than considering $\mathbf{Z}$ we want to consider the ring of integers $\mathcal{O}$ in a totally real number field $\mathbf{F}$ of degree n over $\mathbf{Q}$. Let

$$\mathrm{Gal}(\mathbf{F}/\mathbf{Q}) = \{\psi_1, \ldots, \psi_n\}.$$

For $\alpha \in \mathbf{F}$ we set $\alpha^{(j)} = \psi_j(\alpha)$. The Hilbert modular group $\Gamma = \mathrm{PSL}_2(\mathcal{O})$ embeds discretely in $\mathrm{PSL}_2(\mathbf{R})^n$ by

$$\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\pm\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \ldots, \pm\begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix}\right). \tag{1.6}$$

We will use the convention $z = (z_1, \ldots, z_n) \in \mathbf{H}^n$ and $z = (x, y)$ where $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbf{R}^n_+$. It is immediate from the embedding (1.6) that Γ acts on $\mathbf{H}^n$.

We can regard $\mathbf{H}^n$ as a Riemannian manifold with the metric

$$ds^2 = \frac{dx_1^2 + dy_1^2}{y_1^2} + \cdots + \frac{dx_n^2 + dy_n^2}{y_n^2}$$

and it is known that the quotient $\Gamma\backslash\mathbf{H}^n$ has finite volume. The Laplace-Beltrami operator associated with this metric is

$$\Delta = \Delta_1 + \cdots + \Delta_n$$

where

$$\Delta_j = -y_j^2\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right).$$

We want to study functions which are eigenfunctions of all the Laplacians $\Delta_1, \ldots, \Delta_n$. Spectral theory and Hecke theory for automorphic functions on the space $\Gamma\backslash\mathbf{H}^n$ can be developed analogous to the theory described in the first sections, but we will not go into further details.

If we want to consider more general number fields, it is natural to change to an adelic setting. An excellent account is given in [**6**].

Let $\mathbf{A}$ denote the adele groups of $\mathbf{F}$. We define the quotient

$$X(\mathbf{F}) = Z(\mathbf{A})\mathrm{GL}_2(\mathbf{F})\backslash\mathrm{GL}_2(\mathbf{A})/K,$$

where Z is the 2×2 scalar matrices and $K = \prod_v K_v$ with

$$K_v = \begin{cases} \mathrm{O}(2) & \text{if } v \text{ is real} \\ \mathrm{U}(2) & \text{if } v \text{ is complex} \\ \mathrm{GL}_2(\mathcal{O}_v) & \text{if } v \text{ is finite} \end{cases}.$$

Here $\mathcal{O}_v$ is the ring of integers in $\mathbf{F}_v$. The group $\mathrm{GL}_2(\mathbf{A})$ is equipped with a Haar measure which induces a measure on $X(\mathbf{F})$. We note that

$$X(\mathbf{Q}) \cong \mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$$

and the measure on $X(\mathbf{Q})$ coincides with the Poincaré measure.

While it is relatively straight forward (at least philosophically) to go from spectral theory on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$ to spectral theory on $\mathrm{PSL}_2(\mathcal{O})\backslash\mathbf{H}^n$, it is perhaps less obvious how to go from the spectral theory described in the previous sections to the “spectral theory” of functions on $X(\mathbf{F})$, and we will not make an attempt to explain this. We will merely study the space $X(\mathbf{F})$ as a generalization of the quotient $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$.

CHAPTER 2

Mass Equidistribution of Eisenstein Series on GL(2)

In this chapter we motivate the study of quantum unique ergodicity and mention some of the results that have been obtained. This part is inspired by the survey papers [**23**], [**44**], [**45**] and [**46**] by P. Sarnak. Finally we explain the results obtained in Manuscripts A and B.

1. Hamiltonian Mechanics and Quantum Chaos

We briefly review the mathematical setting of Hamiltonian mechanics (see Arnold's book [**1**] for a detailed account). Recall that the Lagrangian for a mechanical system $L(q,\dot{q},t)$ is a function of the generalized coordinates q_i, the generalized velocities $\dot{q}_i$ and time t defined by $L = T - U$, where T and U are the kinetic and potential energy, respectively. In this setting the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

and these are referred to as Lagrange's equations.

We define the generalized momenta by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

The Hamiltonian $H(q,p,t)$ is defined as

$$H(q,p,t) = p \cdot \dot{q} - L(q,\dot{q},t),$$

and it may be identified with the sum of the kinetic and the potential energy for the system. With these definitions the equations of motion are

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{2.1}$$

and

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{2.2}$$

and these are referred to as Hamilton's equations.

If one consider our configuration space to be some Riemannian manifold M – with the metric given in local coordinates by the matrix g_{ij} – each point (q,p) can be regarded as an element in the cotangent bundle T^*M. Thus we will regard the Hamiltonian as a function on T^*M.

From now on we consider the time-independent Hamiltonian defined in local coordinates by

$$H(q,p) = \frac{1}{2} g^{ij}(q) p_i p_j. \tag{2.3}$$

This may be regarded as the Hamiltonian for a free particle moving around in M without friction and it is well known (see [**25**] Section 1.6) that this Hamiltonian gives rise to the cogeodesic flow on T^*M via the equations (2.1) and (2.2). We also know that for all $\lambda > 0$ the cogeodesic flow maps the set

$$E_\lambda = \{(q,p) \in T^*M \mid H(q,p) = \lambda\}$$

to itself. In physics terms this reflects the fact that we consider a particle with constant kinetic energy. If $(q(t), p(t))$ is a solution to Hamilton's equations we see that $\|\dot{q}\|_q = 2H(q,p)$. Thus we are led to consider the unit tangent bundle

$$SM = \coprod_q \{v \in T_qM \mid \|v\|_q = 1\} \subset TM.$$

It is not hard to see that SM is an embedded submanifold of TM and that $\dim SM = 2\dim M - 1$. We can even put a Riemannian structure on SM where the metric is given by

$$dr^2((q_1,v_1),(q_2,v_2)) = dq^2(q_1,q_2) + dv^2(v_1',v_2).$$

Here dq is the metric on M and dv is the distance between v_1' and v_2 (induced by the inner product on $T_{q_2}M$) where v_1' is the vector obtained from v_1 by parallel transport. We let ν denote the measure on SM induced by the Riemannian metric.

The geodesic flow on SM is the one-parameter semi-group $\{S_t\}$ of transformations that translate a linear element $(q,v) \in SM$ a distance of length $t > 0$ along the geodesic determined by (q,v). The geodesic flow preserves the measure ν on SM (see [**48**] for references). This explains how ergodic theory enters the picture, since we have a family of measure preserving maps.

2. Quantum Ergodicity

As in Chapter 1 we let Δ denote the Laplace-Beltrami operator on M. Note that $\frac{\hbar^2}{2}\Delta$ can be regarded as a quantization of $H(q,p)$ defined in (2.3) in the sense that $H(q,p)$ indeed is the leading symbol of $-\frac{1}{2}\Delta$ (by quantization we mean the transition from the Hamiltonian $H(q,p)$ to a differential operator by substituting q and p with suitable differential operators). Thus the equation for the stationary eigenstates (the time-independent Schrödinger equation with an appropriate choice of units) is

$$\frac{\hbar^2}{2}\Delta\psi_k = \lambda_k\psi_k. \tag{2.4}$$

Let M be a compact Riemannian manifold. In that case we know that Δ has pure point spectrum. From the equation (2.4) above we see that the semi classical limit (i.e. $\hbar \to 0$) is the same as the large eigenvalue limit (i.e. $k \to \infty$). Now assume that our quantum system has a classical analogue. Quantum chaos is the study of the quantized system when the classical system is "chaotic". Here "chaotic" means that the geodesic flow $\{S_t\}$ is ergodic (i.e. $S_t^{-1}(B) = B$ implies $\nu(B) = 0$ or $\nu(B) = 1$ for all $t > 0$). Sometimes one requires more in the definition of "chaotic" but we will only focus on the ergodicity condition. Assume that the geodesic flow on M is ergodic. This holds for example if M has constant, negative sectional curvature. Let $\{\varphi_k\}$ be an orthonormal basis for $L^2(M)$ of eigenfunctions of Δ with eigenvalues λ_k (listed with multiplicity and in increasing order). Y. Colin de Verdière [**8**], A. Schnirelman [**47**] and S. Zelditch [**53**] have proved that there exists a subsequence $\{\varphi_{k_j}\}$ of full density such that the probability measure

$$d\mu_{k_j} = |\varphi_{k_j}|^2 d\mu \to d\mu \tag{2.5}$$

in the weak-$*$ topology, where μ is the normalized Riemannian volume on M. Here full density means that

$$\frac{\#\{j \in \mathbf{N} \mid \lambda_{k_j} \leq N\}}{\#\{k \in \mathbf{N}_0 \mid \lambda_k \leq N\}} \to 1$$

as $N \to \infty$.

This result is known as "quantum ergodicity". Z. Rudnick and P. Sarnak [**40**] conjectured that if M is a compact Riemannian manifold with constant, negative sectional curvature then

$$d\mu_k \to d\mu \tag{2.6}$$

in the weak-$*$ topology. In other words it is not necessary to omit a subsequence of the eigenfunctions in (2.5). This is known as the quantum unique ergodicity conjecture and it is an aspect of this conjecture we will be concerned with. Note that (2.6) means that the measures $d\mu_k$ becomes equidistributed in the limit $k \to \infty$.

Recall from quantum mechanics that for $A \subset M$ the integral $\int_A |\varphi_j|^2 d\mu$ is interpreted as the probability for a particle in state φ_j to be in A. If our classical system was not chaotic, we would not expect the behavior in (2.6). Indeed we would expect that in the semi classical limit the support of the weak-$*$ limit of the measures $|\varphi_j|^2 d\mu$ would be the classical path of the particle.

Discussion of other significant conjectures concerning quantum chaos can be found in the four papers mentioned in the beginning of the chapter.

Let us consider compact hyperbolic surfaces of the form $\Gamma\backslash\mathbf{H}$ (as in Chapter 1). To approach the quantum unique ergodicity conjecture one must assume that Γ is arithmetic. Such arithmetic surfaces are derived from quaternion algebras over totally real number fields. In this case E. Lindenstrauss [**27**] has proved the conjecture.

Now consider $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$. In the light of (1.2) the problem of quantum unique ergodicity still makes sense for the modular group. We simply consider a complete set of orthonormal eigenfunctions $\{\varphi_k\}$ of Δ, i.e. an orthonormal basis for the discrete spectrum. This version of the quantum unique ergodicity conjecture has been proved by K. Soundararajan [**50**].

We mention that a similar result for holomorphic Hecke eigenforms has been proved by R. Holowinsky and K. Soundararajan [**16**] (the problem was originally studied bu W. Luo and P. Sarnak [**30**]). They proved that if f is a holomorphic Hecke eigenform of (even) weight k on the modular group then

$$|f(z)|^2 y^{k-2} dxdy \to \frac{3}{\pi} y^{-2} dxdy$$

as $k \to \infty$.

3. Mass Equidistribution of Eisenstein Series on GL(2)

W. Luo and P. Sarnak [**29**] have proved a continuous spectrum analogue of the quantum unique ergodicity conjecture – recall the spectral expansion (1.4) – in the following sense: Let $A, B \subset \mathrm{PSL}_2(\mathbf{Z})\backslash\mathbf{H}$ be compact and Jordan measurable, and assume that $\mu(B) \neq 0$. Let $d\mu_t = |E(z, 1/2 + it)|^2 d\mu$. Then

$$\frac{\mu_t(A)}{\mu_t(B)} \to \frac{\mu(A)}{\mu(B)} \tag{2.7}$$

as $t \to \infty$, where $E(z, s)$ is the Eisenstein series for $\mathrm{PSL}_2(\mathbf{Z})$ (recall that there is only one cusp for this group). Moreover the result is quantitative in the sense that they actually prove the asymptotics

$$\frac{1}{\log t}\int_{\Gamma\backslash\mathbf{H}} F(z)d\mu_t \to \frac{3}{\pi}\int_{\Gamma\backslash\mathbf{H}} F(z)d\mu \tag{2.8}$$

as $t \to \infty$ – (2.7) follows easily from (2.8). The main results in Manuscripts A and B are generalizations of this result. The strategy used in these papers is the same. However, the proofs are more complicated due to the fact that we work with more general number fields.

Though we won't address it further it should be mentioned that D. Jakobson [**24**] has proved a result analogous to (2.7) for a microlocal lift of the distribution $d\mu_t$ to the unit tangent bundle SM of $\Gamma\backslash\mathbf{H}$ which is isomorphic to $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathrm{PSL}_2(\mathbf{R})$. A microlocal lift is a certain family of measures on SM such that $d\mu_t$ are the pushforward measures of the lifted measures under the natural projection map – see [**53**] and [**24**] for correct definitions.

We briefly outline the proof of (2.8) – this should give the reader a better picture than if chose to work with the more general settings in Manuscripts A and B.

The idea is to use the decomposition (1.3) – i.e. that $L^2(\Gamma\backslash\mathbf{H})$ is the direct sum of the space spanned by cusp forms and the space of incomplete Eisenstein series – and then establish the equidistribution for functions that span these spaces ((2.9) and (2.10) below). Once we know that (2.8) follows from standard approximation arguments.

It is important that we have an arithmetic surface – certainly the arguments below do not apply to a general non-cocompact Fuchsian group. The fact that the Fourier coefficients of the Eisenstein series and the cusp forms are "arithmetic" is crucial.

For an incomplete Eisenstein series $F(z,h)$ we have

$$\frac{1}{\log t}\int_{\Gamma\backslash\mathbf{H}} F(z,h)d\mu_t \to \frac{3}{\pi}\int_{\Gamma\backslash\mathbf{H}} F(z,h)d\mu \tag{2.9}$$

as $t \to \infty$. For a primitive cusp forms φ we get

$$\int_{\Gamma\backslash\mathbf{H}} \varphi(z)d\mu_t \to 0 \tag{2.10}$$

as $t \to \infty$. This indeed corresponds to the desired equidistribution as

$$\int_{\Gamma\backslash\mathbf{H}} \varphi(z)d\mu = 0.$$

The idea in the proof of (2.9) is to unfold the incomplete Eisenstein series and then use the Fourier expansion of the Eisenstein series $E(z,s)$. The main terms of the integral can then be expressed in terms of Γ-factors (which are controlled by Stirling's formula) and the Riemann zeta-function $\zeta(s)$. The result (2.9) then follows from known estimates for $\zeta(s)$. The two non-trivial estimates needed are a subconvexity estimate for $\zeta(s)$ in t-aspect (due to Weyl) as well as a sufficiently good estimate for $\zeta'(1+it)/\zeta(1+it)$ (this follows from the classical Vinogradov zero-free region for ζ). Here "subconvexity estimate in t-aspect" means an estimate of the form

$$|\zeta(1/2+it)| \ll |t|^{\alpha},$$

where the exponent α is smaller than the exponent obtained from the Phragmén-Lindelöf principle and the functional equation (see e.g. [**22**]).

The proof of (2.10) is based in the Rankin-Selberg method. It is well known that the integral considered can be expressed as a product of Γ-factors, the Riemann zeta-function and the standard L-function $L(s,\varphi)$. The result then follows from a subconvexity estimate for $L(s,\varphi)$ in t-aspect due to [**32**].

We mention that S. Koyama [**26**] has proved an analogue of (2.8) for Eisenstein series on $\mathrm{SL}_2(\mathcal{O})\backslash\mathbf{H}_3$, where $\mathcal{O}$ is the ring of integers in a quadratic imaginary field with class number one and $\mathbf{H}_3$ is the 3-dimensional upper half-space. Manuscript B generalizes this result.

In Manuscript A we generalize 2.8 to Eisenstein series on the Hilbert modular group over a totally real field $\mathbf{F}$ of degree n with narrow class number one. In order to give a self-contained account we also give an expository treatment of the theory of Hecke operators on non-holomorphic Hilbert modular forms. The setting in this manuscript is entirely classical (as in [**11**]), i.e. "non-adelic". On the Hilbert modular group we consider the (family of) Eisenstein series considered by I. Efrat [**11**]

$$E(z,s,m)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\prod_{j=1}^{n}\mathrm{Im}(\gamma^{(j)}z_j)^{s_j},$$

where $s_j = s+i\rho_j(m)$, $m\in\mathbf{Z}^{n-1}$. Here $\rho_j(m)$ is a real number such that $\beta\mapsto\prod_{j=1}^{n}|\beta^{(j)}|^{i\rho_j(m)}$ is a Hecke character on $\mathbf{F}$. The spectral theory described in Chapter 1 generalizes to this setting. We extend (2.8) and prove that for $F\in C_c(\Gamma\backslash\mathbf{H}^n)$

$$\frac{1}{\log t}\int_{\Gamma\backslash\mathbf{H}^n}F(z)|E(z,s,m)|^2d\mu\to\frac{\pi^n nR}{2D\zeta_{\mathbf{F}}(2)}\int_{\Gamma\backslash\mathbf{H}^n}F(z)d\mu \tag{2.11}$$

as $t\to\infty$. Here $\zeta_{\mathbf{F}}$ is the Dedekind zeta-function, D is the discriminant and R is the regulator of $\mathbf{F}$.

In Manuscript B we generalize (2.11) to Eisenstein series on GL_2 over a general number field $\mathbf{F}$ with r_1 real places and r_2 complex places and class number h. We let $\mathcal{W}$ denote the number of roots of unity in $\mathbf{F}^\times$. In this case it is more convenient to work with an adelic setting.

We consider the Eisenstein series $E(g,s,\chi)$ on $X(\mathbf{F})$ (defined in Chapter 1, Section 4) defined by (B is the upper triangular matrices)

$$E(g,s,\chi)=\sum_{\gamma\in B(\mathbf{F})\backslash\mathrm{GL}_2(\mathbf{F})}f(\gamma g),$$

where χ is an everywhere unramified character on $\mathbf{A}^\times/\mathbf{F}^\times$. The function $f:\mathrm{GL}_2(\mathbf{A})/K\to\mathbf{C}$ is identical 1 on K and satisfies the condition that ($|\cdot|_{\mathbf{A}}$ denotes the idele norm)

$$f\left(\left(\begin{smallmatrix}y_1 & x\\ & y_2\end{smallmatrix}\right)g\right)=\frac{\chi(y_1)|y_1|_{\mathbf{A}}^s}{\chi(y_2)|y_2|_{\mathbf{A}}^s}f(g)$$

for $g\in\mathrm{GL}_2(\mathbf{A})$, $y_1,y_2\in\mathbf{A}^\times$ and $x\in\mathbf{A}$. By the Iwasawa decomposition this determines f completely.

For $F\in C_c(X(\mathbf{F}))$ we prove that

$$\frac{1}{\log t}\int_{X(\mathbf{F})}F(g)|E(g,1/2+it,\chi)|^2d\mu\to\frac{2^{r_2}\pi^{r_1}nhR}{\zeta_{\mathbf{F}}(2)\mathcal{W}D}\int_{X(\mathbf{F})}F(g)d\mu \tag{2.12}$$

as $t\to\infty$.

We mention that the subconvexity estimate in t-aspect for the standard L-function, which is necessary to prove (2.11) and (2.12), has just recently been established by A. Diaconu and P. Garrett [**10**] and P. Michel and A. Venkatesh [**33**]. A subconvexity estimate for the Hecke L-function is also needed [**49**]. Finally an estimate for the logarithmic derivative of the Hecke L-function on the line $\mathrm{Re}(s)=1$ is required. This follows from the zero-free region derived in [**7**].

CHAPTER 3

Distribution of Angles in Hyperbolic Lattices

In this chapter we review the classical Gauss circle problem and its hyperbolic analogues. We also explain the results obtained in Manuscript C.

1. The Gauss Circle Problem

A classical problem in analytic number theory is to count the number of Gaussian integers lying inside a disc of radius R with center 0 in the complex plane. This is known as the Gauss circle problem. We let

$$\mathfrak{E}(R) = |\pi R^2 - \#\{(m, n) \in \mathbf{Z}^2 \mid m^2 + n^2 \leq R^2\}|.$$

Since the circumference of the disc of radius R is $2\pi R$ it is elementary to show that

$$\mathfrak{E}(R) \ll R.$$

This is illustrated in Figure 1. We see that $\mathfrak{E}(6)$ can be estimated by the number of hatched squares.

A more refined estimate can be obtained by studying

$$\sum_{(m,n)\in\mathbf{Z}^2} F(\sqrt{m^2 + n^2}),$$

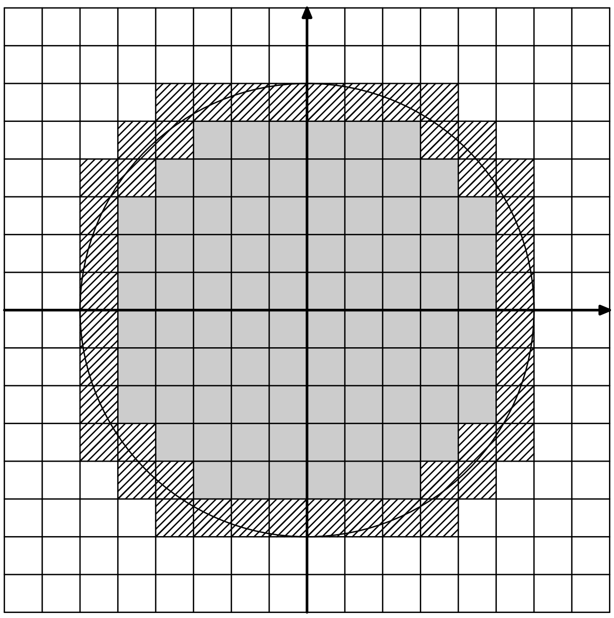

FIGURE 1. Area approach to the Gauss circle problem.

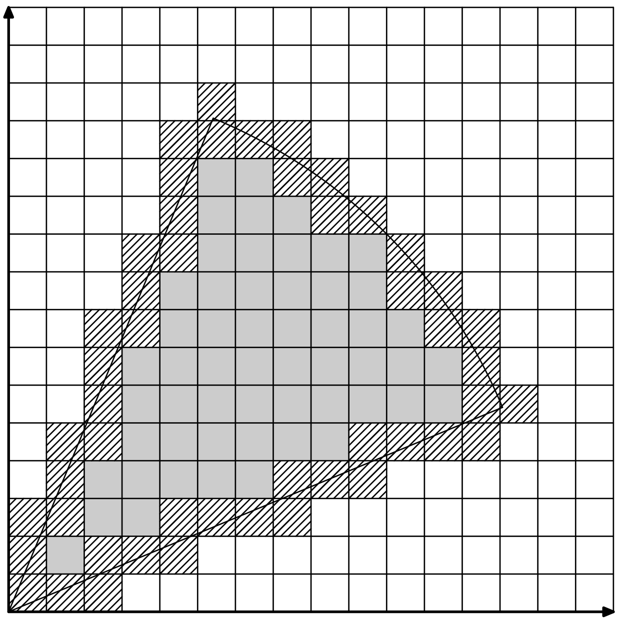

FIGURE 2. Counting Gaussian integers in angular sectors.

where F is a suitable smooth approximation to the characteristic function on the interval $[0, R]$. Using the Poisson summation formula we can obtain an estimate for $\mathfrak{E}$ by estimating the Fourier coefficients of $(x, y) \mapsto F(\sqrt{x^2 + y^2})$ (which naturally should be studied in polar coordinates). Using standard estimates for the J-Bessel functions one can prove that

$$\mathfrak{E}(R) \ll R^{3/4}.$$

The current record for $\mathfrak{E}(R)$ is due to M. N. Huxley [**20**] who proved

$$\mathfrak{E}(R) \ll R^{131/208+\varepsilon}$$

for any $\varepsilon > 0$, and it is believed that

$$\mathfrak{E}(R) \ll R^{1/2+\varepsilon}.$$

It has been proved by Hardy [**14**] that

$$\mathfrak{E}(R) = \Omega(R^{1/2}(\log R)^{1/4} \log \log R).$$

The result has been improved by Hafner [**13**].

We can also refine the Gauss circle problem by counting Gaussian integers in angular sectors. Let $I \subset [0, 2\pi]$ be an interval and let $\theta(x, y)$ denote the angle (in $[0, 2\pi]$) between the x-axis and the line through $(x, y) \in \mathbf{R}^2$ (we may set $\theta(0, 0) = 0$). Since the circumference of the angular sector

$$\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq \mathbf{R}^2,\ \theta(x, y) \in I\}$$

is $O(R)$ it is elementary to show that

$$\{(m, n) \in \mathbf{Z}^2 \mid m^2 + n^2 \leq \mathbf{R}^2,\ \theta(m, n) \in I\} = \frac{|I|R^2}{2\pi} + O(R)$$

where $|I|$ is the length of the interval I (see Figure 2).

2. The Hyperbolic Lattice Point Problem

In the Gauss circle problem we may view $\mathbf{R}^2$ as a Riemannian manifold with the Euclidean metric and $\mathbf{Z}^2$ as a discrete subgroup of the isometry group (acting by translation). We see that

$$\#\{(m,n) \in \mathbf{Z}^2 \mid m^2+n^2 \leq R^2\} = \#\{g \in \mathbf{Z}^2 \mid |g(0,0)| \leq R\}.$$

Rather than considering $\mathbf{R}^2$ we consider the upper half plane $\mathbf{H}$ with the Poincaré metric and a Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$ of finite covolume. The hyperbolic lattice point problem is the problem of estimating

$$N_\Gamma(R, z_0, z_1) = \#\{\gamma \in \Gamma \mid \rho(\gamma z_1, z_0) \leq R\}.$$

The only "satisfactory" answer would be

$$N_\Gamma(R, z_0, z_1) \sim \frac{\pi e^R}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})} \tag{3.1}$$

since the hyperbolic area of a disc of radius R is

$$4\pi \sinh^2(R/2) \sim \pi e^R.$$

However, the hyperbolic lattice point problem is fundamentally more difficult than the Gauss circle problem since the circumference of the hyperbolic circle is

$$2\pi \sinh R \sim \pi e^R.$$

Thus the boundary is of the same size as the area of the entire disc. Therefore a simple area approach to (3.1) will not work. It turns out that (3.1) is the correct asymptotics. This has been proved by (among others) J. Delsarte [**9**], A. Good [**12**], H. Huber [**17**], [**18**] and [**19**] and S. J. Patterson [**37**].

One approach to the hyperbolic lattice-point problem is to use spectral theory of the automorphic Laplacian. We briefly describe an idea due to A. Selberg (see Theorem 12.1 in [**21**]), which (at least to some extend) resembles the method outlined in Section 1 which gave the $R^{3/4}$ estimate in the Gauss circle problem. Consider the automorphic kernel $K : \mathbf{H} \times \mathbf{H} \to \mathbf{C}$ defined by

$$K(z,w) = \sum_{\gamma \in \Gamma} k(u(z, \gamma w)),$$

where $k \in C_c^\infty([0,\infty))$ and u is the point pair invariant

$$u(z,w) = \frac{\cosh \rho(z,w) - 1}{2}.$$

We want to consider a suitable approximation to the characteristic function on $[0,(X-2)/4]$ and then use the fact that K has a spectral expansion of the form (see Theorem 7.4 in [**21**] – also compare with (1.4))

(3.2)
$$K(z,w) = \sum_j h(t_j)\varphi_j(z)\overline{\varphi_j(w)} + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) E_{\mathfrak{a}}(z, 1/2+ir)\overline{E_{\mathfrak{a}}(w, 1/2+ir)}dr,$$

where the first sum is over the eigenvalues and the last sum is over the cusps of Γ, and h is the Harish-Chandra-Selberg transform of k. Note that (3.2) plays (roughly) the same role as the Poisson summation formula in Section 1. With a suitable choice of k one obtains

$$N_\Gamma(R, z_0, z_1) = \sqrt{\pi} \sum_{1/2 < t_j \leq 1} \frac{\Gamma(t_j - 1/2)}{\Gamma(t_j+1)} \varphi_j(z_0)\overline{\varphi_j(z_1)} e^{t_j R} + O(e^{2R/3}). \tag{3.3}$$

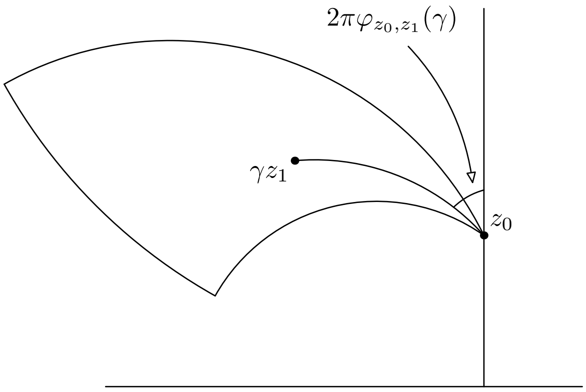

FIGURE 3. Hyperbolic lattice-points in angular sectors.

This proves (3.1). Note that the asymptotics of $N_\Gamma(R, z_0, z_1)$ depends on the small eigenvalues of Γ. The estimate in (3.3) is the best known. It has been proved by R. Phillips and Z. Rudnick [**38**] that the error term must be at least $O(e^{1/2})$. It is believed that the optimal error term (in many cases) is $O(e^{1/2+\varepsilon})$.

3. Distribution of Angles in Hyperbolic Lattices

A natural question to ask now is whether the angles in the hyperbolic lattices are equidistributed. Let $\varphi_{z_0,z_1}(\gamma)$ denote the normalized angle between the vertical geodesic from z_0 to ∞ and the geodesic between z_0 and γz_1 (see Figure 3). For an interval $I \subset \mathbf{R}/\mathbf{Z}$ we define

$$N_\Gamma^I(R, z_0, z_1) = \#\{\gamma \in \Gamma \mid \rho(z_0, \gamma z_1) \leq R,\ \varphi_{z_0,z_1}(\gamma) \in I\}.$$

It has been proved by A. Good [**12**] and P. Nicholls [**36**] that

$$N_\Gamma^I(R, z_0, z_1) \sim \frac{\pi|I|}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})} e^R. \tag{3.4}$$

From Weyl's criterion we know that (3.4) follows if

$$\sum_{\substack{\gamma\in\Gamma \\ \rho(z_0,\gamma z_1)\leq R}} e(n\varphi_{z_0,z_1}(\gamma)) = o(N_\Gamma(z_0, z_1, R)), \tag{3.5}$$

as $R \to \infty$ for all $n \in \mathbf{N}$.

Indeed this is the approach in [**12**], where the estimate in (3.5) is proved using spectral theory.

In Manuscript C we prove (3.4) with an error term. By conjugation we may assume that $z_0 = i$. We let $(r(z), \varphi(z))$ denote the hyperbolic polar coordinates of z, i.e. $r(z) = \rho(z, i)$ and $\varphi(z)$ is half the angle between the vertical geodesic from i to ∞ and the geodesic between i and z. The idea is to consider the series

$$G_n(z, s) = \sum_{\gamma\in\Gamma} \frac{e(n\varphi(\gamma z)/\pi)}{(\cosh(r(\gamma z)))^s}, \tag{3.6}$$

which is convergent for $\mathrm{Re}(s) > 1$ (for any $z \in \mathbf{H}$). Furthermore $G_n(z,s)$ satisfies the equation

$$(\Delta - s(1-s))G_n(z,s) = s(s+1)G_n(z,s+2) + \sum_{\gamma\in\Gamma} \frac{n^2 e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z))(\cosh(r(\gamma z)))^s}.$$

Using the resolvent $R(s) = (\Delta(s) - s(1-s))^{-1}$ we see that

$$G_n(z,s) = R(s)\left(s(s+1)G_n(z,s+2) + \sum_{\gamma\in\Gamma} \frac{n^2 e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z))(\cosh(r(\gamma z)))^s}\right).$$

By standard spectral theory arguments this implies that $G_n(z,s)$ has a meromorphic continuation to $\mathrm{Re}(s) > 1/2$ with potential poles at as $s = t_j$ where $t_j(1-t_j)$ is a small eigenvalue of Δ. Since there is a spectral gap between the 0-eigenvalue and the first eigenvalue of the Laplacian the pole at $s = 1$ is isolated. This pole has residue

$$\frac{2\pi\delta_{n,0}}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})}.$$

We now consider a suitable smooth approximation ψ to the characteristic function on $[0,1]$ and let $(M\psi)(s)$ denote the Mellin transform of ψ, i.e.

$$(M\psi)(s) = \int_0^\infty \psi(t)t^{s-1}dt.$$

From the Mellin inversion formula it follows that

$$\sum_{\gamma\in\Gamma} e(n\varphi(\gamma z)/\pi)\psi\left(\frac{\cosh(r(\gamma z))}{T}\right) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=2} G_n(z,s)(M\psi)(s)T^s ds. \tag{3.7}$$

We want to move the complex line integral on the right hand side of (3.7) to a line strictly to the left of $\mathrm{Re}(s) = 1$ but also strictly to the right of $\mathrm{Re}(s) = 1/2$ if there are no small eigenvalues or t_1 if $\lambda_1 \in (0, 1/4)$. By doing so we pick up the residue from the pole at $s = 1$, but no other poles contribute due to the spectral gap. Estimating G_n using standard Sobolev estimates one obtains an effective bound (i.e. explicit in n and T) for the sum

$$\sum_{\substack{\gamma\in\Gamma\\ \cosh(r(\gamma z))\leq T}} e(n\varphi(\gamma z)/\pi).$$

Using the Erdös-Turán inequality, which is an effective version of Weyl's criterion one concludes that there exists $\alpha < 1$ (which can be computed effectively in terms of the first eigenvalue) such that

$$N_\Gamma^I(R, z_0, z_1) = \frac{\pi|I|}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})}e^R + O(e^{\alpha R}). \tag{3.8}$$

If there are no small eigenvalues of Δ we may take $\alpha = 11/12 + \varepsilon$ for any $\varepsilon > 0$.

We may now consider a different problem. Let

$$\mathcal{N}_\Gamma^I(R, z_0, z_1, w) = \#\{\gamma\in\Gamma \mid \rho(z_1, \gamma w) \leq R,\ \varphi_{z_0,w}(\gamma) \in I\}.$$

Again we may ask for the asymptotics of $\mathcal{N}_\Gamma^I(R, z_0, z_1, w)$ as $R \to \infty$. Clearly $\rho(z_1, \gamma w) = \rho(z_0, \gamma w) + O(1)$ but because the boundary of a hyperbolic circle is huge the question is

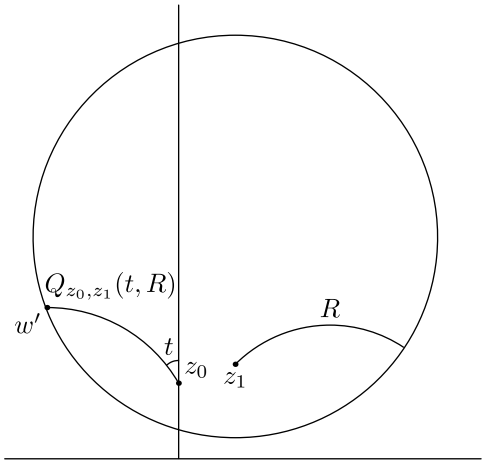

FIGURE 4. Hyperbolic circle with center z_1 and radius R.

non-trivial – the analogous question for lattices in Euclidean space would be trivial. For the principal congruence subgroups $\Gamma(N)$ F. Boca [**5**] proved that for any $\varepsilon > 0$

$$\begin{aligned}&\#\{\gamma \in \Gamma(N) \mid \rho(z_1, \gamma z_1) \leq R,\, 2\pi\varphi_{z_0,z_1}(\gamma) \in J \cup (J+\pi)\} = \\ &\qquad \frac{6e^R}{\pi[\Gamma(1):\Gamma(N)]}\int_J \eta_{z_0,z_1}(t)dt + O(e^{(7/8+\varepsilon)R})\end{aligned} \tag{3.9}$$

where

$$\eta_{z_0,z_1}(\omega) = \frac{2y_0y_1(y_0^2+y_1^2+(x_0-x_1)^2)}{(y_0^2+y_1^2+(x_0-x_1)^2)^2-((y_0^2-y_1^2+(x_0-x_1)^2)\cos(t)+2y_0(x_0-x_1)\sin(t))^2}$$

for an interval $J \subset [0,\pi]$. Note that $2\pi\varphi_{z_0,z_1}(\gamma) \in J \cup (J+\pi)$ means that angles are counted modulo π rather than 2π (i.e. opposite angles are "identified"). The proof of (3.9) is based on the Weil estimate for Kloosterman sums.

In Manuscript C we generalize (3.9) (with an inferior error term though). The idea is to find the hyperbolic distance (this will be denoted $Q_{z_0,z_1}(t,R)$) from z_0 to the intersection between the hyperbolic circle with center at z_1 and radius R determined by the (normalized) angle $t \in [0,1]$ relative to the vertical geodesic through z_0 (see Figure 4). We prove that

$$e^{Q_{z_0,z_1}(t,R)} = \rho_{z_0,z_1}(t)e^R + O(1),$$

where

$$\rho_{z_0,z_1}(\omega) = \frac{2y_0y_1}{(y_0^2+y_1^2+(x_0-x_1)^2)(1-\cos(2\pi\omega))+2y_0^2\cos(2\pi\omega)+2(x_1-x_0)y_0\sin(2\pi\omega)}.$$

Using (3.8) we can make a Riemann sum approximation to prove that there exists $\alpha' < 1$ (which can be computed in terms of the first eigenvalue of the Laplacian) such that

$$\mathcal{N}_\Gamma^I(R, z_0, z_1, w) = \frac{\pi e^R}{\mathrm{Vol}(\Gamma\backslash\mathbf{H})}\int_I \rho_{z_0,z_1}(\omega)d\omega + O(e^{\alpha' R}),$$

for any interval $I \subset [0,1]$. We remark that α' need not be as small as α in (3.8).

CHAPTER 4

Pair Correlation for the Fractional Parts of $n^2\alpha$

In this chapter we introduce some of the recent developments in the study of the pair correlation for the fractional parts of $n^2\alpha$ with emphasis on the approach taken in Manuscript D. We remark that most of the proofs in Manuscript D are elementary.

1. Poissonian Behavior

Let $\{a_n\}_1^\infty$ denote an equidistributed sequence in $[0,1)$, with $a_i \neq a_j$ for $i \neq j$. A natural question to ask is whether $\{a_n\}_1^\infty$ has Poissonian behavior, i.e. if the sequence has the same distribution as a sequence of independent random variables with uniform distribution in $[0,1)$. We explain this more precisely. Let $A_N = \{a_1,\ldots,a_N\}$ and let $\widetilde{a}_1,\ldots,\widetilde{a}_N$ denote the elements in A_N in increasing order (we also set $\widetilde{a}_{j-N} = \widetilde{a}_j$ for $j = 1,\ldots,N$). The sequence $\{a_n\}_1^\infty$ has Poissonian behavior if for any $m \geq 1$

$$\frac{1}{N}\sum_{n=1}^{N}\delta(t - N(\widetilde{a}_n - \widetilde{a}_{n-m}))dt \to \frac{t^{m-1}}{(m-1)!}e^{-t}dt$$

as $N \to \infty$ (δ is the Dirac function).

One way to decide if $\{a_n\}_1^\infty$ has Poissonian behavior is to look at the m-level correlation. Let $\mathcal{B}$ be a box in $\mathbf{R}^{m-1}$. We define $R_m(\mathcal{B}, N, \{a_n\}_1^\infty)$ to be

$$N^{-1}\#\{(x_1,\ldots,x_m) \in A_N^m \mid x_i \text{ distinct},\ (x_1 - x_2,\ldots,x_{m-1} - x_m) \in N^{-1}\mathcal{B} + \mathbf{Z}^{m-1}\}.$$

We say that the m-level correlation is Poissonian if for any $\mathcal{B} \subset \mathbf{R}^{m-1}$

$$R_m(\mathcal{B}, N, \{a_n\}_1^\infty) \to \mathrm{Vol}(\mathcal{B})$$

as $N \to \infty$. The m-level correlation is Poissonian for all $m \geq 2$ if and only if $\{a_n\}_1^\infty$ has Poissonian behavior.

Let α be an irrational number and $g : \mathbf{N} \to \mathbf{N}$ a (strictly) increasing function. It is well known that for almost all α (with respect to the Lebesgue measure) the fractional parts of $\alpha g(n)$ are equidistributed. An interesting question is whether the sequence $a_n = \alpha g(n)$ has Poissonian behavior. If

$$\liminf_n \frac{g(n+1)}{g(n)} > 1 \tag{4.1}$$

Z. Rudnick and A. Zaharescu [**43**] has shown that $\{a_n\}_1^\infty$ has Poissonian behavior for almost all α.

The rest of this chapter will be devoted to the sequence $n^2\alpha$. Clearly $g(n) = n^2$ does not satisfy (4.1). However, it has been conjectured by Rudnick, Sarnak and Zaharescu [**42**] that the fractional parts of $n^2\alpha$ have Poissonian behavior for almost all α. For the pair (or 2-level) correlation more is known. This will be the topic of the next section.

2. Pair Correlation for the Fractional Parts of $n^2\alpha$

Approximation problems for the fractional parts of $n^2\alpha$ have been studied by various people. It is a classical result due to H. Weyl [**51**] that for all irrational α the sequence $n^d\alpha$ is equidistributed modulo 1 for any positive integer d. For $t \in \mathbf{R}$ we define

$$\|t\| = \inf_{n\in\mathbf{Z}} |t - n|$$

and this defines a norm on $\mathbf{R}/\mathbf{Z}$. A. Zaharescu [**52**] has shown that for $\theta < 2/3$ and any $\alpha \in \mathbf{R}$

$$\|n^2\alpha\| < n^{-\theta}$$

has infinitely many solutions. The sequence is also interesting because the spacings between the elements correspond to the spacings between the energy levels of the "boxed oscillator" in quantum mechanics [**3**].

We will focus on the pair correlation for the fractional parts of $n^2\alpha$. We define

$$\begin{aligned} R_2(x, N, \alpha) &= R_2([-x, x], N, \{n^2\alpha\}_1^\infty) \\ &= N^{-1}\#\left\{(m, n) \mid m, n \le N,\ n \ne m,\ \|m^2\alpha - n^2\alpha\| \le \frac{x}{N}\right\}. \end{aligned}$$

The goal is to understand for which α the pair correlation is Poissonian, i.e. for which α

$$R_2(x, N, \alpha) \to 2x$$

as $N \to \infty$ for any $x > 0$.

It has been proved by Z. Rudnick and P. Sarnak [**41**] that for $d \ge 2$ the pair correlation for the fractional parts of $n^d\alpha$ is Poissonian for almost all α. Subsequently J. Marklof and A. Strömbergsson [**31**] and D. R. Heath-Brown [**15**] have given different proofs in the case $d = 2$. However, one does not know of any specific α for which it holds. It is not true that the pair correlation for the fractional parts of $n^d\alpha$ is Poissonian for any irrational α. A condition on the Diophantine approximation is necessary in order for the pair correlation to be Poissonian. We say that an irrational number α is of type κ if

$$|\alpha - p/q| \gg \frac{1}{q^\kappa}.$$

for all $p \in \mathbf{Z}$ and $q \in \mathbf{N}$.

We say that α is "Diophantine" if α is of type $2 + \varepsilon$ for all $\varepsilon > 0$. Note that all real, irrational algebraic numbers are Diophantine (Roth's theorem) and that almost all α are Diophantine. We will now explain why a Diophantine condition is needed (the argument below is taken from [**41**]).

Assume that there are infinitely many $(p, q) \in \mathbf{Z} \times \mathbf{N}$ such that

$$|\alpha - p/q| \le \frac{1}{10q^3}.$$

This implies that

$$\|m^2\alpha - n^2\alpha\| = \left\|\frac{(m^2 - n^2)p}{q} + \frac{t(m^2 - n^2)}{10q^3}\right\|$$

for some t with $|t| \le 1$. For $m, n \le q$ we see that

$$\|m^2\alpha - n^2\alpha\| \le \frac{1}{10q}$$

if $q \mid m^2 - n^2$ and

$$\|m^2\alpha - n^2\alpha\| \geq \frac{9}{10q}$$

if $q \nmid m^2 - n^2$. Thus there are no normalized differences $q\|m^2\alpha - n^2\alpha\|$ in $(1/10, 9/10)$ along the subsequence determined by the q's. From this we see that if the pair correlation for $n^2\alpha$ is Poissonian then α must be at least of type 3.

Rudnick and Sarnak conjectured that if α is Diophantine then the pair correlation for the fractional parts of $n^2\alpha$ is Poissonian. Heath-Brown was able to show (using a lattice point strategy) that for α of type 9/4

$$R_2(x, N, \alpha) = 2x + O(x^{\frac{7}{8}}), \tag{4.2}$$

whenever $1 \leq x \leq \log N$, where the constant implied depends on α. This supports the Rudnick-Sarnak conjecture and suggests that perhaps the condition on the Diophantine approximation in the conjecture can be relaxed to some extend. In Manuscript D it is suggested that it is sufficient that α is of type $3 - \delta$ for some $\delta > 0$.

We remark that for the m-level correlation (for general m) for the fractional parts of $n^2\alpha$ to be Poissonian it is probably not sufficient to assume that α is Diophantine. In [**42**] it is suggested that one must also assume that the numerators of the convergents of α are "almost" square free. However, this last condition does not seem to be necessary for the pair correlation.

3. Divisor Problems Related to the Rudnick-Sarnak Conjecture

In Manuscript D we suggest an arithmetic line of attack for the Rudnick-Sarnak conjecture that is based on the study of the function

$$\tau_{M,N}(m) = \#\{(a,b) \in \mathbf{N}^2 \mid a \leq M,\ b \leq N,\ ab = m\},$$

where $m \in \mathbf{N}$ and $M, N \geq 1$. We will always assume that $M \asymp N$.

Let $K, M, N \geq 1$ with $K \geq N^{\eta}$ for some fixed $\eta > 0$. Assume also that $q \leq N^{2-\delta}$ for some $\delta > 0$ and $(q, \rho) = 1$. We conjecture that

$$\sum_{r \leq K} \sum_{m \equiv \rho r (q)} \tau_{M,N}(m) \sim \frac{KMN}{q} \tag{4.3}$$

as $N \to \infty$ uniformly in M, K, q and ρ. Manuscript D is mainly devoted to finding evidence for this conjecture.

In Manuscript D we show that this conjecture implies that the pair correlation for the fractional parts of $n^2\alpha$ is Poissonian for any α of type $3 - \delta$ for any $\delta \in (0, 1)$. As mentioned before the pair correlation for the fractional parts of $n^2\alpha$ is not Poissonian if α is not of type 3, and this appears to be the "limit". The conjecture (4.3) seems bold but natural. Indeed it is elementary to show that (4.3) holds if $q \leq N^{1-\delta}$.

Let τ denote the usual divisor function. From the Dirichlet divisor problem we know that

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

It is folklore that one expects that

$$\sum_{\substack{n \leq x \\ n \equiv r (q)}} \tau(n) \sim \frac{x}{q^2} \log x \sum_{d \mid (q,r)} \sum_{c \mid \frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) \tag{4.4}$$

as $x \to \infty$ for $q \le x^{1-\delta}$ for some $\delta > 0$. Average results supporting this conjecture have been considered by Banks, Heath-Brown and Shparlinski [**2**] and Blomer [**4**]. Also it is a classical result due to Linnik and Vinogradov [**28**] that

$$\sum_{\substack{m \le x \\ m \equiv r(q)}} \tau(m) \ll \frac{\varphi(q) x \log x}{q^2} \tag{4.5}$$

for $q \le x^{1-\delta}$ and $(r,q) = 1$, where the constant implied depends on $\delta > 0$ only.

Let us consider some simple heuristics to explain why we should expect (4.4). We see that

$$\begin{aligned} \sum_{\substack{m \le x \\ m \equiv r(q)}} \tau(m) &= \sum_{a \le x} \#\{b \in \mathbf{N} \mid b \le x/a,\ ab \equiv r(q)\} \\ &= \sum_{d \mid (r,q)} \sum_{\substack{a' \le x/d \\ (a', q/d)=1}} \#\{b \in \mathbf{N} \mid b \le x/(a'd),\ b \equiv \overline{a'} r/d (q/d)\}. \end{aligned}$$

Now x/a' can be much smaller than q but we should expect "on average" (when summing over a') that

$$\#\{b \in \mathbf{N} \mid b \le x/(a'd),\ b \equiv \overline{a'} r/d (q/d)\} \approx \frac{x}{a'q}.$$

We continue with this assumption and consider

$$\frac{x}{q} \sum_{d \mid (r,q)} \sum_{\substack{a' \le x/d \\ (a', q/d)=1}} \frac{1}{a'}. \tag{4.6}$$

In Manuscript D we show that

$$\sum_{\substack{a \le x \\ (a,q)=1}} \frac{1}{a} \sim \frac{\varphi(q)}{q} \log x.$$

for $q \le x^{1-\delta}$. Using this we see that (4.6) is roughly

$$\frac{x \log x}{q^2} \sum_{d \mid (r,q)} d \varphi\left(\frac{q}{d}\right).$$

This "explains" (4.4).

If we adapt (4.4) to $\tau_{M,N}$ we should expect that

$$\sum_{n \equiv r(q)} \tau_{M,N}(n) \sim \frac{MN}{q^2} \sum_{d \mid (q,r)} \sum_{c \mid \frac{q}{d}} dc \mu\left(\frac{q}{dc}\right). \tag{4.7}$$

In Manuscript D we show that (4.7) holds for most values of q and r if (q,r) is small. Indeed we have

$$\sum_{(r,q)=k} \left(\sum_{m \equiv r(q)} \tau_{M,N}(m) - \frac{MN}{q^2} \sum_{d \mid k} \sum_{c \mid \frac{q}{d}} dc \mu\left(\frac{q}{dc}\right) \right)^2 \ll \frac{N^{\max(\frac{7}{2}+\varepsilon, 4-\delta)}}{q}$$

uniformly for $q \le N^{2-\delta}$ for all $\varepsilon > 0$. From this we can deduce that (4.3) holds on average in the sense that for $K \ge N^\eta$ and $\sqrt{N} \le q \le N^{2-\delta}$ we have

$$\frac{1}{\varphi(q)} \sum_{(\rho,q)=1} \left(\frac{q}{KMN} \sum_{r \le K} \sum_{m \equiv \rho r(q)} \tau_{M,N}(m) - 1 \right)^2 \ll N^{-\min(1/2,\delta,2\eta)+\varepsilon}$$

for $\varepsilon > 0$.

The function $\tau_{M,N}$ is complicated. There is another function of interest

$$\tau_N(m) = \#\{d \in \mathbf{N} \mid d \le N,\ d \mid m\},$$

which is simpler. It is elementary to show that

$$\sum_{m \le x} \tau_N(m) = x \log N + O(N + x).$$

The estimate corresponding to (4.5) holds. More precisely we have

$$\sum_{\substack{m \equiv r(q) \\ m \le x}} \tau_N(m) \ll \frac{\varphi(q) x \log N}{q^2}.$$

for $(r,q) = 1$, $N \ge q^\kappa$ and $x \ge q^{1+\delta}$. This is proved using a result due to M. Nair and G. Tenenbaum [**34**]. For $\tau_{M,N}$ one should expect that

$$\sum_{m \equiv r(q)} \tau_{M,N}(m) \ll \frac{\varphi(q) MN}{q^2}. \tag{4.8}$$

for $(r,q) = 1$ and $N^{2-\delta} \le q$. This conjecture was also mentioned by Heath-Brown [**15**]. Though the function $\tau_{M,N}$ is complicated (4.8) seems (in the words of Heath-Brown) more "pure" than the Linnik-Vinogradov estimate (4.5) since in that problem we also have to deal with the Dirichlet divisor problem. A serious obstacle (at least when it comes to using the ideas of Nair and Tenenbaum) seems to be that $\tau_N(mn) \le \tau_N(m)\tau(n)$, $(m,n) = 1$ is a "good" estimate, while $\tau_{M,N}(mn) \le \tau_{M,N}(m)\tau(n)$, $(m,n) = 1$ is a "bad" estimate. It appears that the best estimate which can be derived using the Nair-Tenenbaum strategy is

$$\sum_{m \equiv r(q)} \tau_{M,N}(m) \ll \frac{N^2 (\log N)^\varepsilon}{q}$$

for $q \le N^{2-\delta}$, $(r,q) = 1$. This is obtained by estimating $\tau_{M,N}$ with the Hooley Δ-function

$$\Delta(n) = \max_{u \in \mathbf{R}} \#\{d \in \mathbf{N} \mid e^u < d \le e^{u+1},\ d \mid n\}.$$

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Manuscripts

MANUSCRIPT A

Quantum Unique Ergodicity of Eisenstein Series on the Hilbert Modular Group over a Totally Real Field

QUANTUM UNIQUE ERGODICITY OF EISENSTEIN SERIES ON THE HILBERT MODULAR GROUP OVER A TOTALLY REAL FIELD

JIMI L. TRUELSEN

ABSTRACT. W. Luo and P. Sarnak have proved the quantum unique ergodicity property for Eisenstein series on $\mathrm{PSL}(2, \mathbf{Z})\backslash\mathbf{H}$. Their result is quantitative in the sense that they find the precise asymptotics of the measure considered. We extend their result to Eisenstein series on $\mathrm{PSL}(2, \mathcal{O})\backslash\mathbf{H}^n$, where $\mathcal{O}$ is the ring of integers in a totally real field of degree n over $\mathbf{Q}$ with narrow class number one, using the Eisenstein series considered by I. Efrat. We also give an expository treatment of the theory of Hecke operators on non-holomorphic Hilbert modular forms.

1. INTRODUCTION

Let $\mathbf{H}$ denote the upper half-plane and Γ be a Fuchsian group of the first kind. We equip the surface $\Gamma\backslash\mathbf{H}$ with the measure induced by the Poincaré measure $d\mu = \frac{dxdy}{y^2}$ on $\mathbf{H}$. If Γ is hyperbolic we know that the quotient $\Gamma\backslash\mathbf{H}$ is compact and that the Laplace-Beltrami operator Δ associated with this surface, given in local coordinates by $-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, has pure point spectrum

$$0 = \lambda_0 < \lambda_1 \leq \dots$$

and that $\lambda_n \to \infty$ as $n \to \infty$. Inspired by quantum chaos (see [19] and [20] for excellent surveys) Z. Rudnick and P. Sarnak [18] conjectured that

$$|\varphi_j|^2 d\mu \to \frac{1}{\mu(\Gamma\backslash\mathbf{H})} d\mu, \tag{1.1}$$

where $\{\varphi_j\}$ is an orthonormal basis for $L^2(\Gamma\backslash\mathbf{H})$ of eigenfunctions of Δ with $\Delta\varphi_j = \lambda_j\varphi_j$, and the convergence is in the weak-$*$ topology. This is known as the quantum unique ergodicity conjecture. It has been established by Y. Colin de Verdière [3], A. Shnirelman [21] and S. Zelditch [28] that (1.1) holds for a subsequence of full density.

If $\Gamma = \mathrm{PSL}(2, \mathbf{Z})$ the quotient $\Gamma\backslash\mathbf{H}$ is no longer compact, and Δ does not have pure point spectrum. However, by the Weyl law it is known that

$$\#\{j \in \mathbf{N}_0 \mid |t_j| \leq T\} \sim \frac{\mu(\Gamma\backslash\mathbf{H})}{4\pi} T^2,$$

where $\lambda_j = 1/4 + t_j^2$ are the eigenvalues of Δ. Thus the analogue of the quantum unique ergodicity conjecture is

$$|\varphi_j|^2 d\mu \to \frac{3}{\pi} d\mu$$

where $\{\varphi_j\}$ is a complete set of orthonormal eigenfunctions of Δ. It was proved in [14] that if the φ_j's are Hecke eigenforms then the conjecture is true for a (large) subsequence of the full sequence and K. Soundararajan [26] has proved that the conjecture holds for the full sequence.

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In [14] a continuous spectrum analogue of the quantum unique ergodicity conjecture was proved. More precisely it was proved that for $A, B \subset \Gamma\backslash\mathbf{H}$ compact and Jordan measurable, such that $\mu(B) \neq 0$, we have the limit

$$\frac{\int_A |E(z, 1/2+it)|^2 d\mu}{\int_B |E(z, 1/2+it)|^2 d\mu} \to \frac{\mu(A)}{\mu(B)}$$

as $t \to \infty$, where $E(z, s)$ is the Eisenstein series on $\mathrm{PSL}(2, \mathbf{Z})$. The authors even found explicit asymptotics for the measure $|E(z, 1/2+it)|^2 d\mu$ (in terms of integration of a continuous function with compact support). In this paper we generalize this result to Eisenstein series $E(z, s, m)$ (it will be defined in Section 11) on $\Gamma\backslash\mathbf{H}^n$, where $\Gamma = \mathrm{PSL}(2, \mathcal{O})$ and $\mathcal{O}$ is the ring of integers in a totally real field K of degree n over $\mathbf{Q}$ with narrow class number one. Note that instead of just one Eisenstein series as in the case of $\mathrm{PSL}(2, \mathbf{Z})$ we have a family of Eisenstein series parametriced by $m \in \mathbf{Z}^{n-1}$.

We investigate the asymptotic behaviour of the measure $d\mu_{m,t} = |E(z, 1/2+it, m)|^2 d\mu$, where μ is the measure on $\Gamma\backslash\mathbf{H}^n$ induced by the measure $\frac{dx_1 \ldots dx_n dy_1 \ldots dy_n}{y_1^2 \cdots y_n^2}$ on $\mathbf{H}^n$:

Theorem 1.1. *For $F \in C_c(\Gamma\backslash\mathbf{H}^n)$ we have that*

$$\frac{1}{\log t} \int_{\Gamma\backslash\mathbf{H}^n} F(z) d\mu_{m,t}(z) \to \frac{\pi^n n R}{2D\zeta_K(2)} \int_{\Gamma\backslash\mathbf{H}^n} F(z) d\mu(z)$$

as $t \to \infty$, where ζ_K denotes the Dedekind zeta-function and D and R denote the discriminant and regulator of K, respectively.

From this one easily deduces that:

Theorem 1.2. *Let $A, B \subset \Gamma\backslash\mathbf{H}^n$ be compact and Jordan measurable, and assume that $\mu(B) \neq 0$. Then*

$$\frac{\mu_{m,t}(A)}{\mu_{m,t}(B)} \to \frac{\mu(A)}{\mu(B)}$$

as $t \to \infty$.

To prove Theorem 1.1 we follow the same strategy as in [14]. The idea in the proof is to find the asymptotics of $\int_{\Gamma\backslash\mathbf{H}^n} f d\mu_{m,t}$, where f is either an incomplete Eisenstein series or a Hecke eigenform, and then use the spectral decomposition of $L^2(\Gamma\backslash\mathbf{H}^n)$. Estimates for various L-functions play a crucial role in the proof, and we will collect these results, as we go along. It should be mentioned that a similar result was shown in the case of a quadratic imaginary field with class number one in [13] using a subconvexity estimate (in the t-aspect) for the standard L-function proved in [17].

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2. Notation and Terminology

Let K be a totally real field of degree n over $\mathbf{Q}$ and narrow class number one (these are the standard assumptions which are usually made to work with a non-adelic setup in textbooks such as [1] and [6]) and let $\mathcal{O}$ denote the ring of integers in K. Here narrow class number one means that $\mathcal{O}$ is a principal ideal domain and that each non-zero ideal in $\mathcal{O}$ has a generator which is totally positive (this term is explained below).

Let

$$\mathrm{Gal}(K/\mathbf{Q}) = \{\psi_1, \ldots, \psi_n\} \tag{2.1}$$

with ψ_1 equal to the identity map on K. In this way we may regard $\mathcal{O}$ as a lattice in $\mathbf{R}^n$, by the injection $\mathcal{O} \hookrightarrow \mathbf{R}^n$ defined by $a \mapsto (a^{(1)}, \ldots, a^{(n)})$, where $a^{(j)} = \psi_j(a)$. Note that this embedding depends on the choice of ordering of the elements in $\mathrm{Gal}(K/\mathbf{Q})$ given in (2.1).

We let $\mathcal{O}^\times$ denote the group of units in $\mathcal{O}$ and $\mathcal{O}^* = \mathcal{O} - \{0\}$. The elements in $\mathcal{O}^*$ for which all the embeddings are positive (such elements are called totally positive) will be denoted $\mathcal{O}_+$. We let $\mathcal{O}_+^\times = \mathcal{O}_+ \cap \mathcal{O}^\times$ which clearly is a multiplicative group.

We let $\mathcal{D}$ denote the different, i.e. the inverse ideal of

$$\mathcal{D}^{-1} = \{v \in K \mid \mathrm{Tr}(v\mathcal{O}) \subset \mathbf{Z}\}.$$

It is a well known fact that $\mathcal{D}^{-1} \supset \mathcal{O}$ is a fractional ideal, and since K has narrow class number one there exists $\omega \in \mathcal{O}_+$ such that $\mathcal{D} = (\omega) = \omega\mathcal{O}$ and $\mathcal{D}^{-1} = \omega^{-1}\mathcal{O}$.

It is well known that $\mathcal{O}$ is a free abelian group of rank n, and $\mathcal{O}^\times/\{\pm 1\}$ is a free abelian group of rank $n-1$. In addition we know that for each $u \in \mathcal{O}^\times$ we have $|u^{(1)} \dots u^{(n)}| = 1$. We will assume that $\varepsilon_1, \dots, \varepsilon_{n-1} \in \mathbf{R}_+$ together with -1 generate $\mathcal{O}^\times$. For later use let

$$\begin{pmatrix} e_{1,1} & \cdots & e_{1,n-1} & 1/n \\ \cdots & \cdots & \cdots & \cdots \\ e_{n,1} & \cdots & e_{n,n-1} & 1/n \end{pmatrix} = \begin{pmatrix} \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log|\varepsilon_{n-1}^{(1)}| & \cdots & \log|\varepsilon_{n-1}^{(n)}| \\ 1 & \cdots & 1 \end{pmatrix}^{-1}. \tag{2.2}$$

Note that we have the relations

$$\sum_{j=1}^{n} e_{j,q} = 0, \tag{2.3}$$

and

$$\sum_{j=1}^{n} e_{j,q'} \log|\varepsilon_q^{(j)}| = \delta_{q,q'} \tag{2.4}$$

for $q, q' = 1, \dots, n-1$.

We let $\mathbf{H}$ denote the upper half-plane of $\mathbf{C}$, i.e.

$$\mathbf{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}.$$

We will use the convention $z = (z_1, \dots, z_n) \in \mathbf{H}^n$ and $z = (x, y)$ where $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbf{R}_+^n$. Furthermore we will use the notation $dx = dx_1 \dots dx_n$ and $dy = dy_1 \dots dy_n$.

We set $\Gamma = \mathrm{PSL}(2, \mathcal{O}) \subset \mathrm{PSL}(2, \mathbf{R})$. This group is often referred to as the Hilbert modular group. The group Γ does not in general embed discretely in $\mathrm{PSL}(2, \mathbf{R})$, but it does embed discretely in $\mathrm{PSL}(2, \mathbf{R})^n$ by the action on $\mathbf{H}^n$ defined by

$$\pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \dots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}} \right)$$

which clearly is an extension of the classical action of $\mathrm{PSL}(2, \mathbf{Z})$ on $\mathbf{H}$ by Möbius transformations. For $\gamma = \pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathcal{O})$ we define $\gamma^{(j)} = \pm \left(\begin{smallmatrix} a^{(j)} & b^{(j)} \\ c^{(j)} & d^{(j)} \end{smallmatrix}\right)$.

If we regard $\mathbf{H}^n$ as a Riemannian manifold with the metric

$$ds^2 = \frac{dx_1^2 + dy_1^2}{y_1^2} + \dots + \frac{dx_n^2 + dy_n^2}{y_n^2}$$

the Laplace-Beltrami operator associated with this metric is

$$\Delta = \Delta_1 + \dots + \Delta_n$$

where $\Delta_j = -y_j^2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right)$. In the natural way the metric on $\mathbf{H}^n$ transfers to the quotient $\Gamma \backslash \mathbf{H}^n$. We also see that the Δ_j's induce symmetric and positive differential operators on $C_b^\infty(\Gamma\backslash\mathbf{H}^n)$ which admit self-adjoint extensions (the Friedrichs extension). It is known that the quotient $\Gamma\backslash\mathbf{H}^n$ has finite volume and as in the case $n=1$ we will often regard functions on $\Gamma\backslash\mathbf{H}^n$ as functions on the space $\mathbf{H}^n$ which are invariant under Γ. The measure on $\Gamma\backslash\mathbf{H}^n$ induced by the Riemannian metric is denoted μ and one can check that $d\mu=\frac{dxdy}{y_1^2\cdots y_n^2}$ in local coordinates.

3. The Hecke L-function

In the following it will be convenient to set $\rho_j(m)=\pi\sum_{q=1}^{n-1}m_qe_{j,q}$ for $m\in\mathbf{Z}^{n-1}$. Let χ_m denote the following function on $\mathbf{C}^{*n}$:

$$\chi_m(w)=\exp\left(i\pi\sum_{q=1}^{n-1}m_q\sum_{j=1}^{n}e_{j,q}\log|w_j|\right)=\prod_{j=1}^{n}|w_j|^{i\rho_j(m)}. \tag{3.1}$$

Clearly we can regard χ_m as a multiplicative function on $\mathcal{O}^*$ by the usual embedding. For $\beta\in\mathcal{O}_+$ we note that $\chi_m(\beta)$ only depends on the ideal (β), so in this way we can regard χ_m as a multiplicative function on the non-zero ideals in $\mathcal{O}$ (a so-called Grössencharacter). Note also that for m even, χ_m is trivial on $\mathcal{O}^\times$. We can now define the Hecke L-function. It is defined by the series

$$\zeta(s,m)=\sum_{\substack{\mathfrak{a}\subset\mathcal{O}\\ \mathfrak{a}\neq 0}}\frac{\chi_m(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s},$$

which converges absolutely for $\mathrm{Re}(s)>1$, and it can also be written as an Euler product over the prime ideals $\mathfrak{p}$, i.e.

$$\zeta(s,m)=\prod_{\mathfrak{p}}\left(1-\frac{\chi_m(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s}\right)^{-1}.$$

The Hecke L-function has a meromorphic continuation to the entire complex plane. Furthermore $\zeta(s,m)$ is entire if $m\neq 0$. The Dedekind zeta function $\zeta(s,0)$ (sometimes also denoted ζ_K) has a simple pole at $s=1$ with residue $\frac{2^{n-1}R}{\sqrt{D}}$ (cf. [1] Section 1.7), and is holomorphic elsewhere. Here $D=\mathcal{N}(\mathcal{D})=|N(\omega)|$ is the discriminant of K and R is the regulator of K, i.e. the absolute value of the determinant

$$\begin{vmatrix}\log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_1^{(n-1)}|\\ \cdots & \cdots & \cdots\\ \log|\varepsilon_{n-1}^{(1)}| & \cdots & \log|\varepsilon_{n-1}^{(n-1)}|\end{vmatrix}.$$

First we will make a convexity bound for the Hecke L-function on the line $\mathrm{Re}(s)=\sigma$, where $\frac{1}{2}\leq\sigma\leq 1$. It is well known (see [1] Theorem 1.7.2) that the Hecke L-function $\zeta(s,m)$ satisfies the functional equation

$$\xi(s,m)=\chi_m(\omega)i^{\mathrm{Tr}(\tau)}\xi(1-s,-m) \tag{3.2}$$

where $\xi(s,m)$ denotes the completed L-function defined by

$$\xi(s,m)=D^{s/2}\pi^{-ns/2}\zeta(s,m)\prod_{j=1}^{n}\Gamma\left(\frac{s+\tau_j-i\rho_j(m)}{2}\right),$$

and $\tau=(\tau_1,\ldots,\tau_n)$ is a binary vector depending on m with the property that

$$\chi_m((\beta))=\chi_m(\beta)\prod_{j=1}^{n}\mathrm{sgn}(\beta^{(j)})^{\tau_j} \tag{3.3}$$

for $\beta\in\mathcal{O}^*$.

Stirling's formula, i.e. the asymptotics of the Γ-function on vertical lines, plays a crucial role in the proof of Theorem 1.1. For any $\sigma \in \mathbf{R}$ we have

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}e^{-\pi|t|/2}|t|^{\sigma-1/2} \tag{3.4}$$

and

$$\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)} \sim \log|t| \tag{3.5}$$

as $|t| \to \infty$. Using the Phragmén-Lindelöf principle (see [12] Section 5.A), the functional equation (3.2) and Stirling's formula we easily derive the convexity bound

$$\zeta(\sigma+it,m) \ll |t|^{\frac{n}{2}(1-\sigma)+\varepsilon} \tag{3.6}$$

as $|t| \to \infty$, for any $\varepsilon > 0$ and $\frac{1}{2} \leq \sigma \leq 1$. Note that (3.6) gives the estimate

$$\zeta(1/2+it,m) \ll |t|^{\frac{n}{4}+\varepsilon}$$

for any $\varepsilon > 0$. For later use it turns out that we need something slightly better ($\frac{n}{4} - \varepsilon$ in the exponent will do), i.e. we need a subconvexity estimate for $\zeta(s,m)$ on the critical line. Such an estimate was proved by P. Söhne [24] (generalizing ideas due to D. R. Heath-Brown [8] and [9]):

Theorem 3.1. *Let $\varepsilon > 0$. Then*

$$\zeta(1/2+it,m) \ll |t|^{\frac{n}{6}+\varepsilon}$$

as $|t| \to \infty$.

It is conjectured (and implied by the generalized Riemann hypothesis) that one in fact has

$$\zeta(1/2+it,m) \ll |t|^{\varepsilon}$$

for any $\varepsilon > 0$ as $|t| \to \infty$.

It will also be necessary to estimate the logarithmic derivative of $\zeta(s,m)$ on the line $\mathrm{Re}(s) = 1$. We introduce a von Mangoldt type function on the non-zero ideals in $\mathcal{O}$ defined by

$$\Lambda_m(\mathfrak{a}) = \begin{cases} \chi_m(\mathfrak{a}) \log \mathcal{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^k \\ 0 & \text{otherwise} \end{cases},$$

where $\mathfrak{p}$ denotes a prime ideal. For $\mathrm{Re}(s) > 1$ we see using the Euler product that

$$\begin{aligned} -\frac{\zeta'(s,m)}{\zeta(s,m)} &= -\sum_{\mathfrak{p}} \left(1 - \frac{\chi_m(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s}\right) \frac{d}{ds}\left(\frac{1}{1-\frac{\chi_m(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s}}\right) \\ &= \sum_{\mathfrak{p}} \frac{\chi_m(\mathfrak{p}) \log \mathcal{N}(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s \left(1 - \frac{\chi_m(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s}\right)} \\ &= \sum_{\mathfrak{p}} \log \mathcal{N}(\mathfrak{p}) \sum_{k=1}^{\infty} \frac{\chi_m(\mathfrak{p})^k}{\mathcal{N}(\mathfrak{p})^{sk}} \\ &= \sum_{\mathfrak{a}} \frac{\Lambda_m(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}. \end{aligned}$$

Thus as in the case of the Riemann zeta-function the logarithmic derivative of $\zeta(s,m)$ can be written as a Dirichlet series.

To estimate the logarithmic derivative of the Hecke L-function we need a zero-free region. By considering exponential sums one can obtain a zero-free region for the Hecke

L-function similar to Vinogradov's bound for the Riemann zeta-function (see [27] Chapter 6). This was done by M. Coleman [2]:

Theorem 3.2. *There exist positive constants C and L such that $\zeta(\sigma+it,m)\neq 0$ for $|t|\geq L$ and $\sigma \geq 1-\frac{C}{(\log|t|)^{2/3}(\log\log|t|)^{1/3}}$.*

At present this is the best zero-free region we know, but the generalized Riemann hypothesis asserts that all zeros of $\zeta(s,m)$ in the critical strip $0<\mathrm{Re}(s)<1$ are on the line $\mathrm{Re}(s)=\frac{1}{2}$.

To obtain a sufficiently good estimate for the logarithmic derivative we follow Landau's strategy (cf. [27] Sections 3.9-3.11), which is based on the Borel-Carathéodory theorem. We remark that in order to use this approach it is necessary to estimate the Hecke L-function from below. To this end we consider the following generalization of the Möbius function to non-zero ideals in $\mathcal{O}$ defined by

$$\mu(\mathfrak{p}_1^{\alpha_1}\dots\mathfrak{p}_k^{\alpha_k}) = \begin{cases} (-1)^k & \text{if } \alpha_1,\dots,\alpha_k\leq 1 \\ 0 & \text{otherwise}\end{cases}.$$

The function μ has the following property ("Möbius inversion"):

$$\sum_{\mathfrak{b}\subset\mathfrak{a}}\mu(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{b}=\mathcal{O} \\ 0 & \text{otherwise}\end{cases}, \tag{3.7}$$

and the proof is the same as in the classical case (see [12] Section 1.3). From this it is clear that

$$\frac{1}{\zeta(s,m)} = \sum_{\mathfrak{a}}\frac{\chi_m(\mathfrak{a})\mu(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s} \tag{3.8}$$

for $\mathrm{Re}(s)>1$. Thus

$$\frac{1}{|\zeta(\sigma+it,m)|}\leq \zeta(\sigma,0)$$

for $\sigma>1$.

We have the following result due to Landau:

Proposition 3.3. *Let $s=\sigma+it$ and assume that $\zeta(s,m)=O(e^{\varphi(|t|)})$ for $|t|\geq L$ and $1-\theta(|t|)\leq\sigma\leq 2$ for some positive L, where $\varphi(t)$ and $1/\theta(t)$ are positive increasing functions defined for $t\geq L$ such that $\theta(t)\leq 1$, $\varphi(t)\to\infty$ as $t\to\infty$ and $\varphi(t)/\theta(t) = o(e^{\varphi(t)})$. Assume also that there exists a positive constant C such that $\zeta(s,m)\neq 0$ for $|t|\geq L$ and $\sigma\geq 1-C\frac{\theta(|t|)}{\varphi(|t|)}$. Then*

$$\frac{\zeta'(s,m)}{\zeta(s,m)} = O\left(\frac{\varphi(|t|)}{\theta(|t|)}\right)$$

and

$$\frac{1}{\zeta(s,m)} = O\left(\frac{\varphi(|t|)}{\theta(|t|)}\right)$$

for $|t|\geq L+1$ and $\sigma\geq 1-\frac{C\theta(t)}{4\varphi(t)}$.

Using Theorem 3.2 we can apply Proposition 3.3 with $\varphi(t)=(\log t)^{\frac{2}{3}}$ and $\theta(t) = (\log\log t)^{-\frac{1}{3}}$ to obtain the following:

Corollary 3.4. *There exists a positive number L such that for $|t|\geq L$ we have the estimate*

$$\frac{\zeta'(1+it,m)}{\zeta(1+it,m)} = O((\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}).$$

In the same way we obtain an explicit lower bound for the Hecke L-function:

Corollary 3.5. *There exists a positive number L such that for $|t| \geq L$ we have the estimate*

$$\frac{1}{\zeta(1+it,m)} = O((\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}).$$

4. Hecke Operators

In this section we give an expository treatment of the theory of Hecke operators on non-holomorphic Hilbert modular forms analogous to the treatment of Hecke operators on holomorphic Hilbert modular forms in [1] Section 1.7 and [6] Section 1.15.

We recall the abstract definition of the Hecke ring (see [22]). We set $G = \mathrm{GL}(2,K)$, $\Gamma = \mathrm{SL}(2,\mathcal{O})$ and let $\mathfrak{D} \subset \mathrm{GL}(2,K)$ denote the 2×2 matrices with entries in $\mathcal{O}$ and totally positive determinant. The Hecke algebra $\mathrm{R}(\Gamma,\mathfrak{D})$ is the $\mathbf{C}$-vector space of finite formal sums $\sum_k c_k\Gamma\alpha_k\Gamma$, where $\alpha_k \in \mathfrak{D}$ and $c_k \in \mathbf{C}$. The addition in $\mathrm{R}(\Gamma,\mathfrak{D})$ is the obvious one, while the multiplication is defined as follows. Let $\alpha,\beta \in \mathfrak{D}$. It is well known that there exist distinct cosets $\Gamma\alpha_1,\ldots,\Gamma\alpha_d$ and $\Gamma\beta_1,\ldots,\Gamma\beta_{d'}$, where $\alpha_i,\beta_{i'} \in \mathfrak{D}$, such that $\Gamma\alpha\Gamma = \cup_{i=1}^{d}\Gamma\alpha_i$ and $\Gamma\beta\Gamma = \cup_{i'=1}^{d'}\Gamma\beta_{i'}$. We define $\Gamma\alpha\Gamma\cdot\Gamma\beta\Gamma = \sum_{i,i'}\Gamma\alpha_i\beta_{i'}\Gamma$, which clearly is independent of the choice of the α_i's and $\beta_{i'}$'s. We extend this multiplication in the obvious way, making $\mathrm{R}(\Gamma,\mathfrak{D})$ an algebra.

We can define a homomorphism from $\mathrm{GL}(2,\mathbf{R})_+$ to $\mathrm{PSL}(2,\mathbf{R})$ by mapping $\tau = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbf{R})_+$ to $w \mapsto \frac{aw+b}{cw+d}$ in $\mathrm{PSL}(2,\mathbf{R})$. Thus for $w \in \mathbf{H}$ we simply define

$$\tau w = \frac{aw+b}{cw+d}.$$

Therefore we get a map from $\mathfrak{D}$ to $\mathrm{PSL}(2,\mathbf{R})^n$ and we see that $\mathrm{R}(\Gamma,\mathfrak{D})$ can be regarded as an algebra of operators on $L^2(\Gamma\backslash\mathbf{H}^n)$ (or even the vector space of automorphic functions) if we define $(\Gamma\alpha\Gamma f)(z) = \sum_{i=1}^{d} f(\alpha_i z)$.

Two double cosets $\Gamma\alpha\Gamma$ and $\Gamma\beta\Gamma$ are said to be equivalent if $\alpha = \eta\beta$ where $\eta = \left(\begin{smallmatrix} u & 0\\ 0 & u\end{smallmatrix}\right)$ for some $u \in \mathcal{O}^\times$. Note that if $\alpha = \eta\beta$ then $\alpha^{(j)}z_j = \beta^{(j)}z_j$ for all $j = 1,\ldots,n$. Let $\nu \in \mathcal{O}_+$. Inspired by Hecke operators in the case of holomorphic Hilbert modular forms (see [6]) we define

$$T_\nu f = \frac{1}{\sqrt{|N(\nu)|}} \sum_{\substack{\det\alpha = u\nu\\ u\in\mathcal{O}_+^\times}} \Gamma\alpha\Gamma f. \tag{4.1}$$

Here the sum should be taken over inequivalent double cosets.

We can use the class number one assumption to make this more explicit. Consider $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathfrak{D}$. Write $a = ra'$ and $c = rc'$ where a' and c' are relative prime (i.e. $(a') + (c') = \mathcal{O}$). There exist $b',d' \in \mathcal{O}$ such that $a'd' - b'c' = 1$ and we see that

$$\begin{pmatrix} d' & -b'\\ -c' & a'\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

is upper triangular. Thus for any $\alpha \in \mathfrak{D}$ we can find $\beta \in \mathfrak{D}$, which is upper triangular and satisfies that $\Gamma\alpha = \Gamma\beta$. Using this we can write the Hecke operator as follows

$$T_\nu f(z) = \frac{1}{\sqrt{|N(\nu)|}} \sum_{\substack{ad=u\nu\\ u\in\mathcal{O}_+^\times}} \sum_{b\in\mathcal{O}/(d)} f\left(\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right) z\right). \tag{4.2}$$

The outer sum is finite by unique factorization and the inner sum is finite since $\mathcal{O}/(d)$ is a finite group. Thus $T_\nu f$ is well defined.

If $\nu,\nu' \in \mathcal{O}_+$ and $\nu = u\nu'$ for some $u \in \mathcal{O}_+^\times$ then by definition $T_\nu = T_{\nu'}$. Thus we can define $T_{(\nu)} = T_\nu$. Since we assumed that all ideals have a generator in $\mathcal{O}_+$ there is a Hecke operator associated with each non-zero ideal. Modifying Theorem 3.12.4 in [7] we obtain that the Hecke operators are self-adjoint.

Now we will investigate the properties of the Hecke operators. We have the following proposition:

Proposition 4.1. *Let $\nu_1, \nu_2 \in \mathcal{O}_+$ be relative prime. Then*

$$T_{\nu_1}T_{\nu_2} = T_{\nu_1\nu_2}.$$

Proof. Let $f \in L^2(\Gamma\backslash\mathbf{H}^n)$. We have that

$$\begin{aligned}
\sqrt{|N(\nu_1\nu_2)|}(T_{\nu_1}T_{\nu_2}f)(z) &= \sum_{\substack{a_1d_1=u_1\nu_1\\ u_1\in\mathcal{O}_+^\times}} \sum_{\substack{a_2d_2=u_2\nu_2\\ u_2\in\mathcal{O}_+^\times}} \sum_{\substack{b_1\in\mathcal{O}/(d_1)\\ b_2\in\mathcal{O}/(d_2)}} f\left(\begin{pmatrix} a_2 & b_2\\ 0 & d_2\end{pmatrix}\begin{pmatrix} a_1 & b_1\\ 0 & d_1\end{pmatrix} z\right)\\
&= \sum_{\substack{a_1d_1=u_1\nu_1\\ u_1\in\mathcal{O}_+^\times}} \sum_{\substack{a_2d_2=u_2\nu_2\\ u_2\in\mathcal{O}_+^\times}} \sum_{\substack{b_1\in\mathcal{O}/(d_1)\\ b_2\in\mathcal{O}/(d_2)}} f\left(\begin{pmatrix} a_1a_2 & b_1a_2+d_1b_2\\ 0 & d_1d_2\end{pmatrix} z\right)\\
&= \sum_{\substack{ad=u\nu_1\nu_2\\ u\in\mathcal{O}_+^\times}} \sum_{b\in\mathcal{O}/(d)} f\left(\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} z\right)\\
&= \sqrt{|N(\nu_1\nu_2)|}(T_{\nu_1\nu_2}f)(z)
\end{aligned}$$

where we have used the Chinese remainder theorem, i.e. that

$$\mathcal{O}/((d_1)\cap(d_2)) \cong \mathcal{O}/(d_1)\oplus\mathcal{O}/(d_2)$$

which holds since $(\nu_1)+(\nu_2)=\mathcal{O}$. □

We need the following important lemma:

Lemma 4.2. *Let $p \in \mathcal{O}_+$ be a prime element. Then for any positive integers k, k' we have*

$$T_{p^k}T_{p^{k'}} = \sum_{d=0}^{\min\{k,k'\}} T_{p^{k+k'-2d}}.$$

Proof. Let $f \in L^2(\Gamma\backslash\mathbf{H}^n)$. We see that

$$\begin{aligned}
\sqrt{|N(p^{k+k'})|}(T_{p^k}T_{p^{k'}}f)(z) &= \sum_{\substack{l_1+l_2=k\\ l_1'+l_2'=k'}} \sum_{\substack{b\in\mathcal{O}/(p^{l_2})\\ b'\in\mathcal{O}/(p^{l_2'})}} f\left(\begin{pmatrix} p^{l_1} & b\\ 0 & p^{l_2}\end{pmatrix}\begin{pmatrix} p^{l_1'} & b'\\ 0 & p^{l_2'}\end{pmatrix} z\right)\\
&= \sum_{\substack{l_1+l_2=k\\ l_1'+l_2'=k'}} \sum_{\substack{b\in\mathcal{O}/(p^{l_2})\\ b'\in\mathcal{O}/(p^{l_2'})}} f\left(\begin{pmatrix} p^{l_1+l_1'} & bp^{l_2'}+b'p^{l_1}\\ 0 & p^{l_2+l_2'}\end{pmatrix} z\right).
\end{aligned}$$

Removing common factors we get

$$\sum_{d=0}^{\min\{k,k'\}} \sum_{\substack{l_1+l_2=k-d\\ l_1'+l_2'=k'-d\\ \min\{l_2',l_1\}=0}} \sum_{b\in\mathcal{O}/(p^{l_2})} \sum_{b'\in\mathcal{O}/(p^{l_2'+d})} f\left(\begin{pmatrix} p^{l_1+l_1'} & bp^{l_2'}+b'p^{l_1}\\ 0 & p^{l_2+l_2'}\end{pmatrix} z\right).$$

We note that as (b,b') runs over all pairs in $\mathcal{O}/(p^{l_2})\times\mathcal{O}/(p^{l_2'+d})$ the expression $bp^{l_2'}+b'p^{l_1}$ will assume each value in $\mathcal{O}/(p^{l_2+l_2'})$ exactly $|N(p)|^d$ times. Thus

$$\sqrt{|N(p^{k+k'})|}T_{p^k}T_{p^{k'}} = \sum_{d=0}^{\min\{k,k'\}} |N(p^d)|\sqrt{|N(p^{k+k'-2d})|}T_{p^{k+k'-2d}},$$

and this proves the theorem. □

Combining Proposition 4.1 and Lemma 4.2 we get:

Theorem 4.3. *Let $(\nu_1), (\nu_2)$ be non-zero ideals in $\mathcal{O}$. Then*

$$T_{(\nu_1)}T_{(\nu_2)} = \sum_{(d)\supset(\nu_1)+(\nu_2)} T_{(\nu_1)(\nu_2)/(d)^2}.$$

In particular the Hecke operators commute.

From Lemma 4.2 we obtain the following proposition:

Proposition 4.4. *Let $p \in \mathcal{O}_+$ be a prime element. Then for $k \in \mathbf{N}_0$ we have that*

$$T_{p^{2k}} = \sum_{l=0}^{k} (-1)^{k+l} \binom{k+l}{2l} T_p{}^{2l} \tag{4.3}$$

and

$$T_{p^{2k+1}} = \sum_{l=1}^{k+1} (-1)^{k+l+1} \binom{k+l}{2l-1} T_p{}^{2l-1}. \tag{4.4}$$

Proof. We first consider (4.3). The claim certainly holds for $k = 0$ and $k = 1$. Now let $k' \geq 2$ be an integer and assume that the formula holds for $k \leq k'$. Using Lemma 4.2 we get

$$\begin{aligned}
T_{p^{2k'+2}} &= T_{p^{2k'}}T_{p^2} - T_{p^{2k'}} - T_{p^{2k'-2}} \\
&= (T_p^2 - 2)T_{p^{2k'}} - T_{p^{2k'-2}} \\
&= T_p^{2k'+2} - (2k'+1)T_p^{2k'} + (-1)^{k'+1} + \\
&\quad \sum_{l=1}^{k'-1} (-1)^{k'+l+1} \left(2\binom{k'+l}{2l} + \binom{k'+l-1}{2l-2} - \binom{k'+l-1}{2l} \right) T_p{}^{2l} \\
&= \sum_{l=0}^{k'+1} (-1)^{k'+l+1} \binom{k'+l+1}{2l} T_p{}^{2l}.
\end{aligned}$$

By induction this proves (4.3), and (4.4) is proved by similar arguments. □

5. The Fourier Expansion of an Automorphic Form

An automorphic form f is a formal eigenfunction of the Laplacians Δ_j (i.e. f need not be square integrable and we allow f to be identically zero also) which satisfies the growth condition

$$f(z) = o(\exp(2\pi y_j))$$

as $y_j \to \infty$ for all $j = 1, \ldots, n$. This holds in particular if f is square integrable. By construction we have $f(z + l) = f(z)$ for all $l \in \mathcal{O}$. Thus f has a Fourier expansion (see [10]):

Theorem 5.1. *Let f be an automorphic form with Laplace eigenvalues $s_j(1 - s_j)$. Then f admits a Fourier expansion of the form*

$$f(z) = \sum_{l \in \mathcal{O}} a_l(y) e(\mathrm{Tr}(\omega^{-1} l x)),$$

where $e(x) = \exp(2\pi i x)$. Since $f(z)$ is an eigenfunction for the Laplacians $\Delta_1, \dots, \Delta_n$ the l-th Fourier coefficient $a_l(y)$ must satisfy the differential equations

$$\frac{\partial^2 a_l(y)}{\partial y_j^2} + \left(\frac{s_j(1-s_j)}{y_j^2} - 4\pi^2 |(\omega^{-1} l)^{(j)}|^2 \right) a_l(y) = 0 \tag{5.1}$$

for $j = 1, \dots, n$ and hence be of the form

$$a_l(y) = c_l \sqrt{y_1 \dots y_n} \prod_{j=1}^{n} K_{s_j - \frac{1}{2}}(2\pi |(\omega^{-1} l)^{(j)}| y_j)$$

for $l \neq 0$. The zeroth Fourier coefficient can be written as a linear combination of $\prod_{j=1}^n y_j^{s_j}$ and $\prod_{j=1}^n y_j^{1-s_j}$. Furthermore the coefficients c_l satisfy the bound

$$c_l \ll \exp(\varepsilon |N(l)|)$$

for any $\varepsilon > 0$.

Here K_ν denotes the usual Macdonald Bessel function

$$K_\nu(y) = \frac{1}{2} \int_0^\infty \exp(-y(t + 1/t)/2) t^{\nu - 1} dt,$$

which is defined for $y > 0$ and $\nu \in \mathbf{C}$. It is well known that these functions decay exponentially as $y \to \infty$.

Note that if f is automorphic with respect to Γ then $f(z) = f(uz)$ for $u \in \mathcal{O}_+$, where

$$uz = (u^{(1)} z_1, \dots, u^{(n)} z_n),$$

since all such u's are squares of units (by the assumption that K has narrow class number one). This implies that $c_l = c_{lu}$ for $l \in \mathcal{O}$ and $u \in \mathcal{O}_+^\times$.

A non-zero square integrable automorphic form f is called a cusp form if

$$\int_F f(z) dx = 0. \tag{5.2}$$

Here

$$F = \{ t_1 a_1 + \dots + t_n a_n \mid 0 \leq t_j < 1 \}$$

where $a_1, \dots, a_n$ is a $\mathbf{Z}$-basis for $\mathcal{O}$ and each a_j is regarded as a vector in $\mathbf{R}^n$ by the embedding $a_j \mapsto (a_j^{(1)}, \dots, a_j^{(n)})$. We will refer to F as the fundamental mesh for $\mathcal{O}$ and one can check that the definition of cuspidal is independent of the choice of $\mathbf{Z}$-basis. By the exponential decay of the Macdonald Bessel function one can deduce that f must be of exponential decay as $y_j \to \infty$.

Using the Hilbert-Schmidt kernel from [5] Section II.9 one can prove using Lemma I.2.1 in [5] that the vector space of square integrable automorphic forms with given Laplace eigenvalues $\lambda_1, \dots, \lambda_n$ is finite dimensional (see [10] for bounds on the dimensions of the eigenspaces). Now define

$$\iota_j(z) = (z_1, \dots, z_{j-1}, -\overline{z_j}, z_{j+1}, \dots, z_n)$$

for $j = 1, \dots, n$. One easily checks that if f is an automorphic form then so is $f \circ \iota_j$ with the same Laplace eigenvalues. Since the eigenspaces are finite dimensional this means that the eigenvalues of ι_j must be ± 1. We also see that the Hecke operators, the Δ_j's and the ι_j's commute. Furthermore all these operators are self-adjoint. Hence we can choose a basis for the vector space spanned by cusp forms which consists of cusp forms that are also eigenfunctions for all the Hecke operators and all the ι_j's. These are called primitive cusp forms. Note that being an eigenfunction of the ι_j's is simply the same as saying that the function is either even or odd in each x_j.

6. Hecke Eigenvalues and Automorphic Forms

In this section we will study automorphic forms which are common eigenfunctions for all the Hecke operators. We first note that the identities derived in Theorem 4.3 and Proposition 4.4 give similar identities for the Hecke eigenvalues:

Theorem 6.1. *Assume that f is a common eigenfunction for all the Hecke operators, i.e. that*

$$T_{(\nu)}f = \lambda((\nu))f$$

for all $\nu \in \mathcal{O}^*$. Then for $\nu_1, \nu_2 \in \mathcal{O}^*$ we have

$$\lambda((\nu_1))\lambda((\nu_2)) = \sum_{(d)\supset(\nu_1)+(\nu_2)} \lambda((\nu_1\nu_2/d^2)). \tag{6.1}$$

For a prime element $p \in \mathcal{O}$ and $k \in \mathbf{N}_0$ we have that

$$\lambda((p^{2k})) = \sum_{l=0}^{k} (-1)^{k+l} \binom{k+l}{2l} \lambda((p))^{2l} \tag{6.2}$$

and

$$\lambda((p^{2k+1})) = \sum_{l=1}^{k+1} (-1)^{k+l+1} \binom{k+l}{2l-1} \lambda((p))^{2l-1}. \tag{6.3}$$

Using the identities above, we can derive a connection between the Fourier coefficients of $T_{(\nu)}f$ and f, where f is a primitive cusp form:

Theorem 6.2. *Let f be a primitive cusp form with Laplace eigenvalues $s_j(1-s_j)$, and assume that f has the Fourier expansion*

$$f(z) = \sum_{l\in\mathcal{O}^*} c_l \sqrt{y_1 \dots y_n} \left(\prod_{j=1}^{n} K_{s_j - \frac{1}{2}}(2\pi|(\omega^{-1}l)^{(j)}|y_j) \right) e(\mathrm{Tr}(\omega^{-1}lx)).$$

Then the l-th Fourier coefficient of $T_{(\nu)}f$ is

$$\sum_{\substack{d|\gcd(l',\nu) \\ l'\nu=d^2 l}} c_{l'}$$

for $\nu \in \mathcal{O}_+$. In particular $c_{\nu u} = \lambda((\nu))c_u$ for $u \in \mathcal{O}^\times$.

Proof. We apply T_ν on the Fourier expansion

$$\sqrt{|N(\nu)|}T_\nu f(z) = \sum_{l'\in\mathcal{O}^*} c_{l'} \sum_{\substack{ad=u\nu \\ u\in\mathcal{O}_+^\times}} \sqrt{\frac{|a^{(1)}|}{|d^{(1)}|}y_1 \cdots \frac{|a^{(n)}|}{|d^{(n)}|}y_n} \times$$
$$\left(\prod_{j=1}^{n} K_{s_j-\frac{1}{2}}(2\pi|(\omega^{-1}l'a/d)^{(j)}|y_j) \right) \sum_{b\in\mathcal{O}/(d)} e\left(\mathrm{Tr}\left(\omega^{-1}l'(ax+b)/d\right)\right),$$

where by abuse of notation

$$(ax+b)/d = ((a^{(1)}x_1 + b^{(1)})/d^{(1)}, \dots, (a^{(n)}x_n + b^{(n)})/d^{(n)}).$$

Now if $d \mid l'$ then

$$\sum_{b\in\mathcal{O}/(d)} e\left(\mathrm{Tr}\left(\omega^{-1}l'\frac{b}{d}\right)\right) = |N(d)|.$$

If $d \nmid l'$ there exist $b' \in \mathcal{O}/(d)$ such that $\mathrm{Tr}\left(\omega^{-1}l'\frac{b'}{d}\right) \notin \mathbf{Z}$. Thus

$$\sum_{b\in\mathcal{O}/(d)} e\left(\mathrm{Tr}\left(\omega^{-1}l'\frac{b}{d}\right)\right) = \sum_{b\in\mathcal{O}/(d)} e\left(\mathrm{Tr}\left(\omega^{-1}l'\frac{b+b'}{d}\right)\right)$$
$$= e\left(\mathrm{Tr}\left(\omega^{-1}l'\frac{b'}{d}\right)\right)\sum_{b\in\mathcal{O}/(d)} e\left(\mathrm{Tr}\left(\omega^{-1}l'\frac{b}{d}\right)\right).$$

But this implies that

$$\sum_{b\in\mathcal{O}/(d)} e\left(\mathrm{Tr}\left(\omega^{-1}l'\frac{b}{d}\right)\right) = 0.$$

Thus

$$\sqrt{|N(\nu)|}T_{(\nu)}f(z) = \sum_{l'\in\mathcal{O}^*} c_{l'} \sum_{\substack{ad=u\nu\\ u\in\mathcal{O}_+^\times}} \sqrt{\frac{|a^{(1)}|}{|d^{(1)}|}y_1\cdots\frac{|a^{(n)}|}{|d^{(n)}|}y_n}\times$$
$$\left(\prod_{j=1}^n K_{s_j-\frac{1}{2}}(2\pi|(\omega^{-1}l'\nu/d^2)^{(j)}|y_j)\right)\times$$
$$\sum_{b\in\mathcal{O}/(d)} e\left(\mathrm{Tr}\left(\omega^{-1}l'(ax+b)/d\right)\right)$$
$$= \sum_{l'\in\mathcal{O}^*} c_{l'} \sum_{d|\gcd(l',\nu)} |N(d)|\sqrt{\frac{|\nu^{(1)}|}{|d^{(1)}|^2}y_1\cdots\frac{|\nu^{(n)}|}{|d^{(n)}|^2}y_n}\times$$
$$\left(\prod_{j=1}^n K_{s_j-\frac{1}{2}}(2\pi|(\omega^{-1}l'\nu/d^2)^{(j)}|y_j)\right)\times$$
$$e\left(\mathrm{Tr}\left(\omega^{-1}l'\frac{\nu}{d^2}x\right)\right).$$

From this it is clear that the l-th Fourier coefficient is

$$\sum_{\substack{d|\gcd(l',\nu)\\ l'\nu=d^2l}} c_{l'}.$$

□

7. The Fundamental Domain for Γ_∞

Before we can prove the functional equation for the standard L-function we need a fundamental domain for $\mathcal{O}_+^\times\backslash\mathbf{R}_+^n$ and this immediately gives us a fundamental domain for Γ_∞ as well.

Let F denote the interior of the fundamental mesh of the lattice $\mathcal{O}$ in $\mathbf{R}^n$ given by the embedding defined earlier. Let Γ_∞ denote the stabilizer subgroup at ∞, i.e.

$$\Gamma_\infty = \left\{\pm\left(\begin{smallmatrix} u & l\\ 0 & u^{-1}\end{smallmatrix}\right) \mid u\in\mathcal{O}^\times, l\in\mathcal{O}\right\}.$$

From [23] we know the fundamental domain for Γ_∞:

Proposition 7.1. *The set*

$$F_\infty = \{z\in\mathbf{H}^n \mid x\in F, y\in U_\infty\},$$

is a fundamental domain for Γ_∞. *Here* $U_\infty \subset \mathbf{R}_+^n$ *is a fundamental domain for* $\mathcal{O}_+^\times \backslash \mathbf{R}_+^n$. *Explicitly we can choose* U_∞ *to be the preimage of*

$$\mathbf{R}_+ \times [-1,1]^{n-1} \subset \mathbf{R}_+ \times \mathbf{R}^{n-1}$$

under the map (defined on $\mathbf{R}_+^n$*)*

$$y \mapsto \left(\prod_{j=1}^n y_j, \sum_{j=2}^n (e_{j,1} - e_{1,1}) \log \frac{y_j}{\sqrt[n]{\prod_{i=1}^n y_i}}, \dots, \sum_{j=2}^n (e_{j,n-1} - e_{1,n-1}) \log \frac{y_j}{\sqrt[n]{\prod_{i=1}^n y_i}} \right),$$

which is injective.

Let $\widetilde{y}$ denote the image of y under the map above. Note that we have the relations

$$\sum_{j=2}^n \widetilde{y}_j \log |\varepsilon_{j-1}^{(k)}| = \log \frac{y_k}{\sqrt[n]{\widetilde{y}_1}} \tag{7.1}$$

for $k = 2, \dots, n$ which follows since

$$\begin{pmatrix} e_{2,1} & \cdots & e_{2,n-1} \\ \cdots & \cdots & \cdots \\ e_{n,1} & \cdots & e_{n,n-1} \end{pmatrix}^{-1} = \begin{pmatrix} \log |\varepsilon_1^{(2)}| & \cdots & \log |\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log |\varepsilon_{n-1}^{(2)}| & \cdots & \log |\varepsilon_{n-1}^{(n)}| \end{pmatrix} (I_{n-1} + E_{n-1}).$$

Here I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix and E_{n-1} is the $(n-1) \times (n-1)$ matrix with all entries equal to 1. Inserting (7.1) in (3.1) we get the relation

$$\chi_m(y) = \exp\left(i\pi \sum_{q=1}^{n-1} m_q \widetilde{y}_{q+1} \right). \tag{7.2}$$

Note also that by (7.1) the ratios y_j/y_i are bounded.

Later we want to integrate so-called incomplete Eisenstein series. To do so it will be convenient to use the transformation from Proposition 7.1 and for that purpose we need to know the Jacobian determinant:

Lemma 7.2. *The numerical value of the Jacobian determinant of the map in Proposition 7.1 is* R^{-1} *where* R *is the regulator of* K.

Proof. Let Ω denote the Jacobian matrix. Note that

$$\frac{\partial \widetilde{y}_1}{\partial y_j} = \frac{\widetilde{y}_1}{y_j}$$

and

$$\frac{\partial \widetilde{y}_{k+1}}{\partial y_j} = \frac{1}{y_j} \sum_{j'=2}^n \delta_{j,j'} (e_{j',k} - e_{1,k}) - \frac{1}{n y_j} \sum_{j'=2}^n (e_{j',k} - e_{1,k})$$

for $k = 1, \dots, n-1$. Thus the y_j's cancel in the Jacobian determinant and we get

$$\det(\Omega) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ A_1 & e_{2,1} - e_{1,1} + A_1 & \cdots & e_{n,1} - e_{1,1} + A_1 \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1} & e_{2,n-1} - e_{1,n-1} + A_{n-1} & \cdots & e_{n,n-1} - e_{1,n-1} + A_{n-1} \end{vmatrix}$$

where $A_k = -\frac{1}{n} \sum_{j=2}^n (e_{j,k} - e_{1,k})$ for $1 \le k \le n-1$. By recursively subtracting column $j-1$ from column j we do not change the determinant. Expanding by minors in the first row (which has 1 in the first entry and 0 in the other entries) we see that

$$\det(\Omega) = \det([e_{j+1,k} - e_{j,k}]_{1 \le j,k \le n-1}).$$

Now using a similar trick on the matrix

$$\begin{pmatrix} e_{1,1} & \cdots & e_{1,n-1} & 1/n \\ \cdots & \cdots & \cdots & \cdots \\ e_{n,1} & \cdots & e_{n,n-1} & 1/n \end{pmatrix} = \begin{pmatrix} \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log|\varepsilon_{n-1}^{(1)}| & \cdots & \log|\varepsilon_{n-1}^{(n)}| \\ 1 & \cdots & 1 \end{pmatrix}^{-1}$$

recursively subtracting row $k-1$ from row k we see that

$$\pm\frac{\det(\Omega)}{n} = \begin{vmatrix} \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log|\varepsilon_{n-1}^{(1)}| & \cdots & \log|\varepsilon_{n-1}^{(n)}| \\ 1 & \cdots & 1 \end{vmatrix}^{-1}.$$

But the determinant on the right-hand side is $\pm nR$ (see e.g. [25]). □

8. The Standard L-function

In this section we will consider the L-function associated with a primitive cusp form – the so-called standard L-function – and show that it has a functional equation.

For a primitive cusp form φ with Hecke eigenvalues $\lambda(\mathfrak{a})$ we consider the L-function (defined for $\mathrm{Re}(s) > \frac{3}{2}$)

$$L(s,\varphi,m) = \sum_{\mathfrak{a}\neq 0} \frac{\chi_m(\mathfrak{a})\lambda(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}.$$

It should be mentioned that one often uses the notation $L(s,\varphi\otimes\chi_m)$ instead of $L(s,\varphi,m)$.

If we use the relations from Theorem 6.1 we can write $L(s,\varphi,m)$ as the Euler product

$$L(s,\varphi,m) = \prod_{\mathfrak{p}} \frac{1}{1-\frac{\chi_m(\mathfrak{p})\lambda(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^s} + \frac{\chi_m(\mathfrak{p})^2}{\mathcal{N}(\mathfrak{p})^{2s}}}$$

where the product is taken over all prime ideals.

Before we go on we need the following result:

Lemma 8.1. *Let f be a formal eigenfunction of the Laplacians $\Delta_1,\dots,\Delta_n$ with eigenvalues $\lambda_1,\dots,\lambda_n$. Assume that $f(iy)=0$ for all $y\in\mathbf{R}_+^n$ where $iy=(iy_1,\dots,iy_n)$. Assume also that*

$$\frac{\partial f}{\partial x_j}(z_1,\dots,z_{j-1},iy_j,z_j,\dots,z_n) = 0 \tag{8.1}$$

for all $z_{j'}\in\mathbf{H}$ with $j'\neq j$, $y_j\in\mathbf{R}_+$ and $j=1,\dots,n$. Then $f(z)=0$ for all $z\in\mathbf{H}^n$.

Proof. Since f is an eigenfunction of the Δ_j's which are elliptic differential operators we conclude that f must be real analytic. Hence it suffices to prove that

$$\frac{\partial^{|a+b|}f}{\partial x_1^{a_1}\dots\partial x_n^{a_n}\partial y_1^{b_1}\dots\partial y_n^{b_n}}(iy) = 0$$

for all $a=(a_1,\dots,a_n), b=(b_1,\dots,b_n)\in\mathbf{N}_0^n$ and $y\in\mathbf{R}_+^n$ – note that we use the notation $|a|=\sum_{j=1}^n a_j$. But clearly this would follow if we could prove that

$$\frac{\partial^{|a|}f}{\partial x_1^{a_1}\dots\partial x_n^{a_n}}(iy) = 0$$

for all $(a_1,\dots,a_n)\in\mathbf{N}_0^n$ and $y\in\mathbf{R}_+^n$.

If $a_j \in \{0,1\}$ for some j, the result follows immediately from (8.1). Now assume that the result holds if for some j we have $a_j \le q$, $q \ge 2$. Consider $(a_1,\dots,a_n) \in \mathbf{N}_0^n$ such that $\min\{a_1,\dots,a_n\} = q+1$; say $a_1 = q+1$. Then we see that

$$\begin{aligned}
\frac{\partial^{|a|} f(iy)}{\partial x_1^{a_1}\dots\partial x_n^{a_n}} &= \frac{\partial^{|a|-2}}{\partial x_1^{a_1-2}\partial x_2^{a_2}\dots\partial x_n^{a_n}}\left(-\frac{1}{y_1^2}\Delta_1 f(iy) - \frac{\partial^2 f(iy)}{\partial y_1^2}\right)\\
&= -\frac{\lambda_1}{y_1^2}\frac{\partial^{|a|-2} f(iy)}{\partial x_1^{a_1-2}\partial x_2^{a_2}\dots\partial x_n^{a_n}} - \frac{\partial^2}{\partial y_1^2}\frac{\partial^{|a|-2} f(iy)}{\partial x_1^{a_1-2}\partial x_2^{a_2}\dots\partial x_n^{a_n}}\\
&= 0
\end{aligned}$$

by induction. This proves the lemma. □

Now we can extend the holomorphic function $L(s,\varphi,m)$ to an entire function with a functional equation of the usual form:

Theorem 8.2. *Let φ be a primitive cusp form with Laplace eigenvalues $\frac{1}{4}+r_j^2$ and Hecke eigenvalues $\lambda(\mathfrak{a})$. Then $L(s,\varphi,m)$ has an analytic continuation to the entire complex plane and it satisfies the functional equation*

$$\Lambda(s,\varphi,m) = (-1)^{\mathrm{Tr}(\kappa)}\chi_{2m}(\mathcal{D})\Lambda(1-s,\varphi,-m) \tag{8.2}$$

where

$$\Lambda(s,\varphi,m) = D^s\pi^{-ns}L(s,\varphi,m)\times \\ \prod_{j=1}^{n}\Gamma\left(\frac{s+\kappa_j+ir_j-i\rho_j(m)}{2}\right)\Gamma\left(\frac{s+\kappa_j-ir_j-i\rho_j(m)}{2}\right)$$

and $\kappa_j = 0$ if φ is even in x_j and $\kappa_j = 1$ if φ is odd in x_j.

Proof. Consider the function

$$\begin{aligned}
f(z) &= \frac{1}{(2\pi i)^{\mathrm{Tr}(\kappa)}}\frac{\partial^{\mathrm{Tr}(\kappa)}\varphi}{\partial x_1^{\kappa_1}\dots\partial x_n^{\kappa_n}}(z)\\
&= \sum_{l\in\mathcal{O}^*} c_l e(\mathrm{Tr}(lx/\omega))\prod_{j=1}^{n}\left(\frac{l^{\kappa_j}}{\omega^{\kappa_j}}\right)^{(j)}\sqrt{y_j}K_{ir_j}(2\pi|(l/\omega)^{(j)}|y_j),
\end{aligned}$$

which is even in all the x_j-variables. For $\mathrm{Re}(s)$ large (this ensures that we can use the Fourier expansion) consider the integral

$$\begin{aligned}
&\frac{\chi_m(\mathcal{D})}{D^s}\int_{\mathcal{O}_+^\times\backslash\mathbf{R}_+^n} f(iy)\prod_{j=1}^{n} y_j^{s-i\rho_j(m)+\kappa_j-3/2}dy\\
&= \sum_{\mathfrak{a}\subset\mathcal{O}}\frac{\chi_m(\mathfrak{a})\lambda(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}\prod_{j=1}^{n}\int_0^\infty K_{ir_j}(2\pi y_j)y_j^{s-i\rho_j(m)+\kappa_j-1}dy_j\times\\
&\qquad\sum_{\beta\in\mathcal{O}_+^\times\backslash\mathcal{O}^\times} c_\beta\prod_{j=1}^{n}(\mathrm{sgn}(\beta^{(j)}))^{\tau_j}
\end{aligned}$$

$$= L(s,\varphi,m)\prod_{j=1}^{n}\frac{\Gamma\left(\frac{s+\kappa_j+ir_j-i\rho_j(m)}{2}\right)\Gamma\left(\frac{s+\kappa_j-ir_j-i\rho_j(m)}{2}\right)}{4\pi^{s+\kappa_j-i\rho_j(m)}}\times$$
$$\sum_{\beta\in\mathcal{O}_+^\times\backslash\mathcal{O}^\times} c_\beta\prod_{j=1}^{n}(\operatorname{sgn}(\beta^{(j)}))^{\tau_j}$$

where τ is the binary vector satisfying (3.3). Note that we have used the formula (see [1] Lemma 1.9.1)

$$\int_0^\infty K_\nu(2\pi y)y^{s-1}dy = \frac{\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)}{4\pi^s}$$

which is valid for $\mathrm{Re}(s) > |\mathrm{Re}(\nu)|$. That the integral above is convergent follows from the fact that $f(iy) = \frac{(-1)^{\mathrm{Tr}(\kappa)}}{\prod_{j=1}^n y_j^{2\kappa_j}} f(i/y)$ (we use the notation $1/y = (1/y_1,\dots,1/y_n)$).

If we can prove that

$$\sum_{\beta\in\mathcal{O}_+^\times\backslash\mathcal{O}^\times} c_\beta\prod_{j=1}^{n}(\operatorname{sgn}(\beta^{(j)}))^{\tau_j}\neq 0 \tag{8.3}$$

we have the analytic continuation since

$$\int_{\mathcal{O}_+^\times\backslash\mathbf{R}_+^n} f(iy)\prod_{j=1}^{n} y_j^{s-i\rho_j(m)+\kappa_j-3/2}dy$$

is an entire function in s (due to exponential decay of f in the y_j-variables). So let us assume that

$$\sum_{\beta\in\mathcal{O}_+^\times\backslash\mathcal{O}^\times} c_\beta\prod_{j=1}^{n}(\operatorname{sgn}(\beta^{(j)}))^{\tau_j} = 0. \tag{8.4}$$

This implies that the integral considered above vanishes for all s and $m\in\mathbf{Z}^{n-1}$. But using the structure of U_∞ we see that ($\widetilde{f}$ is $y\mapsto f(iy)$ composed with the inverse of the map in Proposition 7.1)

$$\int_{\mathcal{O}_+^\times\backslash\mathbf{R}_+^n} f(iy)\prod_{j=1}^{n} y_j^{s-i\rho_j(m)+\kappa_j-3/2}dy =$$
$$R\int_{-1}^{1}\cdots\int_{-1}^{1}\int_0^\infty \widetilde{f}(\widetilde{y})\widetilde{y}_1^{s-3/2+\mathrm{Tr}(\kappa)/n}\times$$
$$\exp\left(\sum_{q=1}^{n-1}\sum_{j=2}^{n}(\kappa_{q+1}-\kappa_1)\widetilde{y}_j\log|\varepsilon_{j-1}^{(q+1)}|\right)\exp\left(-i\pi\sum_{q=1}^{n-1}m_q\widetilde{y}_{q+1}\right)d\widetilde{y},$$

where we have used (7.2). Since this holds for all m we must have $f(iy)=0$ for all $y\in\mathbf{R}_+^n$. We also have that f is a formal eigenfunction of the Δ_j's and since f is even in all the x_j-variables condition (8.1) in Lemma 8.1 is also satisfied. Thus we conclude that f is identically 0. But by the Fourier expansion of f this implies that $c_l=0$ for all $l\in\mathcal{O}^*$ which contradicts that φ is a primitive cusp form and hence non-zero.

Now we prove the functional equation. As remarked earlier $f(iy) = \frac{(-1)^{\mathrm{Tr}(\kappa)}}{\prod_{j=1}^n y_j^{2\kappa_j}} f(i/y)$. From this one easily deduces that

$$\int_{\mathcal{O}_+^\times \backslash \mathbf{R}_+^n} f(iy) \prod_{j=1}^n y_j^{s-i\rho_j(m)+\kappa_j-3/2} dy = (-1)^{\mathrm{Tr}(\kappa)} \int_{\mathcal{O}_+^\times \backslash \mathbf{R}_+^n} f(i/y) \prod_{j=1}^n y_j^{s-i\rho_j(m)-\kappa_j-3/2} dy$$
$$= (-1)^{\mathrm{Tr}(\kappa)} \int_{\mathcal{O}_+^\times \backslash \mathbf{R}_+^n} f(iy) \prod_{j=1}^n y_j^{i\rho_j(m)-s+\kappa_j-1/2} dy$$

where we have used that the map $y \mapsto 1/y$ maps a fundamental domain of $\mathcal{O}_+^\times \backslash \mathbf{R}_+^n$ to another fundamental domain. Now (8.2) follows immediately from the calculation above since $\sum_{j=1}^n \rho_j(m) = 0$. □

Using the Phragmén-Lindelöf principle and the functional equation (8.2) one obtains that

$$L(1/2+it, \varphi, m) \ll |t|^{\frac{n}{2}+\varepsilon}$$

for any $\varepsilon > 0$ as $|t| \to \infty$. This is not enough for our purpose, but any improvement in the exponent will do. In the case $K = \mathbf{Q}$ T. Meurman [15] proved that

$$L(1/2+it, \varphi) \ll \sqrt{r} e^{\pi r/2} |t|^{\frac{1}{3}+\varepsilon},$$

where $\frac{1}{4}+r^2$ is the Laplace eigenvalue and the constant implied only depends on ε. Recently P. Michel and A. Venkatesh [16] and A. Diaconu and P. Garrett [4] proved the estimate that we need in general:

Theorem 8.3. *There exists some $\delta > 0$ such that*

$$L(1/2+it, \varphi, m) \ll |t|^{\frac{n}{2}-\delta}$$

as $|t| \to \infty$.

The generalized Riemann hypothesis implies much more, namely that you can take any $\varepsilon > 0$ in the exponent (the Lindelöf hypothesis for the standard L-function). It should be mentioned that the techniques in [17] probably are adequate to provide the subconvexity estimate in Theorem 8.3.

9. The Eisenstein Series

In the case where $K = \mathbf{Q}$ we have the Eisenstein series

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s.$$

In our case of the Hilbert modular group over general K our candidate for the Eisenstein series would be

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \prod_{j=1}^n \mathrm{Im}(\gamma^{(j)} z_j)^{s_j}. \tag{9.1}$$

Now for this to be well defined we need every term to be independent of the choice of γ in the coset $\Gamma_\infty \backslash \Gamma$. This puts some constraints on the choices of the s_j's. In fact, for (9.1) to be well defined it is necessary and sufficient that

$$|u^{(1)}|^{2s_1} \dots |u^{(n)}|^{2s_n} = 1 \tag{9.2}$$

for all $u \in \mathcal{O}^\times$. The condition (9.2) is certainly equivalent to

$$s_1 \log |\varepsilon_j^{(1)}| + \dots + s_n \log |\varepsilon_j^{(n)}| = i\pi m_j \tag{9.3}$$

for $j = 1, \dots, n-1$ where $m_j \in \mathbf{Z}$. Let $m = (m_1, \dots, m_{n-1}) \in \mathbf{Z}^{n-1}$ be a fixed vector. If we fix the parameter $s \in \mathbf{C}$ and solve the system of equations

$$\begin{pmatrix} \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_1^{(n)}| \\ \cdots & \cdots & \cdots \\ \log|\varepsilon_{n-1}^{(1)}| & \cdots & \log|\varepsilon_{n-1}^{(n)}| \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ \cdots \\ s_n \end{pmatrix} = \begin{pmatrix} i\pi m_1 \\ \cdots \\ i\pi m_{n-1} \\ ns \end{pmatrix},$$

we get the solution (cf. (2.2))

$$s_j = s + i\pi \sum_{q=1}^{n-1} m_q e_{j,q} = s + i\rho_j(m)$$

for $j = 1, \dots, n$. From now on we will view s_j as a function of m and s. Thus in conclusion we define the Eisenstein series for Γ as

$$E(z,s,m) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \prod_{j=1}^{n} \operatorname{Im}(\gamma^{(j)} z_j)^{s_j}, \tag{9.4}$$

which is absolutely convergent for $\operatorname{Re}(s) > 1$ (cf. [5] p. 42). It was proved in [5] that $E(z,s,m)$ has a meromorphic continuation to the entire s-plane, and that $E(z,s,m)$ is holomorphic on the line $\operatorname{Re}(s) = 1/2$.

One can verify that the Eisenstein series is an automorphic form with Laplace eigenvalues $s_j(1-s_j)$ and thus it admits a Fourier expansion. When we calculate the Fourier coefficients it will be convenient to consider the following generalization of the divisor function

$$\sigma_{s,m}(l) = \sum_{\substack{(c) \subset \mathcal{O} \\ c \mid l}} \chi_{2m}(c) |N(c)|^s.$$

Note that $\sigma_{s,m}$ only depends on the ideal (l). The Fourier coefficients are known from [5] Section II.2:

Theorem 9.1. *For $l \in \mathcal{O}$ let $a_l(y,s,m)$ denote the l-th Fourier coefficient of $E(z,s,m)$. For $l \neq 0$ we have that*

$$a_l(y,s,m) = \frac{2^n \pi^{ns} \sigma_{1-2s,-m}(l)}{\chi_m(\mathcal{D}) D^s \zeta(2s,-2m)} \prod_{j=1}^{n} \frac{\sqrt{y_j} K_{s_j - \frac{1}{2}}(2\pi |(l/\omega)^{(j)}| y_j) |l^{(j)}|^{s_j - \frac{1}{2}}}{\Gamma(s_j)}.$$

The zeroth Fourier coefficient is given by

$$a_0(y,s,m) = \left(\prod_{j=1}^{n} y_j \right)^s \chi_m(y) + \varphi(s,m) \left(\prod_{j=1}^{n} y_j \right)^{1-s} \chi_{-m}(y)$$

where

$$\varphi(s,m) = \frac{\zeta(2s-1,-2m)\pi^{\frac{n}{2}}}{\zeta(2s,-2m)\sqrt{D}} \prod_{j=1}^{n} \frac{\Gamma(s_j - \frac{1}{2})}{\Gamma(s_j)}.$$

Note that $\varphi(s,m)$ is unitary for $\operatorname{Re}(s) = \frac{1}{2}$.

As in the classical case we also need to consider incomplete Eisenstein series, i.e. automorphic functions on $\Gamma \backslash \mathbf{H}^n$ formed as Poincaré series which fail to be eigenfunctions of the automorphic Laplacian. Let $h \in C_b^\infty(\mathbf{R}_+)$ and assume that $h(y)y^p \to 0$ as $y \to \infty$ and $h(y)y^{-p} \to 0$ as $y \to 0$ for all $p \in \mathbf{N}$. For $m \in \mathbf{Z}^{n-1}$ we define

$$F(z,h,m) = \sum_{\gamma \in \Gamma^\infty \backslash \Gamma} h\left(\prod_{j=1}^n \operatorname{Im}(\gamma^{(j)} z_j)\right) \prod_{j=1}^n \operatorname{Im}(\gamma^{(j)} z_j)^{i\rho_j(m)}. \tag{9.5}$$

We will refer to $F(z,h,m)$ as the incomplete Eisenstein series induced by h with parameter m. One easily checks that the incomplete Eisenstein series decay faster than any polynomial in the cusp. In particular they are square integrable since they are bounded. Choosing explicit representatives we see that

$$\begin{aligned} F(z,h,0) = h\left(\prod_{j=1}^n y_j\right) + h\left(\prod_{j=1}^n \frac{y_j}{x_j^2+y_j^2}\right) + \\ \frac{1}{2} \sum_{\substack{c,d \in \mathcal{O}^\times \backslash \mathcal{O}^* \\ \gcd(c,d)=1}} h\left(\prod_{j=1}^n \frac{y_j}{(c^{(j)}x_j + d^{(j)})^2 + (c^{(j)}y_j)^2}\right). \end{aligned} \tag{9.6}$$

The following proposition reflects the fact that the Hecke L-function $\zeta(s,m)$ has a pole at $s=1$ if $m=0$ but is regular at $s=1$ if $m \neq 0$:

Proposition 9.2. *For $m \neq 0$ we have*

$$\int_{\Gamma \backslash \mathbf{H}^n} F(z,h,m)\,d\mu(z) = 0.$$

We also have

$$\int_{\Gamma \backslash \mathbf{H}^n} F(z,h,0)\,d\mu(z) = 2^{n-1} R\sqrt{D} \int_0^\infty \frac{h(w)}{w^2}\,dw.$$

Proof. The last statement follows immediately from change of variables using the injective map from Proposition 7.1 and Lemma 7.2.

The first statement follows from a similar argument. Using again the map from Proposition 7.1 and the relation (7.1) we are lead to consider the integral (which only differs from the integral we wish to compute by scaling with a factor of R)

$$\begin{aligned} &\int_{\mathbf{R}_+ \times [-1,1]^{n-1}} \frac{h(\widetilde{y}_1)}{\widetilde{y}_1^2} \exp\left(i\pi \sum_{q=1}^{n-1} m_q \sum_{i=2}^n \widetilde{y}_i \sum_{j=1}^n e_{j,q} \log|\varepsilon_{i-1}^{(j)}|\right) d\widetilde{y} \\ &= \int_{\mathbf{R}_+ \times [-1,1]^{n-1}} \frac{h(\widetilde{y}_1)}{\widetilde{y}_1^2} \exp\left(i\pi \sum_{q=1}^{n-1} m_q \widetilde{y}_{q+1}\right) d\widetilde{y}. \end{aligned}$$

From this the statement is obvious. □

The space spanned by incomplete Eisenstein series will be denoted $\mathcal{E}(\Gamma\backslash\mathbf{H}^n)$. Using the transformation from Proposition 9.2 it is clear that the orthogonal complement to $\mathcal{E}(\Gamma\backslash\mathbf{H}^n)$ is the set of functions $f \in L^2(\Gamma\backslash\mathbf{H}^n)$ for which

$$\int_F f(z)\,dx = 0, \tag{9.7}$$

i.e. the zeroth Fourier coefficient vanishes. As in the classical case $K = \mathbf{Q}$ the space $\mathcal{E}(\Gamma\backslash\mathbf{H}^n)^\perp$ is the closure of the space spanned by cusp forms $\mathcal{C}(\Gamma\backslash\mathbf{H}^n)$ (see [5] Theorem II.9.8). Thus we have the decomposition:

$$L^2(\Gamma\backslash\mathbf{H}^n) = \overline{\mathcal{C}(\Gamma\backslash\mathbf{H}^n)} \oplus \overline{\mathcal{E}(\Gamma\backslash\mathbf{H}^n)}. \tag{9.8}$$

Note that the functions in $\mathcal{C}(\Gamma\backslash\mathbf{H}^n)$ are orthogonal to the constant functions.

10. Quantum Unique Ergodicity

We wish to investigate the behaviour of the measure

$$d\mu_{m,t} = |E(z,1/2+it,m)|^2 d\mu$$

as $t\to\infty$. This is the large eigenvalue limit, since the Laplace eigenvalue of $E(z,1/2+it,m)$ is $nt^2+n/4+\sum_{j=1}^n \rho_j(m)^2$.

In the subsequent sections we will prove the following two results:

Theorem 10.1. *Consider an incomplete Eisenstein series $F(z,h,k)$. Then we have that*

$$\frac{1}{\log t}\int_{\Gamma\backslash\mathbf{H}^n} F(z,h,k)d\mu_{m,t}(z) \to \frac{\pi^n nR}{2D\zeta(2,0)}\int_{\Gamma\backslash\mathbf{H}^n} F(z,h,k)d\mu(z) \tag{10.1}$$

as $t\to\infty$. Note in particular that for $k\neq 0$

$$\frac{1}{\log t}\int_{\Gamma\backslash\mathbf{H}^n} F(z,h,k)d\mu_{m,t}(z) \to 0 \tag{10.2}$$

as $t\to\infty$, cf. Proposition 9.2.

It is interesting that the asymptotics in (10.1) do not depend on m. The constant $\frac{\pi^n nR}{2D\zeta(2,0)}$ can also be given in terms of the volume, since (see [6])

$$\mu(\Gamma\backslash\mathbf{H}^n) = \frac{2\zeta(2,0)D^{\frac{3}{2}}}{\pi^n}. \tag{10.3}$$

Note that since $\zeta(2)=\frac{\pi^2}{6}$ the result above reduces to the result found by W. Luo and P. Sarnak in [14] for $K=\mathbf{Q}$. The results differ by a factor of 16 – they obtain the asymptotics

$$\int_{\Gamma\backslash\mathbf{H}^n} F(z,h)d\mu_t(z) \sim \frac{48}{\pi}\log t\int_{\Gamma\backslash\mathbf{H}^n} F(z,h)d\mu(z) \tag{10.4}$$

as $t\to\infty$. This difference is due to a disagreement regarding the value of the integral (12.3) below, which exactly accounts for the factor of 16. In this connection two other errors in [14] should be mentioned. A factor of 2 is missing in the Fourier expansion of the Eisenstein series on page 211. This error is cancelled though since a factor of $\frac{1}{2}$ is missing in front of the logarithmic derivatives of $\Gamma(s/2\pm it)$ on page 216.

We also obtain the asymptotics for primitive cusp forms:

Theorem 10.2. *Let φ be a primitive cusp form. Then*

$$\int_{\Gamma\backslash\mathbf{H}^n} \varphi(z)d\mu_{m,t}(z) \to 0 \tag{10.5}$$

as $t\to\infty$.

Combining Theorem 10.1 and Theorem 10.2 we can now prove Theorem 1.1:

Proof of Theorem 1.1. Let $\varepsilon>0$ be given and set $\Theta=\frac{\pi^n nR}{2D\zeta(2,0)}$. One can prove that the functions which are a sum of a finite number of primitive cusp forms and incomplete Eisenstein series are dense in the space of continuous functions which vanish in the cusp $C_0(\Gamma\backslash\mathbf{H}^n)$ equipped with the sup norm. Hence let $F\in C_c(\Gamma\backslash\mathbf{H}^n)$ and choose primitive cusp forms $g_1,\dots,g_k$, functions $h_1,\dots,h_l\in C_c^\infty(\mathbf{R}_+)$ and parameters $m_1,\dots,m_l$ such that

$$\|F-G\|_\infty \le \frac{\varepsilon}{2M\mu(\Gamma\backslash\mathbf{H}^n)},$$

where $G(z)=\sum_{j=1}^k g_j(z)+\sum_{i=1}^l F(z,h_i,m_i)$ and M is a constant depending on the field K – in the case $K=\mathbf{Q}$ one can choose $M=4$. Now since cusp forms decay exponentially in the cusp it follows from (9.6) that we can choose a non-negative $h \in C^\infty(\mathbf{R}_+)$ of sufficiently rapid decay such that

$$|F(z) - G(z)| \leq F(z, h, 0) < \frac{\varepsilon}{2\mu(\Gamma\backslash\mathbf{H}^n)}$$

for all $z \in \Gamma\backslash\mathbf{H}^n$. Thus by Theorem 10.1

$$\limsup_{t\to\infty} \frac{1}{\Theta \log t} \left| \int_{\Gamma\backslash\mathbf{H}^n} (F(z) - G(z)) d\mu_{m,t}(z) \right| < \frac{\varepsilon}{2}.$$

Theorem 10.1 and Theorem 10.2 give us that

$$\lim_{t\to\infty} \frac{1}{\Theta \log t} \int_{\Gamma\backslash\mathbf{H}^n} G(z) d\mu_{m,t}(z) = \int_{\Gamma\backslash\mathbf{H}^n} G(z) d\mu(z).$$

Hence

$$\limsup_{t\to\infty} \left| \frac{1}{\Theta \log t} \int_{\Gamma\backslash\mathbf{H}^n} F(z) d\mu_{m,t}(z) - \int_{\Gamma\backslash\mathbf{H}^n} F(z) d\mu(z) \right| < \varepsilon. \tag{10.6}$$

This proves the theorem, since (10.6) holds for any $\varepsilon > 0$. □

Finally, this enables us to prove the main theorem:

Proof of Theorem 1.2. Let $F, G, f, g \in C_c(\Gamma\backslash\mathbf{H}^n)$ be chosen such that

$$F \geq 1_A \geq f \geq 0$$

and

$$G \geq 1_B \geq g \geq 0,$$

where 1_A denotes the indicator function. Then

$$\frac{\int_{\Gamma\backslash\mathbf{H}^n} f(z) d\mu_{m,t}(z)}{\int_{\Gamma\backslash\mathbf{H}^n} G(z) d\mu_{m,t}(z)} \leq \frac{\mu_{m,t}(A)}{\mu_{m,t}(B)} \leq \frac{\int_{\Gamma\backslash\mathbf{H}^n} F(z) d\mu_{m,t}(z)}{\int_{\Gamma\backslash\mathbf{H}^n} g(z) d\mu_{m,t}(z)}.$$

By Theorem 1.1 we see that

$$\frac{\int_{\Gamma\backslash\mathbf{H}^n} f(z) d\mu(z)}{\int_{\Gamma\backslash\mathbf{H}^n} G(z) d\mu(z)} \leq \liminf_{t\to\infty} \frac{\mu_{m,t}(A)}{\mu_{m,t}(B)} \leq \limsup_{t\to\infty} \frac{\mu_{m,t}(A)}{\mu_{m,t}(B)} \leq \frac{\int_{\Gamma\backslash\mathbf{H}^n} F(z) d\mu(z)}{\int_{\Gamma\backslash\mathbf{H}^n} g(z) d\mu(z)}.$$

Since this holds for all F, G, f and g the result follows. □

11. Proof of Theorem 10.1

Consider $F(z, h, k) \in \mathcal{E}(\Gamma\backslash\mathbf{H}^n)$. By standard unfolding arguments we see that

$$\begin{aligned}\int_{\Gamma\backslash\mathbf{H}^n} F(z, h, k) d\mu_{m,t} &= \int_{\Gamma\backslash\mathbf{H}^n} F(z, h, k) |E(z, 1/2 + it, m)|^2 \frac{dxdy}{y_1^2 \cdots y_n^2} \\ &= \int_{U_\infty} h\left(\prod_{j=1}^n y_j\right) \int_F |E(z, 1/2 + it, m)|^2 \frac{dxdy}{\prod_{j=1}^n y_j^{2-i\rho_j(k)}}.\end{aligned}$$

Using the Fourier expansion of the Eisenstein series we get

$$\frac{1}{\sqrt{D}}\int_F |E(z,1/2+it,m)|^2 dx = 2\prod_{j=1}^n y_j + 2\mathrm{Re}\left(\prod_{j=1}^n y_j^{1+2it}\chi_{2m}(y)\overline{\varphi\left(1/2+it,m\right)}\right) + \frac{4^n\pi^n\prod_{j=1}^n y_j}{D|\zeta(1+2it,-2m)|^2}\sum_{l\in\mathcal{O}^*}|\sigma_{-2it,-m}(l)|^2\times \prod_{j=1}^n \frac{\left|K_{it+i\rho_j(m)}\left(2\pi|(\omega^{-1}l)^{(j)}|y_j\right)\right|^2}{|\Gamma(1/2+it+i\rho_j(m))|^2}.$$

Now write

$$\int_{\Gamma\backslash\mathbf{H}^n} F(z,h,k)d\mu_{m,t} = F_1(t)+F_2(t)$$

where

$$F_1(t) = 2\sqrt{D}\int_{U_\infty} h\left(\prod_{j=1}^n y_j\right)\times \left(\prod_{j=1}^n y_j + \mathrm{Re}\left(\prod_{j=1}^n y_j^{1+2it}\chi_{2m}(y)\overline{\varphi\left(1/2+it,m\right)}\right)\right)\frac{dy}{\prod_{j=1}^n y_j^{2-i\rho_j(k)}}$$

and

$$\begin{aligned} F_2(t) &= \frac{4^n\pi^n}{\sqrt{D}|\zeta(1+2it,-2m)|^2}\sum_{l\in\mathcal{O}^*}\int_{U_\infty} h\left(\prod_{j=1}^n y_j\right)|\sigma_{-2it,-m}(l)|^2\times \prod_{j=1}^n \frac{\left|K_{it+i\rho_j(m)}\left(2\pi|(\omega^{-1}l)^{(j)}|y_j\right)\right|^2}{|\Gamma(1/2+it+i\rho_j(m))|^2}\frac{dy}{\prod_{j=1}^n y_j^{1-i\rho_j(k)}} \\ &= \frac{4^n\pi^n}{\sqrt{D}|\zeta(1+2it,-2m)|^2}\sum_{l\in\mathcal{O}_+^\times\backslash\mathcal{O}^*}\int_{\mathbf{R}_+^n} h\left(\prod_{j=1}^n y_j\right)|\sigma_{-2it,-m}(l)|^2\times \prod_{j=1}^n \frac{\left|K_{it+i\rho_j(m)}\left(2\pi|(\omega^{-1}l)^{(j)}|y_j\right)\right|^2}{|\Gamma(1/2+it+i\rho_j(m))|^2}\frac{dy}{\prod_{j=1}^n y_j^{1-i\rho_j(k)}}. \end{aligned}$$

It is clear that $F_1(t)$ is a bounded function of t.

Before we go on we need to consider a new L-function. For a purely imaginary we associate to $\sigma_{a,m}$ an L-function which can be computed in terms of $\zeta(s,m)$:

$$\begin{aligned} \sum_{\mathfrak{a}\neq 0}\frac{\chi_{m'}(\mathfrak{a})|\sigma_{a,m}(\mathfrak{a})|^2}{\mathcal{N}(\mathfrak{a})^s} &= \prod_{\mathfrak{p}}\sum_{k=0}^\infty \frac{\chi_{m'}(\mathfrak{p})^k\sigma_{a,m}(\mathfrak{p}^k)\sigma_{-a,-m}(\mathfrak{p}^k)}{\mathcal{N}(\mathfrak{p})^{ks}} \\ &= \prod_{\mathfrak{p}}\sum_{k=0}^\infty \frac{\chi_{m'}(\mathfrak{p})^k}{\mathcal{N}(\mathfrak{p})^{ks}}\frac{1-\chi_{2m}(\mathfrak{p})^{k+1}\mathcal{N}(\mathfrak{p})^{a(k+1)}}{1-\chi_{2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^a}\frac{1-\chi_{-2m}(\mathfrak{p})^{k+1}\mathcal{N}(\mathfrak{p})^{-a(k+1)}}{1-\chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-a}} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{\mathfrak{p}} \frac{1}{(1-\chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-a})(1-\chi_{2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{a})} \times \\
&\sum_{k=0}^{\infty}(2\chi_{m'}(\mathfrak{p})^k\mathcal{N}(\mathfrak{p})^{-sk} - \chi_{m'}(\mathfrak{p})^k\chi_{2m}(\mathfrak{p})^{k+1}\mathcal{N}(\mathfrak{p})^{(a-s)k+a} - \\
&\chi_{m'}(\mathfrak{p})^k\chi_{-2m}(\mathfrak{p})^{k+1}\mathcal{N}(\mathfrak{p})^{-(a+s)k-a}) \\
&= \prod_{\mathfrak{p}} \frac{1}{(1-\chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-a})(1-\chi_{2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{a})} \times \\
&\left(\frac{2}{1-\chi_{m'}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}} - \frac{\chi_{2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{a}}{1-\chi_{m'+2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{a-s}} - \frac{\chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-a}}{1-\chi_{m'-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-a-s}}\right) \\
&= \prod_{\mathfrak{p}} \frac{1+\chi_{m'}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}}{(1-\chi_{m'}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})(1-\chi_{m'+2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{a-s})(1-\chi_{m'-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-a-s})} \\
&= \frac{\zeta(s,m')^2\zeta(s-a,m'+2m)\zeta(s+a,m'-2m)}{\zeta(2s,2m')}.
\end{aligned}
$$

To deal with $F_2(t)$ we consider the Mellin transform Mh of h, i.e.

$$
(Mh)(r) = \int_0^{\infty} h(w)w^{-r-1}dw.
$$

Note that we have the opposite sign convention in the definition of the Mellin transform than the usual one. However, this is also the convention used in [14], and it is the practical one since we then avoid considering $\zeta(-s,m)$ on the left half-plane. By the Mellin inversion formula we have

$$
h(w) = \frac{1}{2\pi i}\int_{(\sigma)}(Mh)(r)w^r dr
$$

for all $\sigma \in \mathbf{R}$. Thus using the L-function we considered earlier we can rewrite the integral $F_2(t)$ as

$$
\begin{aligned}
F_2(t) &= \frac{(4\pi)^n}{2\pi i\sqrt{D}|\zeta(1+2it,-2m)|^2} \sum_{l\in\mathcal{O}_+^{\times}\backslash\mathcal{O}^*} \int_{\mathbf{R}_+^n}\int_{(2)}(Mh)(r)|\sigma_{-2it,-m}(l)|^2 \times \\
&\prod_{j=1}^{n} \frac{\left|K_{it+i\rho_j(m)}\left(2\pi|(\omega^{-1}l)^{(j)}|y_j\right)\right|^2}{|\Gamma(1/2+it+i\rho_j(m))|^2} y_j^{i\rho_j(k)+r-1}drdy \\
&= \frac{(4\pi)^n}{2\pi i\sqrt{D}|\zeta(1+2it,-2m)|^2\prod_{j=1}^{n}|\Gamma(1/2+it+i\rho_j(m))|^2}\int_{(2)}(Mh)(r) \times \\
&\sum_{l\in\mathcal{O}_+^{\times}\backslash\mathcal{O}^*} |\sigma_{-2it,-m}(l)|^2 \int_{\mathbf{R}_+^n}\prod_{j=1}^{n}\left|K_{it+i\rho_j(m)}\left(2\pi|(l/\omega)^{(j)}|y_j\right)\right|^2 y_j^{i\rho_j(k)+r-1}dydr
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(4\pi)^n}{2\pi i 2^{3n}\sqrt{D}|\zeta(1+2it,-2m)|^2\prod_{j=1}^n|\Gamma(1/2+it+i\rho_j(m))|^2}\int_{(2)}(Mh)(r)\times \\
&\quad \sum_{l\in\mathcal{O}_+^\times\backslash\mathcal{O}^*}|\sigma_{-2it,-m}(l)|^2\prod_{j=1}^n\frac{|\omega^{(j)}|^{i\rho_j(k)+r}\Gamma((i\rho_j(k)+r)/2)^2}{\pi^{i\rho_j(k)+r}|l^{(j)}|^{i\rho_j(k)+r}\Gamma(i\rho_j(k)+r)}\times \\
&\quad \Gamma((i\rho_j(k)+r)/2+it+i\rho_j(m))\Gamma((i\rho_j(k)+r)/2-it-i\rho_j(m))dr \\
&= \frac{(4\pi)^n}{2\pi i 2^{3n}\sqrt{D}|\zeta(1+2it,-2m)|^2\prod_{j=1}^n|\Gamma(1/2+it+i\rho_j(m))|^2}\int_{(2)}B_k(r,t,h)dr
\end{aligned}
$$

where

$$
\begin{aligned}
B_k(r,t,h) = (Mh)(r)&\frac{\zeta(r,-k)^2\zeta(r+2it,-k-2m)\zeta(r-2it,-k+2m)}{\zeta(2r,-2k)\pi^{nr}}\times \\
&\prod_{j=1}^n\frac{|\omega^{(j)}|^{i\rho_j(k)+r}\Gamma((i\rho_j(k)+r)/2)^2}{\Gamma(i\rho_j(k)+r)}\times \\
&\Gamma((i\rho_j(k)+r)/2+it+i\rho_j(m))\Gamma((r+i\rho_j(k))/2-it-i\rho_j(m)).
\end{aligned}
$$

Note that we have used the fact that for any $b\in\mathbf{R}$ we have the formula (see [11] Section B.4)

$$
\int_0^\infty |K_{ib}(2\pi t)|^2 t^{s-1}dt = \frac{\Gamma(s/2+ib)\Gamma(s/2-ib)\Gamma(s/2)^2}{2^3\pi^s\Gamma(s)}. \tag{11.1}
$$

Clearly $(Mh)(r)$ is bounded for $\frac{1}{2}\leq \mathrm{Re}(r)\leq 2$ and Γ decays exponentially in vertical strips by Stirling's formula. Furthermore $\zeta(\sigma+it,k)$ is polynomially bounded in t for $\frac{1}{2}\leq\sigma\leq 2$. Hence we can move the integration from the vertical line $\mathrm{Re}(r)=2$ to the vertical line $\mathrm{Re}(r)=\frac{1}{2}$ perhaps picking up residues from poles at $r=1$ and $r=1\pm 2it$:

$$
\begin{aligned}
F_2(t) = &\frac{(\pi/2)^n\int_{(1/2)}B_k(r,t,h)dr}{2\pi i\sqrt{D}|\zeta(1+2it,-2m)|^2\prod_{j=1}^n|\Gamma(1/2+it+i\rho_j(m))|^2}+ \\
&\frac{(\pi/2)^n\mathrm{res}_{r=1}B_k(r,t,h)}{\sqrt{D}|\zeta(1+2it,-2m)|^2\prod_{j=1}^n|\Gamma(1/2+it+i\rho_j(m))|^2}+O(t^{-10})
\end{aligned}
$$

where the $O(t^{-10})$ term comes from the possible residues from poles at $r=1\pm 2it$, since $(Mh)(\sigma+it)$ is of rapid decay as $t\to\infty$. Let us evaluate the first term. Since Stirling's formula is no good near the real axis in our case, we have to work around that. Note that for $a,b\in\mathbf{R}$ we have

$$
e^{-|a+b|}e^{-|a-b|}\leq e^{-2|a|}.
$$

If $|a+b|\geq 1$ and $a\neq 0$ we also have that

$$
\frac{1}{|a+b|}\leq\frac{1+|b|}{|a|}.
$$

We can now evaluate the first term. Since we are only interested in the asymptotics as $t\to\infty$ we can assume that $t\geq 1$. Using the subconvexity estimate from Theorem 3.1 and

Stirling's formula we see that ($C_1, C_2, C_3 > 0$ are suitable constants)

$$\int_{(1/2)} |B_k(r,t,h)|dr \leq e^{-\pi tn} t^{-\frac{n}{6}+\varepsilon} C_1 \int_{-\infty}^{\infty} |(Mh)(1/2+iw)| \left(1+|w|\right)^{\frac{2n}{3}+\varepsilon} dw +$$
$$e^{-\pi tn} t^{-\frac{n}{4}+\varepsilon} C_2 \int_{2(t+\rho_j(m)-1)-\rho_j(k)}^{2(t+\rho_j(m)+1)-\rho_j(k)} |(Mh)(1/2+iw)| dw +$$
$$e^{-\pi tn} t^{-\frac{n}{4}+\varepsilon} C_3 \int_{-2(t+\rho_j(m)+1)-\rho_j(k)}^{-2(t+\rho_j(m)-1)-\rho_j(k)} |(Mh)(1/2+iw)| dw.$$

Since Mh is of rapid decay the first term dominates, and we obtain the estimate

$$\int_{(1/2)} B_k(r,t,h)dr \ll e^{-t\pi n} |t|^{-\frac{n}{6}+\varepsilon}.$$

By Corollary 3.5 and Stirling's formula we see that

$$\frac{\int_{(1/2)} B_k(r,t,h)dr}{|\zeta(1+2it,-2m)|^2 \prod_{j=1}^n |\Gamma(1/2+it+i\rho_j(m))|^2} \ll |t|^{-\frac{n}{6}+\varepsilon}$$

for any $\varepsilon > 0$.

Now we turn to the residue term. Since $\zeta(s,k)$ is regular at $s=1$ for $k \neq 0$ the residue term will vanish in this case and we are done. Assume therefore that $k=0$. We know that

$$\zeta(s,0) = \frac{\zeta_{-1}}{s-1} + \zeta_0 + O(s-1)$$

and hence

$$\zeta(s,0)^2 = \frac{\zeta_{-1}^2}{(s-1)^2} + \frac{2\zeta_{-1}\zeta_0}{s-1} + O(1)$$

as $s \to 1$ where $\zeta_{-1} = \frac{2^{n-1}R}{\sqrt{D}}$ and ζ_0 is some constant. Now introduce $G(r,t,h)$ defined by

$$B_0(r,t,h) = \zeta(r,0)^2 G(r,t,h).$$

We see that

$$\mathrm{res}_{r=1} B_0(r,t,h) = G(1,t,h)\zeta_{-1}\left(2\zeta_0 + \zeta_{-1}\frac{G'(1,t,h)}{G(1,t,h)}\right).$$

Note that

$$G(1,t,h) = \frac{(Mh)(1)|\zeta(1-2it,2m)|^2}{\zeta(2,0)\pi^n} D\Gamma(1/2)^{2n} \prod_{j=1}^n |\Gamma(1/2+it+i\rho_j(m))|^2$$

and

$$\frac{G'(1,t,h)}{G(1,t,h)} = \frac{\zeta'(1+2it,-2m)}{\zeta(1+2it,-2m)} + \frac{\zeta'(1-2it,2m)}{\zeta(1-2it,2m)} +$$
$$\frac{1}{2}\sum_{j=1}^n \left(\frac{\Gamma'(1/2+it+i\rho_j(m))}{\Gamma(1/2+it+i\rho_j(m))} + \frac{\Gamma'(1/2-it-i\rho_j(m))}{\Gamma(1/2-it-i\rho_j(m))}\right) + C$$

where C is a constant that does not depend on t. Since

$$(Mh)(1) = \frac{2^{1-n}}{\sqrt{D}R} \int_{\Gamma\backslash\mathbf{H}^n} F(z,h,0)d\mu(z)$$

by Proposition 9.2 we see using Corollary 3.4 and Stirling's formula that

$$\frac{1}{\log t} F_2(t) \to \frac{\pi^n nR}{2D\zeta(2,0)} \int_{\Gamma\backslash\mathbf{H}^n} F(z,h,0)d\mu(z)$$

as $t \to \infty$.

12. Proof of Theorem 10.2

Let φ be a primitive cusp form with eigenvalues $\frac{1}{4} + r_j^2$ of the Laplacians Δ_j and Hecke eigenvalues $\lambda(\mathfrak{a})$.

We wish to investigate the asymptotic behaviour of the integral

$$\int_{\Gamma\backslash\mathbf{H}^n} \varphi(z) d\mu_{m,t} = \int_{\Gamma\backslash\mathbf{H}^n} \varphi(z) E(z, 1/2 + it, m) E(z, 1/2 - it, -m) d\mu \tag{12.1}$$

where we have used the fact that $\overline{E(z, s, m)} = E(z, \overline{s}, -m)$. To this end we consider the integral

$$I(s) = \int_{\Gamma\backslash\mathbf{H}^n} \varphi(z) E(z, 1/2 + it, m) E(z, s, -m) d\mu \tag{12.2}$$

for $\mathrm{Re}(s) > 1$. We unfold the integral and get using the Fourier expansions of cusp forms and Eisenstein series that

$$\begin{aligned}
I(s) &= \int_{F_\infty} \varphi(z) E(z, 1/2 + it, m) \prod_{j=1}^n y_j^{s_j(-m)-2} dx dy \\
&= \frac{2^n \pi^{n(1/2+it)}}{\zeta(1 + 2it, -2m)\chi_m(\mathcal{D}) D^{it}} \int_{U_\infty} \sum_{l \in O^*} \sigma_{-2it,-m}(l) c_l \prod_{j=1}^n y_j^{s_j(-m)-1} |l^{(j)}|^{it + i\rho_j(m)} \times \\
&\quad \frac{K_{it+i\rho_j(m)}(2\pi |(l/\omega)^{(j)}| y_j) K_{ir_j}(2\pi |(l/\omega)^{(j)}| y_j)}{\Gamma(1/2 + it + i\rho_j(m))} dy \\
&= \frac{2^n \pi^{n(1/2+it)} (\prod_{j=1}^n |\omega^{(j)}|^{s - i\rho_j(m)})}{\zeta(1 + 2it, -2m)\chi_m(\mathcal{D}) D^{it} \prod_{j=1}^n \Gamma(1/2 + it + i\rho_j(m))} \sum_{l \in \mathcal{O}_+^\times \backslash \mathcal{O}^*} \chi_{2m}(l) \times \\
&\quad \mathcal{N}((l))^{it-s} \sigma_{-2it,-m}(l) c_l \int_{\mathbf{R}_+^n} \prod_{j=1}^n K_{it+i\rho_j(m)}(2\pi y_j) K_{ir_j}(2\pi y_j) y_j^{s_j(-m)-1} dy.
\end{aligned}$$

For $a, b \in \mathbf{R}$ consider the meromorphic function on $\mathbf{C}$:

$$\Gamma(s, a, b) = \frac{\Gamma((s + ia + ib)/2)\Gamma((s + ia - ib)/2)\Gamma((s - ia - ib)/2)\Gamma((s - ia + ib)/2)}{2^3 \pi^s \Gamma(s)}.$$

It is well known (see [11] Section B.4) that

$$\int_0^\infty K_{ia}(2\pi t) K_{ib}(2\pi t) t^{s-1} dt = \Gamma(s, a, b). \tag{12.3}$$

So we get

$$I(s) = \frac{2^n \pi^{n(1/2+it)} R(s)}{\zeta(1 + 2it, -2m)\chi_m(\mathcal{D}) D^{it}} \prod_{j=1}^n \frac{|\omega^{(j)}|^{s - i\rho_j(m)} \Gamma(s_j(-m), r_j, t + \rho_j(m))}{\Gamma(1/2 + it + i\rho_j(m))} \sum_{\beta \in \mathcal{O}_+^\times \backslash \mathcal{O}^\times} c_\beta$$

where

$$\begin{aligned}
R(s) &= \sum_{\mathfrak{a}\subset\mathcal{O}} \chi_{2m}(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{it-s}\sigma_{-2it,-m}(\mathfrak{a})\lambda(\mathfrak{a}) \\
&= \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \chi_{2m}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{k(it-s)}\sigma_{-2it,-m}(\mathfrak{p}^k)\lambda(\mathfrak{p}^k) \\
&= \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \chi_{2m}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{k(it-s)}\lambda(\mathfrak{p}^k) \sum_{j=0}^{k} \chi_{-2m}(\mathfrak{p})^j \mathcal{N}(\mathfrak{p})^{-2ijt} \\
&= \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \chi_{2m}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{k(it-s)}\lambda(\mathfrak{p}^k) \frac{1-\chi_{-2m}(\mathfrak{p})^{k+1}\mathcal{N}(\mathfrak{p})^{-2(k+1)it}}{1-\chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-2it}} \\
&= \prod_{\mathfrak{p}} \frac{1}{1-\chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-2it}} \Bigg(\sum_{k=0}^{\infty} \chi_{2m}(\mathfrak{p})^k \mathcal{N}(\mathfrak{p})^{k(it-s)}\lambda(\mathfrak{p}^k) - \\
&\quad \chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-2it} \sum_{k=0}^{\infty} \lambda(\mathfrak{p}^k)\mathcal{N}(\mathfrak{p})^{k(-it-s)} \Bigg) \\
&= \prod_{\mathfrak{p}} \frac{1}{1-\chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-2it}} \times \\
&\quad \Bigg(\frac{1}{1-\lambda(\mathfrak{p})\chi_{2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{it-s} + \chi_{2m}(\mathfrak{p})^2\mathcal{N}(\mathfrak{p})^{2(it-s)}} - \\
&\quad \frac{\chi_{-2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-2it}}{1-\lambda(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-it-s} + \mathcal{N}(\mathfrak{p})^{2(-it-s)}} \Bigg) \\
&= \prod_{\mathfrak{p}} \frac{1-\chi_{2m}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-2s}}{(1-\chi_{2m}(\mathfrak{p})\lambda(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{it-s} + \chi_{2m}(\mathfrak{p})^2\mathcal{N}(\mathfrak{p})^{2(it-s)})} \times \\
&\quad \frac{1}{(1-\lambda(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-it-s} + \mathcal{N}(\mathfrak{p})^{2(-it-s)})} \\
&= \frac{L(s-it,\varphi,2m)L(s+it,\varphi,0)}{\zeta(2s,2m)}.
\end{aligned}$$

From this we see that $I(s)$ has an analytic continuation to the entire s-plane, and we wish to investigate the asymptotic behaviour of $I(1/2-it)$ as $t\to\infty$. From Stirling's formula we deduce that

$$\prod_{j=1}^{n} \frac{\Gamma(1/2-it-i\rho_j(m), r_j, t+\rho_j(m))}{\Gamma(1/2+it+i\rho_j(m))} \ll |t|^{-n/2}$$

as $t\to\infty$. Using Corollary 3.5 the proof of Theorem 10.2 boils down to proving a subconvexity estimate for $L(s,\varphi,2m)$ on the line $\mathrm{Re}(s)=\frac{1}{2}$. More precisely we need the estimate

$$L(1/2+it,\varphi,2m) \ll |t|^{\frac{n}{2}-\delta}$$

as $|t|\to\infty$ for some $\delta>0$, and this follows from Theorem 8.3. Note that if φ is odd then $I(1/2-it)=0$, since $L(1/2,\varphi,0)=0$ by the functional equation.

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MANUSCRIPT B

Quantitative Mass Equidistribution of Eisenstein Series on GL_2

QUANTITATIVE MASS EQUIDISTRIBUTION OF EISENSTEIN SERIES ON GL_2

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Abstract. We consider the Eisenstein series $E(g,s,\chi)$ on GL_2 over a number field and prove quantitatively that the measure $|E(g,1/2+it,\chi)|^2 d\mu$ becomes equidistributed in the limit $t\to\infty$. This generalizes previous results due to W. Luo and P. Sarnak, S. Koyama and the author.

1. Introduction

Let $\mathbf{F}$ be a number field of degree d over $\mathbf{Q}$. We let $\mathbf{A}$ denote the adele ring of $\mathbf{F}$ and $\mathcal{O}$ the ring of integers of $\mathbf{F}$. Let r_1 denote the number of real embeddings and r_2 denote the number of complex embeddings. Hence $d=r_1+2r_2$ and $n=r_1+r_2$ is the number of places at infinity. We let $G(\mathbf{A})=\mathrm{GL}_2(\mathbf{A})$ and $G(\mathbf{F})=\mathrm{GL}_2(\mathbf{F})$ which embeds discretely in $G(\mathbf{A})$. We set $K=\prod_v K_v$ (the product is taken over all places v of $\mathbf{F}$) where $K_v=\mathrm{GL}_2(\mathcal{O}_v)$ if v is finite, $K_v=\mathrm{O}(2)$ if v is real and $K_v=\mathrm{U}(2)$ if v is complex. We let $B\subset G$ denote the subgroup of upper triangular matrices and $Z\subset G$ the subgroup of scalar matrices.

We consider the Eisenstein series $E(g,s,\chi)$ on $G(\mathbf{A})$ defined by

$$E(g,s,\chi)=\sum_{\gamma\in B(\mathbf{F})\backslash G(\mathbf{F})} f(\gamma g),$$

where $\chi=\prod_v\chi_v$ is an everywhere unramified character on $\mathbf{A}^\times/\mathbf{F}^\times$ and $f:G(\mathbf{A})\to\mathbf{C}$ is a product of local functions $f=\prod_v f_v$. The functions f_v satisfy the condition that ($|\cdot|_v$ denotes the norm on $\mathbf{F}_v$)

$$f_v\left(\left(\begin{smallmatrix} y_{1,v} & x_v\\ & y_{2,v}\end{smallmatrix}\right)g_v\right)=\frac{\chi_v(y_{1,v})|y_{1,v}|_v^s}{\chi_v(y_{2,v})|y_{2,v}|_v^s}f_v(g_v)$$

for $g_v\in G(\mathbf{F}_v)$, $y_{1,v},y_{2,v}\in\mathbf{F}_v^\times$ and $x_v\in\mathbf{F}_v$. Furthermore f_v is identical 1 on K_v. By the Iwasawa decomposition this determines f completely.

Let

$$X(\mathbf{F})=Z(\mathbf{A})G(\mathbf{F})\backslash G(\mathbf{A})/K\cong \mathrm{Cl}_{\mathbf{F}}\times\mathrm{SL}_2(\mathcal{O})\backslash(\mathbf{H}_2^{r_1}\times\mathbf{H}_3^{r_2}),$$

where $\mathbf{H}_2$ is the upper half-plane of $\mathbf{C}$, $\mathbf{H}_3$ is $\mathbf{C}\times\mathbf{R}_+$ inside the quaternions and $\mathrm{Cl}_{\mathbf{F}}$ is the ideal class group ($h=|\mathrm{Cl}_{\mathbf{F}}|$ is the class number). The action of $\mathrm{SL}_2(\mathcal{O})$ on $\mathbf{H}_2^{r_1}\times\mathbf{H}_3^{r_2}$ is the one considered in [16]. The space $X(\mathbf{F})$ is equipped with a measure $d\mu$ induced by the Haar measure on $G(\mathbf{A})$ which will be described in Section 4.

Clearly $E(g,s,\chi)$ may be regarded as a function on the quotient space $X(\mathbf{F})$. We set

$$d\mu_{\chi,t}=|E(g,1/2+it,\chi)|^2d\mu.$$

The main result in this paper is the following quantitative mass equidistribution result:

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Theorem 1.1. *Let $F \in C_c(X(\mathbf{F}))$. We have*

$$\frac{1}{\log t}\int_{X(\mathbf{F})} F(g)d\mu_{\chi,t} \to \frac{2^{r_2}\pi^{r_1}nhR}{\zeta_{\mathbf{F}}(2)\mathcal{W}D}\int_{X(\mathbf{F})} F(g)d\mu$$

as $t \to \infty$.

Here D and R is the discriminant and regulator of $\mathbf{F}$, $\mathcal{W}$ is the number of roots of unity in $\mathbf{F}^\times$ and $\zeta_{\mathbf{F}}$ is the Dedekind zeta-function associated with $\mathbf{F}$.

From Theorem 1.1 we easily obtain the following equidistribution result (see [17] for more details):

Theorem 1.2. *Let $A, B \subset X(\mathbf{F})$ be compact and Jordan measurable, and assume that $\mu(B) \neq 0$. Then*

$$\frac{\mu_{\chi,t}(A)}{\mu_{\chi,t}(B)} \to \frac{\mu(A)}{\mu(B)}$$

as $t \to \infty$.

The question was first investigated by Luo and Sarnak [9] in the case $\mathbf{F} = \mathbf{Q}$ and may be viewed as a continuous spectrum analogue of the quantum unique ergodicity conjecture by Rudnick and Sarnak [12] (see [13] and [14] for connections to quantum chaos). Later Koyama [8] considered the case where $\mathbf{F}$ is a quadratic imaginary field with class number 1, and the author [17] considered the case where $\mathbf{F}$ is a totally real field with narrow class number 1. The strategy in this paper is the same as in the precursors and is due to Luo and Sarnak [9]. However, we work with an adelic setting which is more convenient since we want to deal with general number fields.

The idea in the proof of Theorem 1.1 is to use the fact that $L^2(X(\mathbf{F}))$ is the direct sum of the space spanned by cusp forms and the space of P-series (to be defined in Section 4), and then establish the equidistribution for functions that span these spaces (Theorems 1.3 and 1.4 below). Theorem 1.1 then follows from standard approximation arguments.

Theorem 1.3. *For a P-series $P(g, H, \chi')$ we have*

$$\frac{1}{\log t}\int_{X(\mathbf{F})} P(g,H,\chi')d\mu_{\chi,t} \to \frac{2^{r_2}\pi^{r_1}nhR}{\zeta_{\mathbf{F}}(2)\mathcal{W}D}\int_{X(\mathbf{F})} P(g,H,\chi')d\mu$$

as $t \to \infty$.

Note that

$$\frac{1}{\log t}\int_{X(\mathbf{F})} P(g,H,\chi')d\mu_{\chi,t} \to 0$$

as $t \to \infty$ if $\chi' \neq \chi_0$ (here χ_0 denotes the identity character), since

$$\int_{X(\mathbf{F})} P(g,H,\chi')d\mu = 0.$$

This will be shown in Section 4.

For cusp forms we get:

Theorem 1.4. *Let φ be a cusp form. Then*

$$\int_{X(\mathbf{F})} \varphi(g)d\mu_{\chi,t} \to 0$$

as $t \to \infty$.

This indeed corresponds to the desired equidistribution as

$$\int_{X(\mathbf{F})} \varphi(g) d\mu = 0.$$

The idea in the proofs of Theorems 1.3 and 1.4 is to use the fact that the Eisenstein series $E(g, s, \chi)$ admit a Fourier expansion (see [1] Section 3.7) of the form

$$E(g, s, \chi) = \sum_{\alpha \in \mathbf{F}} c_\alpha(g, s),$$

where

$$c_\alpha(g, s) = \frac{1}{\sqrt{D}} \int_{\mathbf{A}/\mathbf{F}} E\left(\begin{pmatrix} 1 & w \\ & 1 \end{pmatrix} g, s, \chi\right) \psi(-\alpha w) dw. \tag{1.1}$$

The character ψ will be defined in Section 2. One must know the Fourier coefficients of the Eisenstein series rather explicitly. In Section 2 we write down the coefficients in explicit form for elements g of the form $\left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right)$. Using standard unfolding techniques one can express the relevant integrals in terms of the Hecke L-function and the standard L-function. The results then follow from subconvexity estimates in t-aspect.

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2. The Fourier Expansion of the Eisenstein Series

First we describe the Haar measure on $\mathbf{F}_v$. If v is a finite place we normalize the Haar measure such that the volume of $\mathcal{O}_v$ is 1. If $\mathbf{F}_v = \mathbf{R}$ the Haar measure is just the Lebesgue measure. Finally for $\mathbf{F}_v = \mathbf{C}$ the Haar measure is 2 times the Lebesgue measure (on $\mathbf{R}^2$).

Let χ be an everywhere unramified character on $\mathbf{A}^\times/\mathbf{F}^\times$. We can write $\chi = \chi_f \chi_\infty$ where χ_∞ is a character on $\mathbf{R}^{\times n_1} \times \mathbf{C}^{\times n_2}$. Fix an ordering of the infinite places such that the r_1 first places are real and the last r_2 places are complex. These correspond to the different embeddings of $\mathbf{F}$ in $\mathbf{R}$ and $\mathbf{C}$. For $\alpha \in \mathbf{F}$ let $\alpha^{(j)}$ denote the embedding of α corresponding to the j-th place. We let $|\cdot|_{\mathbf{C}}$ denote the usual norm on $\mathbf{C}$ squared. We also set $|\cdot|_j = |\cdot|$ if the j-th place is real and $|\cdot|_j = |\cdot|_{\mathbf{C}}$ if the j-th place is complex. Finally we let $|\cdot|_{\mathbf{A}}$ denote the idele norm.

We know that $\chi_\infty = \prod_{j=1}^n \chi_j$. Since χ_j is unramified χ_j must be of the form $|\cdot|_j^{i\rho_j(\chi)}$ for some $\rho_j(\chi) \in \mathbf{R}$. For a finite place v we let $|\cdot|_v$ denote the norm on the completion $\mathbf{F}_v$. It is well known that the ring of integers $\mathcal{O}_v$ (in $\mathbf{F}_v$) is a local ring with maximal ideal $\mathfrak{p}_v$ and we let ω_v denote a generator of $\mathfrak{p}_v$. We also set $q_v = |\mathcal{O}_v/\mathfrak{p}_v|$. Note that $\chi_v(\omega_v)$ is independent of the choice of generator for an unramified character χ_v on $\mathbf{F}_v$.

The rest of this section will be devoted to calculating the Fourier coefficients (1.1). First we look at the constant term $c_\alpha(g, s)$. Unfolding the integral (1.1) we obtain

$$c_0(g, s) = f(g) + \frac{1}{\sqrt{D}} \int_{\mathbf{A}} f\left(w_0 \begin{pmatrix} 1 & w \\ & 1 \end{pmatrix} g\right) dw$$

with $w_0 = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$. The integral factorizes into a product of local integrals

$$\int_{\mathbf{A}} f\left(w_0 \begin{pmatrix} 1 & w \\ & 1 \end{pmatrix} g\right) dw = \prod_v (M_v(s) f_v)(g_v)$$

where

$$(M_v(s) f_v)(g_v) = \int_{\mathbf{F}_v} f_v\left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} g_v\right) dw_v.$$

A straight forward calculation shows that

$$w_0 \begin{pmatrix} 1 & w \\ & 1 \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & y \end{pmatrix} w_0 \begin{pmatrix} 1 & x/y + w/y \\ & 1 \end{pmatrix}. \tag{2.1}$$

We will use this identity several times.

First we calculate the integral for a real place (corresponding to the j-th embedding). We see that

$$\begin{aligned} (M_v(s)f_v)(g_v) &= \int_{\mathbf{R}} f_v \left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} \begin{pmatrix} y_v & x_v \\ & 1 \end{pmatrix} \right) dw_v \\ &= |y_v|^{1-s_j(\chi)} \int_{\mathbf{R}} f_v \left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} \right) dw_v. \end{aligned}$$

Here $s_j(\chi) = s + i\rho_j(\chi)$.

One can write

$$w_0 \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = \begin{pmatrix} \Delta_t^{-1} & -t\Delta_t^{-1} \\ & \Delta_t \end{pmatrix} \begin{pmatrix} t\Delta_t^{-1} & -\Delta_t^{-1} \\ \Delta_t^{-1} & t\Delta_t^{-1} \end{pmatrix} \tag{2.2}$$

where $\Delta_t = \sqrt{1+t^2}$. Note that the last matrix is in $\mathrm{O}(2)$. Thus we see that

$$\int_{\mathbf{R}} f_v \left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} \right) dw_v = \int_{-\infty}^{\infty} \frac{1}{(1+t^2)^{s_j(\chi)}} dt = \sqrt{\pi} \frac{\Gamma(s_j(\chi) - 1/2)}{\Gamma(s_j(\chi))}. \tag{2.3}$$

The evaluation of the last integral can be found in [7].

Now we look at a complex place (corresponding to the j-th embedding). As before we see that

$$(M_v(s)f_v)(g_v) = |y_v|_{\mathbf{C}}^{1-s_j(\chi)} \int_{\mathbf{C}} f_v \left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} \right) dw_v$$

and

$$w_0 \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} = \begin{pmatrix} \Delta_{|z|}^{-1} & -z\Delta_{|z|}^{-1} \\ & \Delta_{|z|} \end{pmatrix} \begin{pmatrix} z\Delta_{|z|}^{-1} & -\Delta_{|z|}^{-1} \\ \Delta_{|z|}^{-1} & \overline{z}\Delta_{|z|}^{-1} \end{pmatrix} \tag{2.4}$$

where the last matrix is in $\mathrm{U}(2)$. Changing to polar coordinates one easily obtains

$$\int_{\mathbf{C}} f_v \left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} \right) dw_v = \frac{2\pi}{s_j(\chi) - 1}.$$

We now turn to the finite places. Essentially the same reduction applies and we obtain

$$(M_v(s)f_v)(g_v) = |y_v|_v^{1-s} \chi_v(y_v)^{-1} \int_{\mathbf{F}_v} f_v \left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} \right) dw_v.$$

We note that

$$\int_{\mathcal{O}_v} f_v \left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} \right) dw_v = 1.$$

From the matrix identity

$$w_0 \begin{pmatrix} 1 & w \\ & 1 \end{pmatrix} = \begin{pmatrix} w^{-1} & -1 \\ & w \end{pmatrix} \begin{pmatrix} 1 & \\ w^{-1} & 1 \end{pmatrix}$$

and the fact that $\mathfrak{p}_v^{-m} - \mathfrak{p}_v^{-m+1}$, $m > 0$ has volume $q_v^m(1 - q_v^{-1})$ we conclude that

$$\int_{\mathfrak{p}_v^{-m} - \mathfrak{p}_v^{-m+1}} f_v \left(w_0 \begin{pmatrix} 1 & w_v \\ & 1 \end{pmatrix} \right) dw_v = (1 - q_v^{-1}) \alpha_v^{-2m} q_v^{(1-2s)m}$$

if $m \geq 1$. Here $\alpha_v = \chi_v(\omega_v)$. Summing these terms we see that

$$\int_{\mathbf{F}_v} f_v\left(w_0\begin{pmatrix}1 & w_v\\ & 1\end{pmatrix}\right)dw_v = 1 + (1-q_v^{-1})q_v^{1-2s}\alpha_v^{-2}(1-q_v^{1-2s}\alpha_v^{-2})^{-1}$$
$$= \frac{1-q_v^{-2s}\alpha_v^{-2}}{1-q_v^{1-2s}\alpha_v^{-2}}.$$

The calculations above show that

$$\prod_{v \text{ finite}} (M_v(s)f_v)(g_v) = \frac{L(2s-1,\chi^{-2})}{L(2s,\chi^{-2})}\prod_{v<\infty}|y_v|_v^{1-s}\chi_v(y_v)^{-1}.$$

Thus we have calculated the constant term:

Proposition 2.1. *Let $g \in \mathrm{GL}_2(\mathbf{A})$ be of the form $g = \left(\begin{smallmatrix} y & x\\ & 1\end{smallmatrix}\right)$. Then the constant term $c_0(g,s)$ is given by*

$$|y|_{\mathbf{A}}^s\chi(y) + \frac{2^{r_2}\pi^{d/2}}{\chi(y)\sqrt{D}}\frac{L(2s-1,\chi^{-2})}{L(2s,\chi^{-2})}|y|_{\mathbf{A}}^{1-s}\prod_{j\leq r_1}\frac{\Gamma(s_j(\chi)-1/2)}{\Gamma(s_j(\chi))}\prod_{j>r_1}\frac{1}{s_j(\chi)-1}.$$

Now we turn to the other terms. We define

$$(W(s)f)(g) = \int_{\mathbf{A}} f\left(w_0\begin{pmatrix}1 & w\\ & 1\end{pmatrix}g\right)\psi(-w)dw = \prod_v (W_v(s)f_v)(g_v),$$

where

$$(W_v(s)f_v)(g_v) = \int_{\mathbf{F}_v} f_v\left(w_0\begin{pmatrix}1 & w_v\\ & 1\end{pmatrix}g_v\right)\psi_v(-w_v)dw_v.$$

Note that $c_\alpha(g,s) = \frac{1}{\sqrt{D}}(W(s)f)\left(\left(\begin{smallmatrix}\alpha & \\ & 1\end{smallmatrix}\right)g\right)$.

The additive characters ψ_v are described in Section 7.1 in [11]. If v is a real place we have $\psi_v(x) = e(x) = e^{2\pi i x}$. If v is a complex place we have $\psi_v(z) = e(2\mathrm{Re}(z))$. Finally we consider a finite place v (that lies over the prime $p \in \mathbf{Z}$). In that case we have $\psi_v(w_v) = e((Q\circ P)(\mathrm{Tr}(w_v)))$, where $P : \mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p$ is the canonical projection map, $Q : \mathbf{Q}_p/\mathbf{Z}_p \to \mathbf{Q}/\mathbf{Z}$ is the map that "removes" the integer part of an element in $\mathbf{Q}_p$, and $\mathrm{Tr} : \mathbf{F}_v \to \mathbf{Q}_p$ is the trace map.

As before we assume that g is of the form $\left(\begin{smallmatrix} y & x\\ & 1\end{smallmatrix}\right)$. First we look at a finite place v. The conductor of ψ_v is $\mathfrak{p}_v^{-d_v}$. One easily checks that $(W_v(s)f_v)\left(\left(\begin{smallmatrix} y_v & x_v\\ & 1\end{smallmatrix}\right)\right) = 0$ if $y_v \notin \mathfrak{p}_v^{-d_v}$. Consider $(W_v(s)f_v)\left(\left(\begin{smallmatrix} y_v & x_v\\ & 1\end{smallmatrix}\right)\right)$ where $y_v \in \mathfrak{p}_v^{-d_v}$. We see that

$$(W_v(s)f_v)\left(\left(\begin{smallmatrix} y_v & x_v\\ & 1\end{smallmatrix}\right)\right) = \frac{|y_v|_v^{1-s}}{\chi_v(y_v)}\int_{\mathbf{F}_v} f_v\left(w_0\left(\begin{smallmatrix} 1 & w_v\\ & 1\end{smallmatrix}\right)\right)\psi_v(-y_vw_v)dw_v\psi_v(x_v).$$

Thus it remains to evaluate the integral $\int_{\mathbf{F}_v} f_v\left(w_0\left(\begin{smallmatrix} 1 & w_v\\ & 1\end{smallmatrix}\right)\right)\psi_v(-y_vw_v)dw_v$. We see that

$$\int_{\mathbf{F}_v} f_v\left(w_0\left(\begin{smallmatrix} 1 & w_v\\ & 1\end{smallmatrix}\right)\right)\psi_v(-y_vw_v)dw_v$$
$$= 1 + \sum_{j=1}^{\mathrm{ord}_{\mathbf{F}_v}(y_v)+d_v}\frac{q_v^j(1-q_v^{-1})}{\alpha_v^{2j}q_v^{2sj}} - \frac{q_v^{\mathrm{ord}_{\mathbf{F}_v}(y_v)+d_v}}{\alpha_v^{2(\mathrm{ord}_{\mathbf{F}_v}(y_v)+d_v+1)}q_v^{2s(\mathrm{ord}_{\mathbf{F}_v}(y_v)+d_v+1)}}$$
$$= \left(1-\frac{\alpha_v^{-2}}{q_v^{2s}}\right)\sum_{j=0}^{\mathrm{ord}_{\mathbf{F}_v}(y_v)+d_v}\alpha_v^{-2j}q_v^{(1-2s)j}$$
$$= \frac{\sigma_v(y_v,1-2s,\chi_v^{-2})}{L_v(2s,\chi_v^{-2})},$$

where $\sigma_v(y_v, s, \chi_v)$ denotes the local divisor function. For the sake of notation we extend the definition of $\sigma_v(y_v, s, \chi_v)$:

$$\sigma_v(y_v, s, \chi_v) = \begin{cases} \sum_{j=0}^{\mathrm{ord}_{\mathbf{F}_v}(y_v)+d_v} \alpha_v^j q_v^{sj} & \text{if } y_v \in \mathfrak{p}_v^{-d_v} \\ 0 & \text{if } y_v \notin \mathfrak{p}_v^{-d_v} \end{cases}.$$

Note that

$$\sigma_v(y_v, s, \chi_v) = \frac{1 - \alpha_v^{(\mathrm{ord}_{\mathbf{F}_v}(y_v)+d_v+1)} q_v^{(\mathrm{ord}_{\mathbf{F}_v}(y_v)+d_v+1)s}}{1 - \alpha_v q_v^s}$$

for $y_v \in \mathfrak{p}_v^{-d_v}$.

Now we consider a real place v (corresponding to the j-th embedding). We see that

$$\begin{aligned} (W_v(s)f_v)\left(\left(\begin{smallmatrix} y_v & x_v \\ & 1 \end{smallmatrix}\right)\right) &= |y_v|^{1-s_j(\chi)} \int_{\mathbf{R}} f_v\left(w_0 \left(\begin{smallmatrix} 1 & w_v \\ & 1 \end{smallmatrix}\right)\right) e(-y_v w_v) dw_v e(x_v) \\ &= |y_v|^{1-s_j(\chi)} \int_{-\infty}^{\infty} \frac{e(-y_v t)}{(1+t^2)^{s_j(\chi)}} dt e(x_v) \\ &= 2\pi^{s_j(\chi)} \sqrt{|y_v|} \frac{K_{s_j(\chi)-1/2}(2\pi|y_v|)}{\Gamma(s_j(\chi))} e(x_v). \end{aligned}$$

The integral above was evaluated in [16].

Finally we look at a complex place v (corresponding to the j-th embedding) and obtain

$$\begin{aligned} (W_v(s)f_v)\left(\left(\begin{smallmatrix} y_v & x_v \\ & 1 \end{smallmatrix}\right)\right) &= |y_v|_{\mathbf{C}}^{1-s_j(\chi)} \int_{\mathbf{C}} f_v\left(w_0 \left(\begin{smallmatrix} 1 & w_v \\ & 1 \end{smallmatrix}\right)\right) e(-2\mathrm{Re}(y_v w_v)) dw_v e(2\mathrm{Re}(x_v)) \\ &= 2|y_v|_{\mathbf{C}}^{1-s_j(\chi)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e(-2\mathrm{Re}(y_v(t_1+it_2)))}{(1+t_1^2+t_2^2)^{2s_j(\chi)}} dt_1 dt_2 e(2\mathrm{Re}(x_v)) \\ &= 2(2\pi)^{2s_j(\chi)} |y_v| \frac{K_{2s_j(\chi)-1}(4\pi|y_v|)}{\Gamma(2s_j(\chi))} e(2\mathrm{Re}(x_v)). \end{aligned}$$

We summarize our results in the following proposition:

Proposition 2.2. *Let $g = \left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right)$. Assume $y_v \in \mathfrak{p}_v^{-d_v}$ for all finite v. Then*

$$\begin{aligned} (W(s)f)(g) = 2\pi^{s_1(\chi)} &\frac{K_{s_1(\chi)-1/2}(2\pi|y_1|)}{\Gamma(s_1(\chi))} \dots 2(2\pi)^{2s_n(\chi)} \frac{K_{2s_n(\chi)-1}(4\pi|y_n|)}{\Gamma(2s_n(\chi))} \\ &\times \frac{\sqrt{|y|_{\mathbf{A}}}}{L(2s, \chi^{-2})} \psi(x) \prod_{v \text{ finite}} \frac{|y_v|_v^{\frac{1}{2}-s}}{\chi_v(y_v)} \sigma_v(y_v, 1-2s, \chi_v^{-2}). \end{aligned}$$

If $y_v \notin \mathfrak{p}_v^{-d_v}$ for some finite v then $(W(s)f)(g) = 0$.

Thus we have calculated the Fourier coefficients.

3. The Fourier Expansion of a Cusp Form

We recall the Fourier expansion of a (spherical) cusp form (see [1] Section 3.5). Let φ be a cusp form. Then φ admits a Fourier expansion of the form

$$\varphi(g) = \sum_{\alpha \in \mathbf{F}^\times} c_\alpha^\varphi(g),$$

where

$$c_\alpha^\varphi(g) = \frac{1}{\sqrt{D}} \prod_v W_v^\varphi\left(\left(\begin{smallmatrix} \alpha_v & \\ & 1 \end{smallmatrix}\right) g_v\right)$$

and $W_v^\varphi(g_v k_v) = W_v^\varphi(g_v)$ for $k_v \in K_v$. For the finite places we know that

$$W_v^\varphi\left(\begin{pmatrix} y_v & x_v \\ & 1 \end{pmatrix}\right) = \begin{cases} \psi_v(x_v) q_v^{-\mathrm{ord}_{\mathbf{F}_v}(y_v)/2} \frac{\beta_{1,v}^{\mathrm{ord}_{\mathbf{F}_v}(y_v)+1+d_v} - \beta_{2,v}^{\mathrm{ord}_{\mathbf{F}_v}(y_v)+1+d_v}}{\beta_{1,v}-\beta_{2,v}} & \text{if } y_v \in \mathfrak{p}_v^{-d_v} \\ 0 & \text{if } y_v \notin \mathfrak{p}_v^{-d_v} \end{cases},$$

where $\beta_{1,v}$ and $\beta_{2,v}$ are the Satake parameters. This is known as Shintani's formula (see [1] Theorem 4.6.5). The finite part of the standard L-function $L(s,\varphi,\chi)$ (we also set $L(s,\varphi) = L(s,\varphi,\chi_0)$) associated with φ (twisted with an everywhere unramified character χ) is

$$L(s,\varphi,\chi) = \prod_{v<\infty} \frac{1}{(1-\beta_{1,v}\chi_v(\omega_v)q^{-s})(1-\beta_{2,v}\chi_v(\omega_v)q^{-s})}.$$

We will not be concerned with the exact behaviour at the infinite places. From [1] Sections 2.8 and 3.5 we know we can assume that there exist constants $C_v(\varphi)$ s.t.

$$W_v^\varphi\left(\begin{pmatrix} y_v & x_v \\ & 1 \end{pmatrix}\right) = \begin{cases} C_v(\varphi)\sqrt{|y_v|} K_{ir_j}(2\pi|y_v|) e(x_v) & \text{if } v \text{ is real} \\ C_v(\varphi)|y_v| K_{2ir_j}(4\pi|y_v|) e(2\mathrm{Re}(x_v)) & \text{if } v \text{ is complex} \end{cases}.$$

4. P-series

We wish to identify the orthogonal complement to the space spanned by cusp forms. Let $H : \mathbf{R}_+ \to \mathbf{C}$ be smooth and compactly supported. Let $f_{H,\chi} : G(\mathbf{A})/K \to \mathbf{C}$ be defined by

$$f_{H,\chi}\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} g\right) = H\left(\frac{|y_1|_{\mathbf{A}}}{|y_2|_{\mathbf{A}}}\right) \frac{\chi(y_1)}{\chi(y_2)} f_{H,\chi}(g)$$

and $f_{H,\chi}(k) = 1$ for $k \in K$.

We define the P-series as

$$P(g,H,\chi) = \sum_{\gamma \in B(\mathbf{F})\backslash G(\mathbf{F})} f_{H,\chi}(\gamma g). \tag{4.1}$$

It is well known (see [4]) that these span a dense subspace of the orthogonal complement (in $L^2(X(\mathbf{F}))$) of the subspace spanned by cusp forms. This follows from the Stone-Weierstrass theorem since the quotient $\mathbf{A}_1^\times/\mathbf{F}^\times$ is compact. Here $\mathbf{A}_1^\times$ denotes the ideles of norm 1.

It is clear that we may view $X(\mathbf{F})$ as a subset of $G(\mathbf{A})$ of matrices in $G(\mathbf{A})$ of the form $\begin{pmatrix} y & x \\ & 1 \end{pmatrix}$ where $y \in \mathbf{A}^\times$ and $x \in \mathbf{A}$. Following Section 1.5 in [5] we see that the measure induced on $X(\mathbf{F})$ is $\frac{dxdy}{|y|_{\mathbf{A}}^2}$.

From Section 3.8 in [1] we know that integration over

$$Z(\mathbf{A})B(\mathbf{F})\backslash G(\mathbf{A})$$

can be replaced by integration over

$$B(\mathbf{A})\backslash G(\mathbf{A}) \times T_1(\mathbf{F})\backslash T_1(\mathbf{A}) \times N(\mathbf{F})\backslash N(\mathbf{A}), \tag{4.2}$$

where $N \subset G$ is matrices of the form $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ and $T_1 \subset G$ is matrices of the form $\begin{pmatrix} y & \\ & 1 \end{pmatrix}$.

From this we see that

$$\begin{aligned} \int_{Z(\mathbf{A})G(\mathbf{F})\backslash G(\mathbf{A})/K} P(g,H,\chi) d\mu &= \int_{\mathbf{A}/\mathbf{F}} \int_{\mathbf{A}^\times/\mathbf{F}^\times} f_{H,\chi}\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) \frac{dydx}{|y|_{\mathbf{A}}^2} \\ &= \sqrt{D} \int_0^\infty H(t) \int_{\mathbf{A}_1^\times/\mathbf{F}^\times} \chi(y) dy \frac{dt}{t^2}. \end{aligned} \tag{4.3}$$

It follows from (4.3) that

$$\int_{Z(\mathbf{A})G(\mathbf{F})\backslash G(\mathbf{A})/K} P(g,H,\chi)d\mu = 0$$

if $\chi \neq \chi_0$ and

$$\int_{Z(\mathbf{A})G(\mathbf{F})\backslash G(\mathbf{A})/K} P(g,H,\chi_0)d\mu = \frac{2^n\pi^{r_2}hR}{\mathcal{W}}\int_0^\infty \frac{H(t)}{t^2}dt,$$

since $\mathrm{Vol}(\mathbf{A}_1^\times/\mathbf{F}^\times) = \frac{2^n\pi^{r_2}hR}{\mathcal{W}\sqrt{D}}$ (see Theorem 7.21 in [11]).

5. Proof of Theorem 1.3

Using (4.2) we deduce that

$$\int_{X(\mathbf{F})} P(g,H,\chi')d\mu_{\chi,t} = \int_{\mathbf{A}^\times/\mathbf{F}^\times} f_{H,\chi'}\left(\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right)\right)\int_{\mathbf{A}/\mathbf{F}} \left|E\left(\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right),s,\chi\right)\right|^2 dx\frac{dy}{|y|_{\mathbf{A}}^2}.$$

From the Fourier expansion we get

$$\int_{\mathbf{A}/\mathbf{F}} \left|E\left(\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right),1/2+it,\chi\right)\right|^2 dx = \left|c_0\left(\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right),1/2+it\right)\right|^2 + \frac{4^d\pi^d|y|_{\mathbf{A}}}{D|L(1+2it,\chi^{-2})|^2}$$
$$\times \sum_{\zeta\in\mathbf{F}^\times}\prod_{v \text{ finite}}\sigma_v(\zeta_v y_v,-2it,\chi_v^{-2})\prod_{j\le r_1}\frac{K_{it+i\rho_j(\chi)}(2\pi|\zeta^{(j)}y_j|)}{\Gamma(1/2+it+i\rho_j(\chi))}\prod_{j>r_1}\frac{K_{2it+2i\rho_j(\chi)}(4\pi|\zeta^{(j)}y_j|)}{\Gamma(1+2it+2i\rho_j(\chi))}.$$

Let MH denote the Mellin transform of H defined by

$$(MH)(r) = \int_0^\infty H(t)t^{-r-1}dt.$$

Since $\left|c_0\left(\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right),1/2+it\right)\right|$ is bounded it is enough to consider

$$\begin{aligned}
F(t) &= \int_{\mathbf{A}^\times/\mathbf{F}^\times} f_{H,\chi'}\left(\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right)\right)\int_{\mathbf{A}/\mathbf{F}} \left|E\left(\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right),s,\chi\right) - c_0\left(\left(\begin{smallmatrix} y & x \\ & 1\end{smallmatrix}\right),1/2+it\right)\right|^2 dx\frac{dy}{|y|_{\mathbf{A}}^2} \\
&= \frac{4^d\pi^d}{D|L(1+2it,\chi^{-2})|^2}\int_{\mathbf{A}^\times} H(|y|_{\mathbf{A}})\chi'(y)|y|_{\mathbf{A}}\prod_{v \text{ finite}}|\sigma_v(y_v,-2it,\chi_v^{-2})|^2 \\
&\quad\times \prod_{j\le r_1}\left|\frac{K_{it+i\rho_j(\chi)}(2\pi|y_j|)}{\Gamma(1/2+it+i\rho_j(\chi))}\right|^2\prod_{j>r_1}\left|\frac{K_{2it+2i\rho_j(\chi)}(4\pi|y_j|)}{\Gamma(1+2it+2i\rho_j(\chi))}\right|^2\frac{dy}{|y|_{\mathbf{A}}^2} \\
&= \frac{4^d\pi^d}{2\pi i D|L(1+2it,\chi^{-2})|^2}\int_{(2)}(MH)(r)\int_{\mathbf{A}^\times}|y|_{\mathbf{A}}^{r-1}\chi'(y)\prod_{v \text{ finite}}|\sigma_v(y_v,-2it,\chi_v^{-2})|^2 \\
&\quad\times \prod_{j\le r_1}\left|\frac{K_{it+i\rho_j(\chi)}(2\pi|y_j|)}{\Gamma(1/2+it+i\rho_j(\chi))}\right|^2\prod_{j>r_1}\left|\frac{K_{2it+2i\rho_j(\chi)}(4\pi|y_j|)}{\Gamma(1+2it+2i\rho_j(\chi))}\right|^2 dydr,
\end{aligned}$$

where we have used the Mellin inversion formula. We now consider the innermost integral as a product of local integrals. First we look at a finite place v. Let $a\in\mathbf{R}$ and χ_v,χ_v' be unramified characters on $\mathbf{F}_v^\times$. We set $\alpha_v = \chi_v(\omega_v)$ and $\alpha_v' = \chi_v'(\omega_v)$. Then

$$\begin{aligned}
\int_{\mathbf{F}_v^\times}|y_v|_v^s\chi_v'(y_v)|\sigma_v(y_v,ai,\chi_v)|^2\frac{dy_v}{|y_v|_v} &= \int_{\mathfrak{p}_v^{-d_v}-\{0\}}|y_v|_v^s\chi_v'(y_v)|\sigma_v(y_v,ai,\chi_v)|^2\frac{dy_v}{|y_v|_v} \\
&= \frac{q_v^{sd_v}}{\alpha_v'^{d_v}\left|1-\frac{\alpha_v}{q_v^{ia}}\right|^2}\sum_{k=0}^\infty\frac{\alpha_v'^k}{q_v^{ks}}\left|1-\alpha_v^{k+1}q_v^{ia(k+1)}\right|^2.
\end{aligned}$$

This is evaluated in the same way as in [17] and we get

$$\prod_{v \text{ finite}} \int_{\mathbf{F}_v^\times} |y_v|_v^s \chi'_v(y_v) |\sigma_v(y_v, ai, \chi_v)|^2 \frac{dy_v}{|y_v|_v} = \frac{\mathcal{N}(\mathcal{D})^s L(s,\chi')^2 L(s-ia,\chi\chi') L(s+ia,\chi^{-1}\chi')}{\chi'_f(\mathcal{D}) L(2s,\chi'^2)}.$$

Now we look at the infinite places. From Section B.4 in [6] we recall the formula

$$\int_0^\infty K_\mu(t) K_\nu(t) t^{s-1} dt = \frac{2^{s-3}}{\Gamma(s)} \prod \Gamma\left(\frac{s \pm \mu \pm \nu}{2}\right)$$

for $\mathrm{Re}(s) > |\mathrm{Re}(\mu)| + |\mathrm{Re}(\nu)|$. From this we see that

$$\int_0^\infty |K_{it+i\rho_j(\chi)}(2\pi t)|^2 t^{r+i\rho_j(\chi')-1} dt = \frac{\Gamma((r+i\rho_j(\chi'\chi^2))/2+it)\Gamma((r+i\rho_j(\chi'\chi^{-2}))/2-it)\Gamma((r+i\rho_j(\chi'))/2)^2}{2^3 \pi^{r+i\rho_j(\chi')} \Gamma(r+i\rho_j(\chi'))}$$

and

$$\int_0^\infty |K_{2it+2i\rho_j(\chi)}(4\pi t)|^2 t^{2r+2i\rho_j(\chi')-1} dt = \frac{\Gamma(r+i\rho_j(\chi\chi')+it)\Gamma(r+i\rho_j(\chi'\chi^{-1})-it)\Gamma(r+i\rho_j(\chi')+1)\Gamma(r+i\rho_j(\chi'))}{2^3 (2\pi)^{2r+2i\rho_j(\chi')} \Gamma(2r+2i\rho_j(\chi'))}.$$

We set

$$\begin{aligned} B(r,t) &= (MH)(r) \frac{L(r,\chi')^2 L(r+2it,\chi^{-2}\chi') L(r-2it,\chi^2\chi') D^r}{4^{r_2 r} \pi^{dr} L(2r,\chi'^2)} \\ &\times \prod_{j \le r_1} \frac{\Gamma((r+i\rho_j(\chi'\chi^2))/2+it)\Gamma((r+i\rho_j(\chi'\chi^{-2}))/2-it)\Gamma((r+i\rho_j(\chi'))/2)^2}{\Gamma(r+i\rho_j(\chi'))} \\ &\times \prod_{j > r_1} \frac{\Gamma(r+1/2+i\rho_j(\chi\chi')+it)\Gamma(r+1/2+i\rho_j(\chi'\chi^{-1})-it)}{\Gamma(2r+2i\rho_j(\chi')+1)} \\ &\times \Gamma(r+i\rho_j(\chi')+1)\Gamma(r+i\rho_j(\chi')). \end{aligned}$$

Thus

$$\begin{aligned} F(t) &= \frac{2^{2r_2-r_1}\pi^d}{2\pi i D \chi'_f(\mathcal{D}) |L(1+2it,\chi^{-2})|^2 \prod_{j>r_1} 4^{i\rho_j(\chi')}} \\ &\times \frac{1}{|\prod_{j\le r_1} \Gamma(1/2+it+i\rho_j(\chi)) \prod_{j>r_1} \Gamma(1+2it+2i\rho_j(\chi))|^2} \int_{(2)} B(r,t) dr. \end{aligned}$$

As in [17] we estimate the complex integral by moving the integration from the line $\mathrm{Re}(r) = 2$ to the curve $\mathrm{Re}(r) = 1/2$ and get

$$\int_{(2)} B(r,t) dr = \int_{(1/2)} B(r,t) dr + 2\pi i \mathrm{res}_{r=1} B(r,t) + O(|t|^{-10}),$$

From [15] it is well known that

$$L(1/2+it,\chi) \ll |t|^{\frac{d}{6}+\varepsilon}.$$

We also recall Stirling's formula

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi} e^{-\pi|t|/2} |t|^{\sigma-1/2}. \tag{5.1}$$

Proceeding as in [17] we obtain the estimate

$$\frac{1}{|L(1+2it,\chi^{-2})\prod_{j\le r_1}\Gamma(1/2+it+i\rho_j(\chi))\prod_{j>r_1}\Gamma(1+2it+2i\rho_j(\chi))|^2}\times$$
$$\int_{(2)} B(r,t)dr \ll |t|^{-\frac{n}{6}+\varepsilon}.$$

for any $\varepsilon > 0$.

Now we look at the residue term. It is well known that $L(s,\chi)$ is entire if $\chi \neq \chi_0$ so in that case the residue term vanishes and we are done. Assume therefore that $\chi = \chi_0$. In this case $L(s,\chi_0) = \zeta_{\mathbf{F}}(s)$ has a pole of order 1 at $s = 1$, i.e. there exist constants C_1 and C_2 such that

$$L(s,\chi_0) = \frac{C_1}{s-1} + C_2 + O(s-1).$$

In fact we know that

$$C_1 = \frac{2^{r_1}(2\pi)^{r_2}hR}{\mathcal{W}\sqrt{D}}.$$

We define $G(r,t)$ by

$$B(r,t) = L(s,\chi_0)^2 G(r,t).$$

We see that

$$\mathrm{res}_{r=1}B(r,t) = G(1,t)C_1\left(2C_2 + C_1\frac{G'(1,t)}{G(1,t)}\right).$$

As in [17] we obtain the estimate

$$\frac{L'(1+it,\chi)}{L(1+it,\chi)} \ll (\log t)^{\frac{2}{3}}$$

using the zero free region derived in [2]. Further more we know that

$$\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)} \sim \log t.$$

Since

$$(MH)(1) = \frac{\mathcal{W}}{2^n\pi^{r_2}hR}\int_{X(\mathbf{F})} P(g,H,\chi_0)d\mu$$

this implies that

$$\frac{F(t)}{\log t} \to \frac{2^{r_2}\pi^{r_1}nhR}{\zeta_{\mathbf{F}}(2)\mathcal{W}D}\int_{X(\mathbf{F})} P(g,H,\chi_0)d\mu$$

as $t \to \infty$ and we are done.

6. Proof of Theorem 1.4

Let φ denote a cusp form with a Fourier expansion as in Section 3. We want to use the Rankin-Selberg method to evaluate the integral

$$\int_{X(\mathbf{F})} \varphi(g)d\mu_{\chi,t} = \int_{X(\mathbf{F})} \varphi(g)E(g,1/2+it,\chi)E(g,1/2-it,\chi^{-1})d\mu,$$

where we have used the identity $\overline{E(g,s,\chi)} = E(g,\overline{s},\chi^{-1})$. Therefore we consider the integral

$$I(s) = \int_{X(\mathbf{F})} \varphi(g)E(g,1/2+it,\chi)E(g,s,\chi^{-1})d\mu$$

for $\mathrm{Re}(s) > 1$ and then later use the fact that the integral has an analytic continuation. We unfold the integral using the series defining the Eisenstein series (C collects constants from the Fourier coefficients of φ and $E(g,s,\chi)$)

$$\begin{aligned}
I(s) &= \int_{\mathbf{A}^\times/\mathbf{F}^\times} f\left(\left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right)\right) \int_{\mathbf{A}/\mathbf{F}} \varphi(g) E\left(\left(\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right), 1/2+it, \chi\right) dx \frac{dy}{|y|_{\mathbf{A}}^2} \\
&= C\pi^{dit} 4^{r_2 it} L(1+2it, \chi^{-2})^{-1} \int_{\mathbf{A}^\times/\mathbf{F}^\times} \chi^{-1}(y)|y|_{\mathbf{A}}^{s+1} \times \\
&\sum_{\alpha\in\mathbf{F}^\times} \prod_{j\le r_1} |\alpha^{(j)}|^{it+i\rho_j(\chi)} \frac{K_{i\rho_j(\chi)+it}(2\pi|\alpha^{(j)} y_j|) K_{ir_j}(2\pi|\alpha^{(j)} y_j|)}{\Gamma(1/2+it+i\rho_j(\chi))} \times \\
&\prod_{j>r_1} |\alpha^{(j)}|_{\mathbf{C}}^{it+i\rho_j(\chi)} \frac{K_{2i\rho_j(\chi)+2it}(4\pi|\alpha^{(j)} y_j|) K_{2ir_j}(4\pi|\alpha^{(j)} y_j|)}{\Gamma(2+2it+2i\rho_j(\chi))} \times \\
&\prod_{v \text{ finite}} \frac{\sigma_v(\alpha_v y_v, -2it, \chi_v^{-2})}{|y_v|_v^{it}\chi_v(y_v)} \frac{\beta_{1,v}^{\mathrm{ord}_{\mathbf{F}_v}(\alpha_v y_v)+1+d_v} - \beta_{2,v}^{\mathrm{ord}_{\mathbf{F}_v}(\alpha_v y_v)+1+d_v}}{\beta_{1,v}-\beta_{2,v}} \frac{dy}{|y|_{\mathbf{A}}^2} \\
&= C\pi^{dit} 4^{r_2 it} L(1+2it, \chi^{-2})^{-1} \int_{\mathbf{A}^\times} \chi^{-2}(y)|y|_{\mathbf{A}}^{s-1-it} \times \\
&\prod_{j\le r_1} |y_j|^{it+i\rho_j(\chi)} \frac{K_{i\rho_j(\chi)+it}(2\pi|y_j|) K_{ir_j}(2\pi|y_j|)}{\Gamma(1/2+it+i\rho_j(\chi))} \times \\
&\prod_{j>r_1} |y_j|_{\mathbf{C}}^{it+i\rho_j(\chi)} \frac{K_{2i\rho_j(\chi)+2it}(4\pi|y_j|) K_{2ir_j}(4\pi|y_j|)}{\Gamma(2+2it+2i\rho_j(\chi))} \times \\
&\prod_{v \text{ finite}} \sigma_v(y_v, -2it, \chi_v^{-2}) \frac{\beta_{1,v}^{\mathrm{ord}_{\mathbf{F}_v}(y_v)+1+d_v} - \beta_{2,v}^{\mathrm{ord}_{\mathbf{F}_v}(y_v)+1+d_v}}{\beta_{1,v}-\beta_{2,v}} dy
\end{aligned}$$

Define

$$\begin{aligned}
&\Gamma(s,a,b) = \\
&\quad \frac{\Gamma((s+ia+ib)/2)\Gamma((s+ia-ib)/2)\Gamma((s-ia-ib)/2)\Gamma((s-ia+ib)/2)}{2^3\pi^s\Gamma(s)}.
\end{aligned}$$

It is well known (see [6] Section B.4) that

$$\int_0^\infty K_{ia}(2\pi t) K_{ib}(2\pi t) t^{s-1} dt = \Gamma(s,a,b). \tag{6.1}$$

Thus the contribution from the local integrals at the real places is (the constant Γ-factor is ignored)

$$\Gamma(s - i\rho_j(\chi), \rho_j(\chi)+t, r_j).$$

Similarly the contribution from the local integrals at the complex places is

$$2^{1+2i\rho_j(\chi)-2s}\Gamma(2s-1-i\rho_j(\chi), 2\rho_j(\chi)+2t, 2r_j).$$

Finally we must evaluate the integrals over the finite places, and we obtain

$$q_v^{d_v(s-it)}\chi_v(\omega_v)^{2d_v} \sum_{k=0}^\infty q_v^{k(it-s)}\chi_v(\omega_v)^{-2k} \frac{\beta_{1,v}^{k+1}-\beta_{2,v}^{k+1}}{\beta_{1,v}-\beta_{2,v}} \frac{1-\chi_v(\omega_v)^{-2(k+1)}q_v^{-2it(k+1)}}{1-\chi_v(\omega_v)^{-2}q_v^{-2it}}$$

which can be written in terms of local L-factors (see [17]) as

$$q_v^{d_v(s-it)}\chi_v(\omega_v)^{2d_v} \frac{L_v(s-it,\varphi,\chi_v)L_v(s+it,\varphi)}{L_v(2s,\chi_v^2)}.$$

From this we conclude that $I(s)$ has a meromorphic continuation to the entire s-plane, which is holomorphic on the line $\mathrm{Re}(s) = \frac{1}{2}$ and

$$I(1/2 - it) \ll \frac{|L(1/2 - 2it, \varphi, \chi)|}{|L(1 + 2it, \chi^{-2})|^2} \prod_{j \leq r_1} \frac{\Gamma(1/2 - it - i\rho_j(\chi), \rho_j(\chi) + t, r_j)}{\Gamma(1/2 + it + i\rho_j(\chi))} \times$$
$$\prod_{j > r_1} \frac{\Gamma(-2it - i\rho_v(\chi), 2\rho_v(\chi) + 2t, 2r_j)}{\Gamma(2 + 2it + 2i\rho_j(\chi))}.$$

From Stirling's formula (5.1) we obtain

$$\prod_{j \leq r_1} \frac{\Gamma(1/2 - it - i\rho_j(\chi), \rho_j(\chi) + t, r_j)}{\Gamma(1/2 + it + i\rho_v(\chi))} \ll |t|^{-\frac{r_1}{2}}$$

and

$$\prod_{j > r_1} \frac{\Gamma(1 - 2it - i\rho_j(\chi), 2\rho_j(\chi) + 2t, 2r_j)}{\Gamma(1 + 2it + 2i\rho_v(\chi))} \ll |t|^{-r_2}.$$

From [3] and [10] we know that there exists some $\delta > 0$ such that

$$L(1/2 + it, \varphi, \chi) \ll (1 + |t|)^{\frac{d}{2} - \delta}$$

and this proves the theorem.

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DISTRIBUTION OF ANGLES IN HYPERBOLIC LATTICES

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Abstract

We prove an effective equidistribution result about angles in a hyperbolic lattice. We use this to generalize a result from the study by Boca.

1. Introduction

Consider the group $G = \mathrm{SL}_2(\mathbb{R})$ that acts on the upper halfplane $\mathbb{H}$ by linear fractional transformations. Let $\Gamma \subset G$ be a cofinite discrete group, and let $d : \mathbb{H} \times \mathbb{H} \to \mathbb{R}_+$ denote the hyperbolic distance. Consider the counting function

$$N_\Gamma(R, z_0, z_1) = \#\{\gamma \in \Gamma \,|\, d(z_0, \gamma z_1) \leq R\}.$$

The *hyperbolic lattice point problem* is the problem of estimating this function as $R \to \infty$. A typical result would be an asymptotic expansion of the form

$$N_\Gamma(R, z_0, z_1) = \frac{\kappa_\Gamma \pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} e^R + O(e^{R(\alpha+\varepsilon)}) \tag{1}$$

for some $\alpha < 1$, where $\kappa_\Gamma = 2$ if $-I \in \Gamma$ and $\kappa_\Gamma = 1$ otherwise. The problem has been considered by numerous people including Delsarte [3], Huber [8–10] (Γ cocompact), Patterson [20] ($\alpha = 3/4$ if there are no small eigenvalues), Selberg (unpublished) and Good [6] ($\alpha = 2/3$ if there are no small eigenvalues). Higher dimensional analogues have also been considered (see e.g [4, 14, 15]), as well as the analogous problem for manifolds with non-constant curvature [7, 16]. For a discussion of the optimal choice of α, we refer to [21], where the authors prove that α must be at least $1/2$ and they indicate that in many cases we should maybe expect (1) to hold with $\alpha = 1/2$.

Let $\varphi_{z_0,z_1}(\gamma)$ be $(2\pi)^{-1}$ times the angle between the vertical geodesic from z_0 to ∞ and the geodesic between z_0 and γz_1 (Fig. 1).

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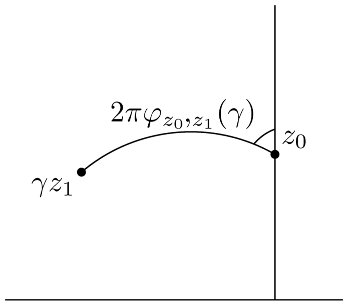

Figure 1. Angle between geodesic rays.

These normalized angles are equidistributed modulo one, i.e. for every interval $I \subset \mathbb{R}/\mathbb{Z}$ we have

$$\frac{N_\Gamma^I(R, z_0, z_1)}{N_\Gamma(R, z_0, z_1)} \to |I| \text{ as } R \to \infty, \tag{2}$$

where

$$N_\Gamma^I(R, z_0, z_1) = \#\{\gamma \in \Gamma \mid d(z_0, \gamma z_1) \le R,\ \varphi_{z_0,z_1}(\gamma) \in I\}, \tag{3}$$

and $|I|$ is the length of the interval. This has been proved by Selberg (unpublished, see comment in [**6**, p. 120]), Nicholls [**19**] and Good [**6**].

In this paper we start by proving (2) with an error term:

Theorem 1.1 *Let $K \subset \mathbb{H}$ be a compact set. There exists a constant $\alpha < 1$ possibly depending on Γ and K such that for all $z_0, z_1 \in K$ and all intervals I in $\mathbb{R}/\mathbb{Z}$*

$$\frac{N_\Gamma^I(R, z_0, z_1)}{N_\Gamma(R, z_0, z_1)} = |I| + O(e^{R(\alpha-1+\varepsilon)}).$$

If we assume that the automorphic Laplacian on $\Gamma\backslash\mathbb{H}$ has no exceptional eigenvalues, i.e. eigenvalues in $]0, 1/4[$, we prove that we can take

$$\alpha = 11/12.$$

If there are exceptional eigenvalues, the exponent could become larger, depending on how close to zero they are. We prove Theorem 1.1 by proving asymptotic expansions for the exponential sums

$$\sum_{\substack{\gamma\in\Gamma \\ d(z_0,\gamma z_1)\le R}} e(n\varphi_{z_0,z_1}(\gamma)), \tag{4}$$

where $n \in \mathbb{Z}$ and $e(x) = \exp(2\pi i x)$. The exponent $11/12$ can certainly be improved. In fact our proof uses a variant of Huber's method [**8**] that does not give the optimal bound even for the expansion (1). In principle, Theorem 1.1 could be proved using the method of Good from [**6**], which gives the best known error term in the hyperbolic lattice point problem (1). The one missing point in [**6**] to prove Theorem 1.1 is the dependence of n in the expansion of the exponential sum (4). Rather than

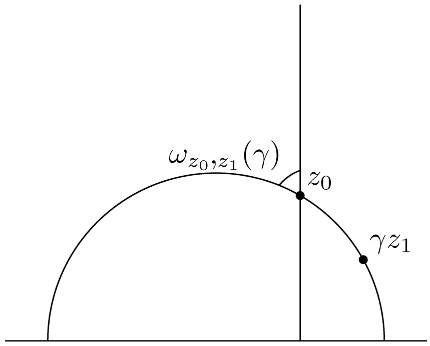

Figure 2. Definition of $\omega_{z_0,z_1}(\gamma)$.

patiently tracking down the n-dependence, we found it more to the point — albeit at the expense of poor error terms — to provide an alternative and more direct proof inspired by [**8**].

Recently, Boca [**2**] considered a related problem: What happens if we order the elements according to $d(z_1, \gamma z_1)$ instead of $d(z_0, \gamma z_1)$? Let $\Gamma(N)$ be the principal congruence group of level N, i.e. the set of 2×2 matrices γ satisfying $\gamma \equiv I \mod N$. Boca identified for these groups the limiting distribution using non-trivial bounds for Kloosterman sums. He proved the following[1]: Let $z_0, z_1 \in \mathbb{H}$ and let $\omega_{z_0,z_1}(\gamma)$ denote the angle in $[-\pi/2, \pi/2]$ between the vertical geodesic through z_0 and the geodesic containing z_0 and γz_1 (Fig. 2). If $z_0 = \gamma z_1$ you can assign $\omega_{z_0,z_1}(\gamma)$ the value 0 — it does not matter what you choose, since there are only a finite number of such γ's.

For any interval $I \subset [-\pi/2, \pi/2]$, we consider the counting function

$$\mathfrak{N}_\Gamma^I(R, z_0, z_1) = \#\{\gamma \in \Gamma \mid d(z_1, \gamma z_1) \leq R,\ \omega_{z_0,z_1}(\gamma) \in I\}.$$

We emphasize that the elements are ordered according to $d(z_1, \gamma z_1)$ instead of $d(z_0, \gamma z_1)$. We shall write $\mathfrak{N}_\Gamma(R, z_0, z_1)$ instead of $\mathfrak{N}_\Gamma^{[-\pi/2,\pi/2]}(R, z_0, z_1)$. Following Boca we define

$$\eta_{z_0,z_1}(t) = \frac{2y_0y_1(y_0^2 + y_1^2 + (x_0 - x_1)^2)}{(y_0^2 + y_1^2 + (x_0 - x_1)^2)^2 - ((y_1^2 - y_0^2 + (x_0 - x_1)^2)\cos(t) + 2y_0(x_0 - x_1)\sin(t))^2}.$$

Then Boca proves the following result:

THEOREM 1.2 *Let* $\Gamma = \Gamma(N)$. *For any interval* $I \subset [-\pi/2, \pi/2]$

$$\frac{\mathfrak{N}_\Gamma^I(R, z_0, z_1)}{\mathfrak{N}_\Gamma(R, z_0, z_1)} = \frac{1}{\pi}\int_I \eta_{z_0,z_1}(t)\,\mathrm{d}t + O(e^{(7/8-1+\varepsilon)R})$$

for any $\varepsilon > 0$.

[1]Readers consulting [**2**] should be warned that our notation differs slightly from Boca's.

In the view of (1) Theorem 1.2 is equivalent to an expansion of $\mathfrak{N}^{I}_{\Gamma(N)}(R, z_0, z_1)$. We generalize and refine Boca's result: with data as above, $I \subset \mathbb{R}/\mathbb{Z}$ and $w \in \mathbb{H}$ we consider the counting function

$$\mathcal{N}^{I}_{\Gamma}(R, z_0, z_1, w) = \#\{\gamma \in \Gamma \,|\, d(z_1, \gamma w) \leq R,\ \varphi_{z_0,w}(\gamma) \in I\}.$$

We emphasize that we order according to $d(z_1, \gamma w)$. As before we shall write $\mathcal{N}_{\Gamma}(R, z_0, z_1, w)$ instead of $\mathcal{N}^{[-1/2,1/2]}_{\Gamma}(R, z_0, z_1, w)$. Besides the more general ordering, our result is more refined in the sense that we can distinguish between angles that differ by π. Consider

$$\rho_{z_0,z_1}(\omega) = \frac{2y_0y_1}{((x_0 - x_1)^2 + y_0^2 + y_1^2)(1 - \cos(2\pi\omega)) + 2y_0^2\cos(2\pi\omega) + 2(x_1 - x_0)y_0\sin(2\pi\omega)}.$$

Then we prove the following result:

THEOREM 1.3 *Let Γ be any cofinite Fuchsian group. There exists $\alpha < 1$ such that for any $I \subset \mathbb{R}/\mathbb{Z}$ we have*

$$\frac{\mathcal{N}^{I}_{\Gamma}(R, z_0, z_1, w)}{\mathcal{N}_{\Gamma}(R, z_0, z_1, w)} = \int_I \rho_{z_0,z_1}(\omega)d\omega + O(e^{(\alpha-1+\varepsilon)R})$$

for any $\varepsilon > 0$.

Note that in the special case of $\Gamma = \Gamma(N)$ and $w = z_1$, this implies Theorem 1.2 (with a poorer error term though), since

$$\eta_{z_0,z_1}(2\pi t) = \rho_{z_0,z_1}(t) + \rho_{z_0,z_1}(t + 1/2).$$

We will prove that Theorem 1.3 follows from Theorem 1.1.

Whereas Boca is using a non-trivial bound for Kloosterman sums, we are utilizing the fact that for *any* group there is a spectral gap between the zero eigenvalue of the Laplacian and the first non-zero eigenvalue. As in Theorem 1.1, α in Theorem 1.3 generally depends on the size of the first non-zero eigenvalue.

We remark that all the results presented here can easily be phrased in terms of points in the orbit Γz_1, rather than elements in Γ, since

$$\#\{z \in \Gamma z_1 \mid d(z_0, z) \leq R\} = \frac{N_{\Gamma}(R, z_0, z_1)}{|\Gamma_{z_1}|},$$

where Γ_{z_1} denotes the stabilizer of z_1.

2. Effective equidistribution of angles

Let $G = \mathrm{SL}_2(\mathbb{R})$. The group G acts on the upper halfplane $\mathbb{H}$ by linear fractional transformations

$$gz = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,\ z \in \mathbb{H}.$$

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be discrete and cofinite. For simplicity we assume that $-I \notin \Gamma$. If $-I \in \Gamma$ we need to multiply all main terms by 2.

For $z \in \mathbb{H}$, we let $r = r(z)$ and $\varphi = \varphi(z)$ be the geodesic polar coordinates of z. These are related to the rectangular coordinates by

$$z = \begin{pmatrix} \cos\varphi(z) & \sin\varphi(z) \\ -\sin\varphi(z) & \cos\varphi(z) \end{pmatrix} \exp(-r(z))i. \tag{5}$$

We note that if $z_0 = x_0 + iy_0$ and we let

$$\gamma_0 = \begin{pmatrix} 1/\sqrt{y_0} & -x_0/\sqrt{y_0} \\ 0 & \sqrt{y_0} \end{pmatrix}$$

then it is straightforward to check that $\gamma_0 z_0 = i$. We see that

$$\varphi_{z_0,z_1}(\gamma) = \varphi_{i,\gamma_0 z_1}(\gamma_0\gamma\gamma_0^{-1}) = \frac{\varphi(\gamma_0\gamma\gamma_0^{-1}(\gamma_0 z_1))}{\pi}$$

and

$$d(z_0, \gamma z_1) = d(i, \gamma_0\gamma\gamma_0^{-1}(\gamma_0 z_1)) = r(\gamma_0\gamma\gamma_0^{-1}(\gamma_0 z_1)).$$

Therefore, after conjugation of the group Γ the counting problems in the introduction may be formulated in terms of $r(\gamma z)$ and $\varphi(\gamma z)$ with $z = \gamma_0 z_1$.

The Laplacian for the G-invariant measure $\mathrm{d}\mu(z) = \mathrm{d}x\mathrm{d}y/y^2$ on $\mathbb{H}$ is given in Cartesian coordinates by

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

In geodesic polar coordinates, the Laplace operator is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\tanh r}\frac{\partial}{\partial r} + \frac{1}{4\sinh^2(r)}\frac{\partial^2}{\partial\varphi^2}. \tag{6}$$

Consider $L^2(\Gamma\backslash\mathbb{H}, \mathrm{d}\mu(z))$ with inner product $\langle f, g\rangle = \int_{\Gamma\backslash\mathbb{H}} f\overline{g}\,\mathrm{d}\mu(z)$ and norm $\|f\|_2 = \sqrt{\langle f, f\rangle}$. The Laplacian induces an operator on $L^2(\Gamma\backslash\mathbb{H}, d\mu(z))$ called the automorphic Laplacian defined as follows: consider the operator defined by $-\Delta f$ on smooth, bounded, Γ-invariant functions satisfying that $-\Delta f$ is also bounded. This operator is densely defined in $L^2(\Gamma\backslash\mathbb{H})$ and is in fact essentially self-adjoint. The closure of this operator is called the automorphic Laplacian. By standard abuse of notation, we also denote this operator by $-\Delta$.

The automorphic Laplacian is self-adjoint and non-negative and has eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_i \leq \cdots$$

with the number of eigenvalues being finite or $\lambda_i \to \infty$. It has a continuous spectrum $[1/4, \infty[$ with multiplicity equal to the number of inequivalent cusps.

By standard operator theory for self-adjoint operators (see e.g. [13]) the resolvent $R(s) = (-\Delta - s(1-s))^{-1}$ is a bounded operator that is meromorphic in s for $s(1-s)$ *off* the spectrum of $-\Delta$. For an eigenvalue λ_i outside the continuous spectrum, the operator $R(s) - P_i/(\lambda_i - s(1-s))$ is analytic at s satisfying $s(1-s) = \lambda_i$ where P_i is the projection to the λ_i eigenspace. In particular, for $\lambda = 0$, we note that

$$R(s) - \frac{P_0}{-(s(1-s))} \tag{7}$$

is analytic for $\Re(s) > 1 - \delta$ for some δ where $P_0 f = \int f(z)\,d\mu(z)/\mathrm{vol}(\Gamma\backslash\mathbb{H})$ is the projection to the 0 eigenspace. (Alternatively one may quote [11, Theorem 7.5] to obtain the same result.)

We define for $\Re(s) > 1$

$$G_n(z,s) = \sum_{\gamma\in\Gamma} \frac{e(n\varphi(\gamma z)/\pi)}{(\cosh(r(\gamma z)))^s}. \tag{8}$$

We recall that

$$\cosh(r(\gamma z)) = 1 + 2u(\gamma z, i), \tag{9}$$

where $u(z,w)$ is the point pair invariant defined by

$$u(z,w) = \frac{|z-w|^2}{4\Im(z)\Im(w)}. \tag{10}$$

Hence

$$\left|\frac{e(n\varphi(z)/\pi)}{(\cosh(r(z)))^s}\right| \le \frac{1}{(1+2u(z,i))^{\Re(s)}}.$$

It therefore follows from [22, Theorem 6.1] and the discussion leading up to it that the sum (10) converges absolutely and uniformly on compact sets and the limit is Γ-automorphic, and bounded in z – in particular square integrable on $\Gamma\setminus\mathbb{H}$.

By applying the Laplace operator to $G_n(z,s)$ a straightforward calculation shows that

$$(-\Delta - s(1-s))G_n(z,s) = s(s+1)G_n(z,s+2) + \sum_{\gamma\in\Gamma} \frac{n^2 e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z))(\cosh(r(\gamma z)))^s}. \tag{11}$$

The sum on the right converges absolutely and uniformly on compacta for $\Re(s) > -1$. Since $G_n(z,s)$ is square integrable, we may invert (11) using the resolvent

$$R(s) = (-\Delta - s(1-s))^{-1}, \tag{12}$$

so we have

$$G_n(z,s) = R(s)\left(s(s+1)G_n(z,s+2) + \sum_{\gamma\in\Gamma} \frac{n^2 e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z))(\cosh(r(\gamma z)))^s}\right). \tag{13}$$

The right-hand side is meromorphic in s for $\Re(s) > 1/2$ since the resolvent is holomorphic for $s(1-s)$ not in the spectrum of the automorphic Laplacian. This gives the meromorphic continuation of $G_n(z,s)$ to $\Re(s) > 1/2$. The only potential poles are at $s = 1$ and $s = s_j$ where $s_j(1-s_j)$ is a small eigenvalue for the automorphic Laplacian. Using the analyticity of (7), we see that the pole at $s=1$ has residue

$$\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\int_{\Gamma\backslash\mathbb{H}}\left(2G_n(z,3)+\sum_{\gamma\in\Gamma}\frac{n^2e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z))\cosh(r(\gamma z))}\right)\mathrm{d}\mu(z). \tag{14}$$

By unfolding the integral we find that this equals

$$\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\int_{\mathbb{H}}\left(2\frac{e(n\varphi(z)/\pi)}{\cosh^3(r(z))}+\frac{n^2e(n\varphi(z)/\pi)}{\sinh^2(r(z))\cosh(r(z))}\right)\mathrm{d}\mu(z). \tag{15}$$

Changing to polar coordinates, we find

$$\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\int_0^\infty\int_0^\pi\left(2\frac{e(n\varphi/\pi)}{\cosh^3(r)}+\frac{n^2e(n\varphi/\pi)}{\sinh^2(r)\cosh(r)}\right)2\sinh(r)\,\mathrm{d}\varphi\,\mathrm{d}r, \tag{16}$$

which equals

$$\frac{2\pi\delta_{n=0}}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}. \tag{17}$$

This follows since

$$\int_0^\infty\frac{2\sinh(r)}{\cosh(r)^3}\mathrm{d}r=1.$$

From a Wiener–Ikehara Tauberian theorem (see e.g. [18, Theorem 3.3.1 and Exercises $3.3.3+3.3.4$]) we may conclude that

$$\sum_{\substack{\gamma\in\Gamma\\ \cosh(r(\gamma z))\le R}}e(n\varphi(\gamma z)/\pi)=2\pi\frac{\delta_{n=0}}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}R+o(R). \tag{18}$$

This implies immediately – via Weyl's criterion – that the angles $\varphi(\gamma z)/\pi$ are equidistributed modulo 1.

Since we intend to obtain a power saving in the remainder term we investigate $G_n(z,s)$ a bit more carefully:

LEMMA 2.1 *Write $s=\sigma+it$. For z in a fixed compact set $K\subset\mathbb{H}$, $|t|>1$ and $\sigma>\sigma_0>1/2$ we have*

$$G_n(z,s)=O(|t|\,(|t|^2+n^2)),$$

where the implied constant may depend on Γ, K and σ_0.

Proof. We recall that [13, V (3.16)]

$$\|R(s)\|_\infty \leq \frac{1}{\operatorname{dist}(s(1-s),\operatorname{spec}(-\Delta))} \leq \frac{1}{|t|\,(2\sigma-1)}, \tag{19}$$

where $\|\cdot\|_\infty$ denotes the operator norm. For $\sigma > 3/2$ we have

$$\|G_n(z,s)\|_2 \leq \|G_0(z,3/2)\|_2 = O(1). \tag{20}$$

For $\sigma > \sigma_0$ we may use (20) and (13) to conclude that

$$\begin{aligned}
\|G_n(z,s)\|_2 &\leq \|R(s)\|_\infty \left(\|s(s+1)G_n(z,s+2)\|_2 + \left\| \sum_{\gamma\in\Gamma} \frac{n^2 e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z))(\cosh(r(\gamma z)))^s} \right\|_2 \right) \\
&\leq \frac{1}{|t|\,(2\sigma-1)} \left(|t|^2 \|G_0(z,3/2)\|_2 + \left\| \sum_{\gamma\in\Gamma} \frac{n^2}{\sinh^2(r(\gamma z))(\cosh(r(\gamma z)))^{1/2}} \right\|_2 \right) \\
&= O(|t|^{-1}(|t|^2+n^2)).
\end{aligned} \tag{21}$$

Using (21) and (11) we find

$$\|\Delta G_n(z,s)\|_2 = O(|t|\,(|t|^2+n^2)). \tag{22}$$

We can now use the Sobolev embedding theorem and elliptic regularity theory to get a pointwise bound.

For any non-empty open set Ω in $\mathbb{R}^2$ we consider the classical Sobolev space $W^{k,p}(\Omega)$ with corresponding norm $\|\cdot\|_{W^{k,p}(\Omega)}$ (see [1, p. 59]). Whenever Ω satisfies the cone condition (see [1, p. 82]) the Sobolev embedding theorem [1, Theorem 4.12]) gives an embedding

$$W^{2,2}(\Omega) \to C_B(\Omega) \tag{23}$$

where $C_B(\Omega)$ is the set of bounded continuous functions on Ω equipped with the sup norm. In particular, for $f \in W^{2,2}(\Omega)$, we have

$$\sup_{z\in\Omega} |f(z)| \leq C\,\|f\|_{W^{2,2}(\Omega)} \tag{24}$$

where C is a constant which depends only on Ω.

By elliptic regularity theory, if $\Delta_E = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Euclidean Laplace operator we have also that if $u \in W^{1,2}(\Omega)$ satisfies $\Delta_E u \in L^2(\Omega)$ (weak derivative) then

$$\|u\|_{W^{2,2}(\Omega')} \leq C'(\|u\|_{L^2(\Omega)} + \|\Delta_E u\|_{L^2(\Omega)}) \tag{25}$$

for all $\Omega' \subset \Omega$, which satisfies that the closure of Ω' is compact and contained in Ω. Here C' is a constant, which depends only on Ω and Ω' (see [12, Theorem 8.2.1]).

We can use this general theory to bound $|G_n(z, s)|$ in the following way: for every z in the compact set K we fix a small open (Euclidean) disc Ω_z centered at z with some radius such that its closure $\overline{\Omega}_z$ is contained in $\mathbb{H}$. Let Ω'_z be the open disc with half the radius. By compactness of K, the cover $\{\Omega'_z\}$ admits a finite subcover, i.e. $K \subset \cup_{i=1}^n \Omega_{z_i}$ for $z_i \in K$. Choose as a fundamental domain for $\Gamma\backslash\mathbb{H}$ a normal polygon F. Since Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, Ω_{z_i} intersects non-trivially with γF for only finitely many (say n_i) $\gamma \in \Gamma$ (see [**17**, 1.6.2 (3)]).

Therefore, for any automorphic function f,

$$\begin{aligned} \|f\|^2_{L^2(\Omega_{z_i})} &:= \int_{\Omega_{z_i}} |f(z)|^2 \, \mathrm{d}x \, \mathrm{d}y \\ &\le n_i \overline{y}_i^2 \int_F |f(z)|^2 \, \mathrm{d}\mu(z) = n_i \overline{y}_i^2 \, \|f\|_2^2 \end{aligned} \tag{26}$$

and

$$\begin{aligned} \|\Delta_E f\|^2_{L^2(\Omega_{z_i})} &:= \int_{\Omega_{z_i}} |\Delta_E f(z)|^2 \, \mathrm{d}x \, \mathrm{d}y \\ &\le n_i \underline{y}_i^{-2} \int_F |\Delta f(z)|^2 \, \mathrm{d}\mu(z) = n_i \underline{y}_i^{-2} \, \|\Delta f\|_2^2 \end{aligned} \tag{27}$$

where $\overline{y}_i < \infty$ and $\underline{y}_i > 0$ are heights over and under Ω_i. It is straightforward to verify that $G_n(z, s)$ is in $W^{1,2}(\Omega_i)$ (since it is continuously differentiable) and that Ω_i has the cone property, so we may use the above inequalities to conclude

$$\begin{aligned} \sup_{z \in K} |G_n(z, t)| &\le \max_{i=1}^{n} \sup_{z \in \Omega'_{z_i}} |G_n(z, s)| \\ &\le \max_i C_i \, \|G_n(z, s)\|_{W^{2,2}(\Omega'_{z_i})} \quad \text{by (24)} \\ &\le \max_i C_i C'_i C''_i (\|G_n(z, s)\|_{L^2(\Omega_{z_i})} + \|\Delta_E G_n(z, s)\|_{L^2(\Omega_{z_i})}) \quad \text{by (25)} \\ &\le \max_i B_i B'_i B''_i (\|G_n(z, s)\|_2 + \|\Delta G_n(z, s)\|_2) \quad \text{by (26) and (27)} \\ &\le C_K (|t| \, (n^2 + |t|^2)) \quad \text{by (21) and (22)} \end{aligned}$$

which concludes the proof.

We note that Lemma 2.1 implies that

$$G_n(z, s) = O(|t|^3) \tag{28}$$

when $|n| \le |t|$, and by applying the Phragmén-Lindelöf theorem we may reduce the exponent to $\max(6(1 - \sigma) + \varepsilon, 0)$ for any $\varepsilon > 0$.

We may now use the meromorphic continuation of $G_n(z, s)$ and Lemma 2.1 to get an asymptotic expansion with error term for the sum in (18). We will assume that there are no exceptional eigenvalues, which implies that $G_n(z, s)$ is regular in $\Re(s) > 1/2$. If this is not the case, $G_n(z, s)$ will still be regular in $\Re(s) > h$ for some $h < 1$. In (33) we then move the line of integration to $\Re(s) > h + \varepsilon$. Proceeding with the obvious changes still gives a non-trivial error term in the end. We shall not dwell on the details.

Let $\psi_U : \mathbb{R}_+ \to \mathbb{R}$, $U \geq U_0$, be a family of smooth non-increasing functions with

$$\psi_U(t) = \begin{cases} 1 & \text{if } t \leq 1 - 1/U \\ 0 & \text{if } t \geq 1 + 1/U, \end{cases} \tag{29}$$

and $\psi_U^{(j)}(t) = O(U^j)$ as $U \to \infty$. For $\Re(s) > 0$ we let

$$M_U(s) = \int_0^\infty \psi_U(t) t^{s-1}\, \mathrm{d}t$$

be the Mellin transform of ψ_U. Then we have

$$M_U(s) = \frac{1}{s} + O\left(\frac{1}{U}\right) \qquad \text{as } U \to \infty \tag{30}$$

and for any $c > 0$

$$M_U(s) = O\left(\frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^c\right) \qquad \text{as } |s| \to \infty. \tag{31}$$

Both estimates are uniform for $\Re(s)$ bounded. The first is a mean value estimate whereas the second is successive partial integration and a mean value estimate. We use here the estimate $\psi_U^{(j)}(t) = O(U^j)$. The Mellin inversion formula now gives

$$\sum_{\gamma\in\Gamma} e(n\varphi(\gamma z)/\pi)\psi_U\left(\frac{\cosh(r(\gamma z))}{R}\right) = \frac{1}{2\pi i}\int_{\Re(s)=2} G_n(z,s) M_U(s) R^s\, \mathrm{d}s. \tag{32}$$

We note that by Lemma 2.1 the integral is convergent as long as $G_n(z, s)$ has polynomial growth on vertical lines. We now move the line of integration to the line $\Re(s) = h$ with $h < 1$ by integrating along a box of some height and then letting this height go to infinity. Using Lemma 2.1 we find that the contribution from the horizontal sides goes to zero. Assume that $s = 1$ is the only pole of the integrand with $\Re(s) \geq 1/2 + \varepsilon$. Then using Cauchy's residue theorem we obtain

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\Re(s)=2} G_n(z,s) M_U(s) R^s\, \mathrm{d}s \\ &\quad = \mathrm{Res}_{s=1}\left(G_n(z,s) M_U(s) R^s\right) + \frac{1}{2\pi i}\int_{\Re(s)=1/2+\varepsilon} G_n(z,s) M_U(s) R^s\, \mathrm{d}s \\ &\quad = \delta_{n=0}\left(\frac{2\pi R}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + O(R/U)\right) + \frac{1}{2\pi i}\int_{\Re(s)=1/2+\varepsilon} G_n(z,s) M_U(s) R^s\, \mathrm{d}s. \end{aligned} \tag{33}$$

If there are other small eigenvalues, there are additional main terms. In bypassing we note that their coefficients will depend on the n-th hyperbolic Fourier coefficients of the eigenfunctions corresponding to small eigenvalues. (See [6, Theorem 4 p. 116].) If we choose $c = 3 + \varepsilon$ and use Lemma

2.1, the last integral is $O(R^{1/2+\varepsilon}U^{3+\varepsilon}(n^2+1))$. The interval with $|\Im(s)| \leq 1$ can easily be dealt with using the bound

$$\|R(s)\|_\infty \leq \max_j \left| \frac{1}{\sigma(1-\sigma)} - \frac{1}{\sigma_j(1-\sigma_j)} \right|,$$

which in turn gives us an estimate for $G_n(z,s)$.

If $n = 0$ we see that by further requiring $\psi_U(t) = 0$ if $t \geq 1$ and $\tilde{\psi}_U(t) = 1$ if $t \leq 1$, we have

$$\sum_{\gamma\in\Gamma} \psi_U\left(\frac{\cosh(r(\gamma z))}{R}\right) \leq \sum_{\substack{\gamma\in\Gamma \\ \cosh(r(\gamma z))\leq R}} 1 \leq \sum_{\gamma\in\Gamma} \tilde{\psi}_U\left(\frac{\cosh(r(\gamma z))}{R}\right).$$

Choosing $U = R^{1/8}$ we therefore obtain:

LEMMA 2.2 *With assumptions as above we have*

$$\#\{\gamma \in \Gamma \,|\, \cosh(r(\gamma z)) \leq R\} = \frac{2\pi R}{vol(\Gamma\backslash\mathbb{H})} + O(R^{7/8+\varepsilon}). \tag{34}$$

We note that this implies (1) with $\alpha = 7/8$. Using this we can now deal with the general case. To get from a smooth cut-off to a sharp one we notice that if $\psi_U(t) = 1$ for $t \leq 1$ then we may bound the difference

$$\sum_{\gamma\in\Gamma} e(n\varphi(\gamma z)/\pi)\psi_U\left(\frac{\cosh(r(\gamma z))}{R}\right) - \sum_{\substack{\gamma\in\Gamma \\ \cosh(r(\gamma z))\leq R}} e(n\varphi(\gamma z)/\pi) = O\left(\sum_{\substack{\gamma\in\Gamma \\ R<\cosh(r(\gamma z))\leq R(1+1/U)}} 1\right)$$

which by Lemma 2.2 is $O(R/U + R^{7/8+\varepsilon})$. Combining the above we find that for $n \neq 0$

$$\sum_{\substack{\gamma\in\Gamma \\ \cosh(r(\gamma z))\leq R}} e(n\varphi(\gamma z)/\pi) = O(R^{1/2+\varepsilon}U^{3+\varepsilon}(n^2+1) + R/U + R^{7/8+\varepsilon}).$$

Using the Erdös–Turán inequality [5, Theorem 3] we find that

$$\frac{\#\{\gamma \in \Gamma \,|\, \cosh(r(\gamma z)) \leq R,\ \varphi(\gamma z)/\pi \in I\}}{\#\{\gamma \in \Gamma \,|\, \cosh(r(\gamma z)) \leq R\ \}} = |I| + O(1/M + R^{-1/2+\varepsilon}U^{3+\varepsilon}M^2 + \log M(1/U + R^{-1/8+\varepsilon}))$$

for any M. Letting $M = U = R^{1/12}$ we arrive at the following (still assuming that there are no small eigenvalues):

THEOREM 2.3 *For all $\varepsilon > 0$ and $I \subset \mathbb{R}/\mathbb{Z}$ we have*

$$\frac{\#\{\gamma \in \Gamma \,|\, \cosh(r(\gamma z)) \leq R,\ \varphi(\gamma z)/\pi \in I\}}{\#\{\gamma \in \Gamma \,|\, \cosh(r(\gamma z)) \leq R\ \}} = |I| + O(R^{-1/12+\varepsilon}).$$

Theorem 1.1 follows easily.

3. Proof of Theorem 3

We wish to find the limiting distribution of the number of lattice points in angular sectors defined from z_0 when ordering the lattice points γw according to the distance to z_1. More precisely, we want to find the asymptotics of

$$\mathcal{N}_\Gamma^I(R, z_0, z_1, w) = \#\{\gamma \in \Gamma | d(z_1, \gamma w) \le R,\ \varphi_{z_0,w}(\gamma) \in I\}. \tag{35}$$

Our strategy for finding the asymptotics is the following: we find the hyperbolic distance from z_0 to the intersection(s) between the hyperbolic circle with center at z_1 and radius R and the geodesic through z_0 determined by an angle $t \in [-\pi, \pi]$ relative to the vertical geodesic through z_0. Once we have an asymptotic expression for this distance we can make a Riemann sum approximation of the counting function (35). The summands can be estimated through Theorem 1.1 leading to a proof of Theorem 1.3.

We may safely assume that $z_0 = i$ – it is easy to extend our results to the general case. We would like to find the distance from i to the relevant intersection point which will be denoted by $w' = x' + iy'$. There are two intersection points, but we choose the one that has negative real part for $t > 0$. This distance will be denoted $Q(z_1, t, R)$.

Now fix z_1, t and R. Let $\alpha \in \mathbb{R}$ and $\delta \in \mathbb{R}_+$ denote the center and the radius, respectively, of the Euclidean half-circle, which is the geodesic through i and w'. From Fig. 3 it is clear that

$$\delta = 1/|\sin(t)|, \quad \alpha = -\cot(t) \tag{36}$$

if $t \neq 0, \pm\pi$. Thus we see that

$$y' = \sqrt{\delta^2 - (x' - \alpha)^2} = \sqrt{1 - x'^2 + 2\alpha x'}. \tag{37}$$

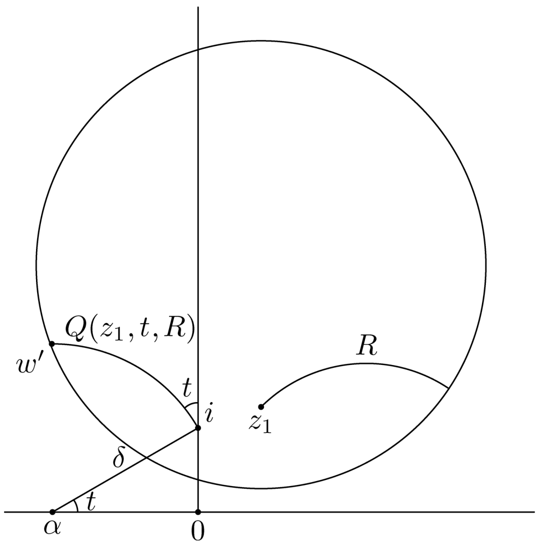

Figure 3. Hyperbolic circle with center z_1 and radius R.

On the other hand, it is well-known that the locus of points on the hyperbolic circle with center at $x_1 + iy_1$ and radius R is determined by the equation

$$|x_1 + iy_1 \cosh(R) - z| = y_1 \sinh(R),$$

which is equivalent to

$$x^2 + y^2 + x_1^2 - 2xx_1 + y_1^2 = 2y_1 y \cosh(R).$$

Using the expression for y' given in (37) we obtain the equation

$$\frac{\beta}{2} + (\alpha - x_1)x' = y_1 \cosh(R)\sqrt{\delta^2 - (x' - \alpha)^2} \tag{38}$$

for x', where $\beta = |z_1|^2 + 1$. By squaring (38) we get the quadratic equation

$$\left(\frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} + 1\right) x'^2 + \left(\frac{\beta(\alpha - x_1)}{y_1^2 \cosh^2(R)} - 2\alpha\right) x' + \frac{\beta^2}{4y_1^2 \cosh^2(R)} - 1 = 0,$$

with the solution

$$x' = \frac{\alpha - \frac{\beta(\alpha - x_1)}{2y_1^2 \cosh^2(R)} - \operatorname{sign}(t)\sqrt{\delta^2 + \frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} - \frac{\beta^2}{4y_1^2 \cosh^2(R)} - \frac{\alpha\beta(\alpha - x_1)}{y_1^2 \cosh^2(R)}}}{1 + \left(\frac{\alpha - x_1}{y_1 \cosh(R)}\right)^2}. \tag{39}$$

Naturally, the quadratic equation has two solutions, but the solution above is the intersection point we are interested in. The distance $Q(z_1, t, R)$ is

$$Q(z_1, t, R) = \log\left(\frac{|w' + i| + |w' - i|}{|w' + i| - |w' - i|}\right). \tag{40}$$

We note that

$$\begin{aligned}
\frac{|w' + i| + |w' - i|}{|w' + i| - |w' - i|} &= \frac{x'^2 + y'^2 + 1 + \sqrt{(x'^2 + y'^2 + 1)^2 - 4y'^2}}{2y'} \\
&= \frac{1 + \alpha x' + \delta|x'|}{y'} \\
&= \frac{1 + \alpha x' - \delta' x'}{y'},
\end{aligned} \tag{41}$$

where $\delta' = 1/\sin(t)$. Using Taylor's formula with remainder we see that

$$\begin{aligned}
&\operatorname{sign}(t)\sqrt{\delta^2 + \frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} - \frac{\beta^2}{4y_1^2 \cosh^2(R)} - \frac{\alpha\beta(\alpha - x_1)}{y_1^2 \cosh^2(R)}} \\
&\quad = \delta' + \frac{\frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} - \frac{\beta^2}{4y_1^2 \cosh^2(R)} - \frac{\alpha\beta(\alpha - x_1)}{y_1^2 \cosh^2(R)}}{2\delta'} + O\left(\frac{\delta}{\cosh^4(R)}\right)
\end{aligned}$$

as $R \to \infty$, where the constant implied depends on z_1. From the above equation and (41) we deduce that

$$x' = \frac{\alpha - \delta'}{1 + \left(\frac{\alpha - x_1}{y_1 \cosh(R)}\right)^2} + \frac{O((1+\delta)e^{-R})}{1 + \left(\frac{\alpha - x_1}{y_1 \cosh(R)}\right)^2} \tag{42}$$

and hence

$$1 + \alpha x' - \delta' x' = 1 + \frac{(\alpha - \delta')^2}{1 + ((\alpha - x_1)/y_1 \cosh(R))^2} + \frac{O(\delta(1+\delta)e^{-R})}{1 + ((\alpha - x_1)/y_1 \cosh(R))^2}. \tag{43}$$

This implies that

$$\sin^2(t)\left(1 + \left(\frac{\alpha - x_1}{y_1 \cosh(R)}\right)^2\right)(1 + \alpha x' - \delta' x') = 2 + 2\cos(t) + O(e^{-R}). \tag{44}$$

Now we look at

$$y'^2\left(1 + \left(\frac{\alpha - x_1}{y_1 \cosh(R)}\right)^2\right).$$

Using Taylor's formula as before we get

$$\begin{aligned}
y'^2\left(1 + \left(\frac{\alpha - x_1}{y_1 \cosh(R)}\right)^2\right)^2 &= \left(1 + \left(\frac{\alpha - x_1}{y_1 \cosh(R)}\right)^2\right)^2 - \Bigg(\alpha - \frac{\beta(\alpha - x_1)}{2y_1^2 \cosh^2(R)} \\
&\quad - \operatorname{sign}(t)\sqrt{\delta^2 + \frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} - \frac{\beta^2}{4y_1^2 \cosh^2(R)} - \frac{\alpha\beta(\alpha - x_1)}{y_1^2 \cosh^2(R)}}\Bigg)^2 \\
&\quad + 2\alpha\left(1 + \left(\frac{\alpha - x_1}{y_1 \cosh(R)}\right)^2\right)\Bigg(\alpha - \frac{\beta(\alpha - x_1)}{2y_1^2 \cosh^2(R)} \\
&\quad - \operatorname{sign}(t)\sqrt{\delta^2 + \frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} - \frac{\beta^2}{4y_1^2 \cosh^2(R)} - \frac{\alpha\beta(\alpha - x_1)}{y_1^2 \cosh^2(R)}}\Bigg) \\
&= \frac{1}{y_1^2 \cosh^2(R)}\Bigg(\frac{\beta^2}{4} + (\alpha - x_1)^2 + \alpha\beta(\alpha - x_1) + 2\alpha^2(\alpha - x_1)^2 \\
&\quad - \delta'(\alpha - x_1)(\beta + 2\alpha(\alpha - x_1))\Bigg) + O\left(\frac{\delta^4}{\cosh^4(R)}\right) \\
&= \frac{(\beta - (\beta - 2)\cos(t) + 2x_1 \sin(t))^2(1 + \cos(t))^2}{4y_1^2 \cosh^2(R)\sin^4(t)} + O\left(\frac{\delta^4}{\cosh^4(R)}\right)
\end{aligned}$$

as $R \to \infty$. From this we conclude that

$$\frac{1+\cos(t)}{2y'\left(1+\left(\frac{\alpha-x_1}{y_1\cosh(R)}\right)^2\right)\sin^2(t)} = \frac{y_1\cosh(R)}{\beta-(\beta-2)\cos(t)+2x_1\sin(t)} + O(e^{-4R})$$

$$= \frac{y_1 e^R}{2(\beta-(\beta-2)\cos(t)+2x_1\sin(t))} + O(e^{-R}). \tag{45}$$

We are interested in $e^{Q(z_1,t,R)}$. Combining (41), (40), (45) and (44) we conclude that

$$e^{Q(z_1,t,R)} = \frac{1+\alpha x'+\delta' x'}{y'} = \frac{2y_1 e^R}{\beta-(\beta-2)\cos(t)+2x_1\sin(t)} + O(1). \tag{46}$$

To finish the proof we use the following elementary lemma which 'integrates' Theorem 1.1 over more general regions:

LEMMA 3.1 *Let $D(R,\theta):\mathbb{R}_+\times\mathbb{R}/\mathbb{Z}\to\mathbb{R}_+$ be a function which satisfies $e^{D(R,\theta)} = k(\theta)e^R + O(e^{\beta R})$ for some $\beta<1$ uniformly in θ. Assume that $k(\theta)\in C^1(\mathbb{R}/\mathbb{Z})$. Then as $R\to\infty$*

$$N^I_{\Gamma,D}(R,z_0,z_1) := \#\{\gamma\in\Gamma \mid d(z_0,\gamma z_1)\le D(R,\varphi_{z_0,z_1}(\gamma)),\ \varphi_{z_0,z_1}(\gamma)\in I\}$$
$$= \frac{\kappa_\Gamma \pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\int_I k(\theta)\,\mathrm{d}\theta\, e^R + O(e^{\delta R})$$

for some $\delta<1$.

Proof. Let $B=B(R)$ be an integer-valued function of R to be determined later. For each integer $j\le B$, we choose ω_j,

$$\omega^j \in \left[a+\frac{(j-1)(b-a)}{B}, a+\frac{j(b-a)}{B}\right]$$

such that

$$k(\omega_j) = \inf\left\{k(\omega)\;\middle|\;\omega\in\left[a+\frac{(j-1)(b-a)}{B}, a+\frac{j(b-a)}{B}\right]\right\}$$

and

$$k(\omega^j) = \sup\left\{k(\omega)\;\middle|\;\omega\in\left[a+\frac{(j-1)(b-a)}{B}, a+\frac{j(b-a)}{B}\right]\right\}.$$

We split the interval in B equal intervals (and compensate for counting the endpoints twice) to get

$$N^I_{\Gamma,D}(R,z_0,z_1) = \sum_{j=0}^{B} N_{\Gamma,D}^{[a+\frac{(j-1)}{B}(b-a),a+\frac{j}{B}(b-a)]}(R,z_0,z_1)$$
$$-\sum_{j=1}^{B-1} N_{\Gamma,D}^{[a+\frac{j}{B}(b-a),a+\frac{j}{B}(b-a)]}(R,z_0,z_1).$$

The last sum is $O(Be^{\alpha R})$ by Theorem 1.1 and the assumption on $D(R,\theta)$. The first sum can be evaluated as follows. By using Theorem 1.1 again we have

$$\begin{aligned}\frac{\kappa_\Gamma \pi (b-a)}{B\,\mathrm{vol}\,(\Gamma\backslash\mathbb{H})}\omega_j e^R - Ce^{\alpha R} &\leq N_{\Gamma,D}^{[a+\frac{(j-1)}{B}(b-a),a+\frac{j}{B}(b-a)]}(R,z_0,z_1)\\ &\leq \frac{\kappa_\Gamma \pi (b-a)e^R}{B\,\mathrm{vol}\,(\Gamma\backslash\mathbb{H})}\omega^j + Ce^{\alpha R}.\end{aligned}$$

Summing this inequality we find the Riemann sums

$$\sum_{j=1}^{B}\omega_j\frac{(b-a)}{B},\quad \sum_{j=1}^{B}\omega^j\frac{(b-a)}{B}.$$

Since k is C^1 these converge to $\int_I k(\theta)\,\mathrm{d}\theta$ with rate $O(1/B)$ as is seen using the mean value theorem. We therefore find that

$$N_{\Gamma,D}^I(R,z_0,z_1) = \frac{\kappa_\Gamma\pi}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\int_I k(\theta)\,\mathrm{d}\theta\, e^R + O(e^R/B) + O(Be^{\alpha R}).$$

Balancing the error terms we get the result.

We can now finish the proof of Theorem 1.3. Let $\rho_{z_0,z_1}(\omega)$ denote the fraction

$$\frac{2y_0y_1}{((x_0-x_1)^2+y_0^2+y_1^2)(1-\cos(2\pi\omega))+2y_0^2\cos(2\pi\omega)+2(x_1-x_0)y_0\sin(2\pi\omega)}.$$

We start with the case $z_0 = i$. Equation (46) allows us to use Lemma 3.1, which gives Theorem 1.3 immediately. The general case can easily be reduced to the case where $z_0 = i$ by conjugation of Γ with the element $\begin{pmatrix}\sqrt{y_0} & x_0/\sqrt{y_0}\\ 0 & 1/\sqrt{y_0}\end{pmatrix}$. This finishes the proof of Theorem 1.3.

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MANUSCRIPT D

Divisor Problems and the Pair Correlation for the Fractional Parts of $n^2\alpha$

DIVISOR PROBLEMS AND THE PAIR CORRELATION FOR THE FRACTIONAL PARTS OF $n^2\alpha$

JIMI L. TRUELSEN

Abstract. Z. Rudnick and P. Sarnak have proved that the pair correlation for the fractional parts of $n^2\alpha$ is Poissonian for almost all α. However, they were not able to find a specific α for which it holds. We show that the problem is related to the problem of determining the number of $(a, b, r) \in \mathbf{N}^3$ such that $a \le M$, $b \le N$, $r \le K$ and $pab \equiv r(q)$ for p and q coprime. With suitable assumptions on the relative size of K, M, N and q one should expect there to be KMN/q such triples asymptotically and we will show that this holds on average.

1. Introduction

For $t \in \mathbf{R}$ and $q \in \mathbf{N}$ let

$$\|t\|_q = \inf_{n \in \mathbf{Z}} |t - qn|,$$

and set $\|\cdot\| = \|\cdot\|_1$. Clearly $\|\cdot\|_q$ defines a norm on $\mathbf{R}/q\mathbf{Z}$. For a sequence $\{a_n\}_1^\infty \subset \mathbf{R}/\mathbf{Z}$, $x > 0$ and $N \in \mathbf{N}$ we define

$$R_2(x, N, \{a_n\}_1^\infty) = N^{-1}\#\left\{(m,n) \in \mathbf{N}^2 \middle| m, n \le N,\ n \ne m,\ \|a_m - a_n\| \le \frac{x}{N}\right\}.$$

We say that the pair correlation for $\{a_n\}_1^\infty$ is Poissonian if for every $x > 0$ we have that

$$\lim_{N\to\infty} R_2(x, N, \{a_n\}_1^\infty) = 2x.$$

Note that the limit is not uniform in x. We will be particularly interested in the case where a_n equals the fractional parts of $n^2\alpha$ for α irrational. The spacings between the elements of this sequence correspond to the spacings between the energy levels of the boxed oscillator in quantum mechanics [2]. We define (by an obvious abuse of notation)

$$R_2(x, N, \alpha) = R_2(x, N, \{n^2\alpha\}_1^\infty) = N^{-1}\#\left\{(m,n) \mid m, n \le N,\ n \ne m,\ \|m^2\alpha - n^2\alpha\| \le \frac{x}{N}\right\}.$$

Clearly we may as well assume that $0 < \alpha < 1$. We will be interested in α with certain Diophantine properties. We say that an irrational number α is of type κ if

$$|\alpha - p/q| \gg \frac{1}{q^\kappa}.$$

for all $p \in \mathbf{Z}$ and $q \in \mathbf{N}$. We say that α is "Diophantine" if α is of type $2 + \varepsilon$ for any $\varepsilon > 0$. In particular all real, irrational algebraic numbers are Diophantine (Roth's theorem – see Theorem 5.7.1 in [10]). Note also that almost all α (with respect to the Lebesgue measure) are Diophantine. To see this we use the identity of sets

$$\{\beta \in \mathbf{R} \mid \beta \text{ is not Diophantine}\} = \bigcup_{n=1}^{\infty}\bigcup_{l\in\mathbf{Z}}\bigcap_{k=1}^{\infty}\bigcup_{q=k}^{\infty}\bigcup_{p=1}^{q}\left[l + \frac{p}{q} - \frac{1}{q^{2+1/n}}, l + \frac{p}{q} + \frac{1}{q^{2+1/n}}\right].$$

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Let $\mathcal{L}$ denote the Lebesgue measure on the real line. We see that

$$\mathcal{L}\left(\bigcup_{p=1}^{q}\left[l+\frac{p}{q}-\frac{1}{q^{2+1/n}}, l+\frac{p}{q}+\frac{1}{q^{2+1/n}}\right]\right)=2q^{-(1+1/n)}.$$

Since

$$\sum_{q=1}^{\infty} q^{-(1+1/n)}<\infty$$

it follows from the Borel-Cantelli lemma that

$$\mathcal{L}\left(\bigcap_{k=1}^{\infty}\bigcup_{q=k}^{\infty}\bigcup_{p=1}^{q}\left[l+\frac{p}{q}-\frac{1}{q^{2+1/n}}, l+\frac{p}{q}+\frac{1}{q^{2+1/n}}\right]\right)=0.$$

Thus the set of non-Diophantine real numbers is a null set.

It is a classical result due to H. Weyl [16] that the sequence $n^d\alpha$ is equidistributed modulo 1 for any integer $d \geq 1$. However, it is not true that the pair correlation for the fractional parts of $n^d\alpha$, $d \geq 2$ is Poissonian for all irrational α (for $d=1$ it is never the case – see Exercise 12.6.3 in [10]). A simple construction shows (see [12] p. 62) that α must be at least of type $d+1$.

Z. Rudnick and P. Sarnak have proved [12, Theorem 1] that the pair correlation for the fractional parts of $n^d\alpha$ is Poissonian for almost all α. Subsequently J. Marklof and A. Strömbergsson [9], and D. R. Heath-Brown [5] have given different proofs in the case $d=2$. However, one does not know of any specific α for which it holds, but Rudnick and Sarnak made the following conjecture:

Conjecture 1.1. *Assume α is Diophantine. Then the pair correlation for the fractional parts of $n^2\alpha$ is Poissonian.*

Furthermore, in [5] Heath-Brown was able to show (using a lattice point strategy) that for α of type $9/4$

$$R_2(x,N,\alpha)=2x+O(x^{7/8}), \tag{1.1}$$

whenever $1 \leq x \leq \log N$, where the constant implied depends on α. This supports Conjecture 1.1 and suggests that perhaps the condition on the Diophantine approximation in the conjecture can be relaxed to some extend.

We remark that the m-level correlation for the fractional parts of $n^2\alpha$ has been studied by Rudnick, Sarnak and Zaharescu in [13] and by Zaharescu in [17]. It is not known if the fractional parts of $n^2\alpha$ for almost all α have Poissonian behavior, i.e. have the same distribution as a sequence of independent and uniformly distributed random variables, but it is expected (cf. the conjecture on page 38 in [13]).

In this paper we will only be concerned with Conjecture 1.1 (not higher level correlations). We suggest a line of attack that is based on the study of the function

$$\tau_{M,N}(m)=\#\{(a,b)\in\mathbf{N}^2 \mid a\leq M,\ b\leq N,\ ab=m\},$$

where $m \in \mathbf{N}$ and $M,N \geq 1$. We also define $\tau_M^*=\tau_{M,M}$. We make the following conjecture:

Conjecture 1.2. *Let $K,M,N \geq 1$ with $M \asymp N$ (i.e. $C_1N \leq M \leq C_2N$) and $K \geq N^\eta$ for some $\eta>0$. Assume also that $q \leq N^{2-\delta}$ for some $\delta>0$ and $(q,\rho)=1$. Then*

$$\sum_{r\leq K}\sum_{m\equiv\rho r(q)}\tau_{M,N}(m)\sim\frac{KMN}{q}$$

as $N\to\infty$ uniformly in M, K, q and ρ. The rate of convergence may depend on η, δ, C_1 and C_2.

Conjecture 1.2 has applications to the pair correlation problem at hand. We will show that:

Proposition 1.3. *Conjecture 1.2 implies that the pair correlation for the fractional parts of $n^2\alpha$ is Poissonian for any α of type $3-\delta$ for any $\delta \in (0,1)$.*

This is an immediate consequence of Proposition 2.3. As mentioned previously the pair correlation for the fractional parts of $n^2\alpha$ is not Poissonian if α is not of type 3. Conjecture 1.2 claims that $3-\delta$ is sufficient.

Conjecture 1.2 seems bold but natural. Indeed the conjecture provably holds if q is smaller than $N^{1-\delta}$ (see Proposition 3.2 below). However, it turns out that we need $q \geq N^{\frac{3}{2}+\delta}$ for our purpose. We can actually obtain partial results for larger q as well based on a lattice point approach using the ideas of Heath-Brown [5]. Before we can state the result we introduce some terminology. We say that a rational number p/q is of type $(e, \mathcal{K})$ if

$$\left|\frac{p}{q} - \frac{u}{v}\right| \geq \frac{1}{\mathcal{K}v^e}$$

for any rational number u/v with $u/v \neq p/q$. One easily checks that if α is an irrational number of type e then there exists $\mathcal{K} > 0$ such that the convergents will be of type $(e, \mathcal{K})$ from some step.

Modifying the proof of (1.1) we prove the following (τ denotes the ordinary divisor function):

Theorem 1.4. *Let $K, M, N \geq 3$ with $M \asymp N$ and let $\gamma \in (0,1)$. Assume that*

$$q^{1+\delta} \leq \left(\frac{N^2}{K}\right)^{1/(1+\gamma)} \tag{1.2}$$

for some $\delta > 0$ and $KN/q \geq 1$. Then

$$\sum_{|r|\leq K} \sum_{pm\equiv r(q)} \tau_{M,N}(m) = \frac{2KMN}{q} + O\left(N(KN/q)^{7/8} + \frac{N^2}{q}\left(\tau(q)^2(\log N)^3 + \frac{K(\log\log N)^2}{(\log N)^{1/4}}\right)\right)$$

uniformly in M, K, p and q for p/q of type $(2+\gamma, \mathcal{K})$.

It is well known (see e.g. [6]) that one expects that

$$\sum_{\substack{n\leq x \\ n\equiv r(q)}} \tau(n) \sim \frac{x}{q^2}\log x \sum_{d|(q,r)} \sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) \tag{1.3}$$

as $x \to \infty$ for $q \leq x^{1-\delta}$ for some $\delta > 0$. Average results supporting this conjecture have been considered by Banks, Heath-Brown and Shparlinski [1], and Blomer [3]. If we adapt (1.3) to $\tau_{M,N}$ we should expect that

$$\sum_{n\equiv r(q)} \tau_{M,N}(n) \sim \frac{MN}{q^2} \sum_{d|(q,r)} \sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right). \tag{1.4}$$

Note that

$$\sum_{d|(q,r)} \sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) = \sum_{d|(q,r)} d\varphi\left(\frac{q}{d}\right). \tag{1.5}$$

It has been proved by Linnik and Vinogradov [8] that

$$\sum_{\substack{m\leq x \\ m\equiv r(q)}} \tau(m) \ll \frac{\varphi(q)x\log x}{q^2} \tag{1.6}$$

for $q \leq x^{1-\delta}$ and $(r,q)=1$, where the constant implied depends on $\delta > 0$ only. In view of Conjecture 1.2 and (1.6) it would be interesting to find upper bounds for

$$\sum_{m \equiv r(q)} \tau_N^*(m).$$

Heath-Brown [5] suggested the following conjecture which is the analogue of (1.6) for τ_N^*:

Conjecture 1.5. *Let $\delta \in (0,1)$. Then*

$$\sum_{m \equiv r(q)} \tau_N^*(m) \ll \frac{\varphi(q)N^2}{q^2}$$

uniformly for $(r,q)=1$ and $q \leq N^{2-\delta}$, where the constant implied depends only on δ.

Using the work of M. Nair and G. Tenenbaum [11] we prove an upper bound for the sum in Conjecture 1.5.

Proposition 1.6. *Let $q \leq N^{2-\delta}$. Then*

$$\sum_{m \equiv r(q)} \tau_N^*(m) \ll \frac{N^2}{\varphi(q)} e^{\sqrt{(2+\varepsilon)(\log\log N)(\log\log\log N)}}$$

uniformly for $(r,q)=1$ for any $\varepsilon > 0$.

Note that the estimate in the proposition above is off by less than a factor of $(\log N)^\varepsilon$ compared to Conjecture 1.5 since $q/\varphi(q) \ll \log\log q$.

The function $\tau_{M,N}$ is complicated. There is another similar function of interest

$$\tau_M(m) = \#\{d \in \mathbf{N} \mid d \leq M,\ d \mid m\}.$$

The function τ_M is in many ways simpler than $\tau_{M,N}$. The estimate corresponding to Conjecture 1.5 holds. More precisely we prove:

Theorem 1.7. *Let $0 < \delta \leq 1$, $0 < \varepsilon < \frac{1}{8}$, $0 < \kappa$ and $2 \leq N$. Assume also that $N \geq q^\kappa$. Then*

$$\sum_{\substack{x < n \leq x+y \\ n \equiv r(q)}} \tau_N(n) \ll \frac{y\varphi(q)\log N}{q^2}$$

uniformly for N, $(r,q)=1$, $x^{\frac{1+4\varepsilon\delta}{1+\delta}} \leq y \leq x$, $x \geq c_0 q^{1+\delta}$, where c_0 and the constant implied depends at most on δ and ε. In particular

$$\sum_{\substack{m \equiv r(q) \\ m \leq x}} \tau_N(m) \ll \frac{\varphi(q)x\log N}{q^2}.$$

Note that with $N = x+y$ we obtain

$$\sum_{\substack{m \equiv r(q) \\ x < m \leq x+y}} \tau(m) \ll \frac{y\varphi(q)\log x}{q^2}.$$

This extension of (1.6) was also obtained by P. Shiu [14].

Finally we show that Conjecture 1.2 and Conjecture 1.5 holds on average. Indeed we start by proving that (1.4) holds for most values of q and r if (q,r) is small:

Theorem 1.8. *Let $\delta > 0$ and assume $M \asymp N$. Then*

$$\sum_{(r,q)=k} \left(\sum_{m \equiv r(q)} \tau_{M,N}(m) - \frac{MN}{q^2} \sum_{d|k} \sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) \right)^2 \ll \frac{N^{\max(\frac{7}{2}+\varepsilon, 4-\delta)}}{q}$$

uniformly for $q \leq N^{2-\delta}$.

From Theorem 1.8 we can deduce the following:

Theorem 1.9. *Let $M, N \geq 1$ with $M \asymp N$, $q \in \mathbf{N}$ and $K \geq N^\eta$ for some $\eta > 0$. Assume also that $q \leq N^{2-\delta}$ for some $\delta > 0$. Then*

$$\sum_{(\rho,q)=1} \left(\sum_{r \leq K} \sum_{m \equiv \rho r(q)} \tau_{M,N}(m) - \frac{KMN}{q} \right)^2 \ll \frac{K^2 N^4}{q} \left(q^\varepsilon \left(\frac{1}{q} + \frac{1}{K} \right)^2 + N^{\max(-1/2+\varepsilon, -\delta)} \right)$$

for any $\varepsilon > 0$.

In Proposition 3.2 we show that Conjecture 1.2 holds for $q \leq N^{1-\delta}$. Thus we can safely restrict our attention to the case where $q \geq \sqrt{N}$. We have the following corollary, which states that Conjecture 1.2 is true on average:

Corollary 1.10. *Let $M, N \geq 1$ with $M \asymp N$, $q \in \mathbf{N}$ and $K \geq N^\eta$ for some $\eta > 0$. Assume also that $\sqrt{N} \leq q \leq N^{2-\delta}$ for some $\delta > 0$. Then*

$$\frac{1}{\varphi(q)} \sum_{(\rho,q)=1} \left(\frac{q}{KMN} \sum_{r \leq K} \sum_{m \equiv \rho r(q)} \tau_{M,N}(m) - 1 \right)^2 \ll N^{-\min(1/2, \delta, 2\eta)+\varepsilon}$$

for any $\varepsilon > 0$.

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2. Reducing the Question to an Arithmetic Problem

Set

$$S(x, N, \alpha) = \frac{\#\left\{(a,b) \in \mathbf{N} \times \mathbf{Z} \mid 1 \leq a < 2N,\ 1 \leq |b| \leq N - |N-a|,\ 2 \mid a+b,\ \|ab\alpha\| \leq \frac{x}{N}\right\}}{N}$$

By factoring $m^2 - n^2$ into $a = m+n$ and $b = m-n$ we see that

$$\begin{aligned} 0 &\leq S(x,N,\alpha) - R_2(x,N,\alpha) \\ &\leq \frac{2}{N} + \frac{2}{N}\#\left\{ n \in \mathbf{N} \mid n \leq N,\ \|n^2\alpha\| \leq \frac{x}{N} \right\} \\ &\to 0 \end{aligned}$$

as $N \to \infty$ (the difference between S and R_2 is that in S we do not exclude all the cases corresponding to m or n equal to 0). This follows since the fractional parts of $n^2\alpha$ becomes equidistributed in the unit interval. Thus if we want to study Poissonian behavior we may as well study $S(x, N, \alpha)$ rather than $R_2(x, N, \alpha)$.

From the elementary theory of continued fractions (see [10] Chapter 7) we know that the convergents p_n/q_n of α satisfy

$$\left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}} \tag{2.1}$$

and

$$q_{n+1}p_n - p_{n+1}q_n = \pm 1. \tag{2.2}$$

Define $\mathfrak{R}(y, N, p, q)$ by

$$\#\left\{(a,b) \in \mathbf{N} \times \mathbf{Z} \mid 1 \leq a < 2N,\ 1 \leq |b| \leq N - |N-a|,\ 2 \mid a+b,\ \|abp\|_q \leq y\right\}.$$

We have the following:

Proposition 2.1. *Let α be irrational with convergents p_n/q_n. The pair correlation for the fractional parts of $n^2\alpha$ is Poissonian if and only if there for all fixed $x > 0$ exists $\kappa > 0$ and a sequence $\{n_k\}_1^\infty$ such that $N^{3+\kappa} \ll q_{n_N} q_{n_N+1}$ and*

$$\mathfrak{R}\left(\frac{xq_{n_N}}{N}, N, p_{n_N}, q_{n_N}\right) \sim 2xN$$

as $N \to \infty$.

Proof. Note that

$$\|ab\alpha\| \le \frac{x}{N}$$

if and only if

$$\left\| abp_n + abq_n\left(\alpha - \frac{p_n}{q_n}\right)\right\|_{q_n} \le \frac{xq_n}{N}.$$

Now

$$\left| abq_n\left(\alpha - \frac{p_n}{q_n}\right)\right| \le \frac{N^2}{q_{n+1}},$$

and this implies that

$$\mathfrak{R}\left(\frac{xq_n}{N} - \frac{N^2}{q_n}, N, p_n, q_n\right) \le NS(\alpha, N, x) \le \mathfrak{R}\left(\frac{xq_n}{N} + \frac{N^2}{q_n}, N, p_n, q_n\right).$$

Assume $\frac{N^2}{q_{n+1}} = o\left(\frac{xq_n}{N}\right)$ as $N \to \infty$. Since $\mathfrak{R}(y, N, p, q)$ is an increasing function of y we conclude that for any $\varepsilon > 0$

$$\mathfrak{R}\left(\frac{(x-\varepsilon)q_n}{N}, N, p_n, q_n\right) \le NS(\alpha, N, x) \le \mathfrak{R}\left(\frac{(x+\varepsilon)q_n}{N}, N, p_n, q_n\right)$$

for N sufficiently large. From this the result follows easily. $\square$

Now we have an arithmetic version of Conjecture 1.1. However, the constraints on a and b in the definition of $\mathfrak{R}(y, N, p, q)$ are a bit complicated. We can split $\mathfrak{R}(y, N, p, q)$ into some nicer pieces. The hope is that we can say something about these. This is where Conjecture 1.2 enters the picture as we will see below. First we make a (technical) conjecture:

Conjecture 2.2. *Let $\mathcal{K}, \lambda, \zeta, c > 0$ be constants with $\lambda < 1$. Let $N \ge 1$ and assume $p/q \in \mathbf{Q}$ (with $(p,q) = 1$) is of type $(2+\lambda, \mathcal{K})$ and $N^{\frac{3}{2+\lambda}} \le q \le N^{\frac{3(1+\lambda)}{2+\lambda}}$. Then*

$$\sum_{|r| \le \frac{\zeta q}{N}} \sum_{pm \equiv r(q)} \tau_{cN,N}(m) \sim 2\zeta cN$$

as $N \to \infty$ uniformly in q and p.

Clearly Conjecture 1.2 implies Conjecture 2.2. From the next proposition we may therefore conclude that Conjecture 1.2 implies Conjecture 1.1.

Proposition 2.3. *Assume Conjecture 2.2 holds with some $\lambda \in (0,1)$ and for any $\mathcal{K} > 1$. For α of type $2+\beta$ with $\beta < \lambda$ the pair correlation for the fractional parts of $n^2\alpha$ is Poissonian. In particular Conjecture 1.1 holds.*

Proof. Let us first ignore the technical conditions on N, p and q. We see that

$$\begin{aligned}\mathfrak{R}\left(\frac{xq}{N}, N, p, q\right) &= \#\left\{(a,b) \in \mathbf{N}^2 \mid a, b \le N,\ 2 \mid a+b,\ \|abp\|_q \le xq/N\right\} + \\ &\quad 2\#\left\{(a,b) \in \mathbf{N}^2 \mid N < a < 2N,\ 0 < b \le 2N - a,\ 2 \mid a+b,\ \|abp\|_q \le xq/N\right\}.\end{aligned} \tag{2.3}$$

We will see that the two terms are of the same size (if we assume Conjecture 2.2). We start by considering the first term. We define

$$\mathfrak{T}_M(m) = \#\{(a,b) \in \mathbf{N}^2 \mid a,b \le M,\ a \equiv b(2),\ ab = m\}.$$

Note that

$$\mathfrak{T}_M(m) = \begin{cases} \tau_M^*(m) & \text{if } m \equiv 1(2) \\ 0 & \text{if } m \equiv 2(4) \\ \tau_{M/2}^*(m/4) & \text{if } m \equiv 0(4) \end{cases}. \tag{2.4}$$

Using this notation we see that

$$\#\left\{(a,b) \in \mathbf{N}^2 \mid a,b \le N,\ 2 \mid a+b,\ \|abp\|_q \le xq/N\right\} = \sum_{|r| \le \frac{xq}{N}} \sum_{m \equiv r\overline{p}(q)} \mathfrak{T}_N(m),$$

where $\overline{p}$ is the inverse of p modulo q. Using (2.4) we can write this as

$$\sum_{|r| \le \frac{xq}{N}} \sum_{m \equiv r\overline{p}(q)} \mathfrak{T}_N(m) = \sum_{|r| \le \frac{xq}{N}} \Big(\sum_{2k+1 \equiv r\overline{p}(q)} \tau_N^*(2k+1) + \sum_{4l \equiv r\overline{p}(q)} \tau_{N/2}^*(l) \Big).$$

Furthermore we see that

$$\sum_{|r| \le \frac{xq}{N}} \sum_{4l \equiv r\overline{p}(q)} \tau_{N/2}^*(l) = \begin{cases} \sum_{|r| \le \frac{xq}{N}} \sum_{l \equiv r\overline{4p}(q)} \tau_{N/2}^*(l) & \text{if } q \equiv 1(2) \\ \sum_{|r| \le \frac{xq}{2N}} \sum_{l \equiv r\overline{2p}(q/2)} \tau_{N/2}^*(l) & \text{if } 2 \parallel q \\ \sum_{|r| \le \frac{xq}{4N}} \sum_{l \equiv r\overline{p}(q/4)} \tau_{N/2}^*(l) & \text{if } 4 \mid q \end{cases}. \tag{2.5}$$

By (2.5) Conjecture 2.2 implies that

$$\sum_{|r| \le \frac{xq}{N}} \sum_{4l \equiv r\overline{p}(q)} \tau_{N/2}^*(l) \sim \frac{1}{2} xN.$$

Note also that

$$\sum_{2k+1 \equiv r\overline{p}(q)} \tau_N^*(2k+1) = \sum_{m \equiv r\overline{p}(q)} \tau_N^*(m) - \sum_{2m \equiv r\overline{p}(q)} \tau_N^*(2m).$$

If q is odd then

$$\sum_{2m \equiv r\overline{p}(q)} \tau_N^*(2m) = 2 \sum_{m \equiv r\overline{2p}(q)} \tau_{N,N/2}(m) - \sum_{m \equiv r\overline{2p}(q)} \tau_{N/2}^*(m).$$

Thus Conjecture 2.2 implies

$$\sum_{|r| \le \frac{xq}{N}} \sum_{2k+1 \equiv r\overline{p}(q)} \tau_N^*(2k+1) \sim \frac{1}{2} xN.$$

Now assume q is even. This implies that p and $\overline{p}$ are odd. Thus m is even if and only if r is even. Hence

$$\sum_{|r| \le \frac{xq}{N}} \sum_{2m \equiv r\overline{p}(q)} \tau_N^*(2m) = \sum_{|r| \le \frac{xq}{2N}} \left(2 \sum_{m \equiv r\overline{p}(q/2)} \tau_{N,N/2}(m) - \sum_{m \equiv r\overline{p}(q/2)} \tau_{N/2}^*(m) \right) \sim \frac{3}{2} xN.$$

Thus we conclude that

$$\sum_{|r| \le \frac{xq}{N}} \sum_{m \equiv r\overline{p}(q)} \mathfrak{T}_N(m) \sim xN.$$

Now we consider the second term in (2.3) and we set

$$\mathfrak{S} = \#\left\{(a,b) \in \mathbf{N}^2 \mid N < a < 2N,\ 0 < b \le 2N - a,\ 2 \mid a+b,\ \|abp\|_q \le xq/N\right\}.$$

For $k, l_1, l_2 \in \mathbf{N}$ such that $l_1 + l_2 \leq 2^k$ we define

$$T_{l_1,l_2}(k) = \#\left\{(a,b) \in \mathbf{N}^2 \mid a \in I(k,l_1), b \in J(k,l_2),\ 2 \mid a+b,\ \|abp\|_q \leq xq/N\right\},$$

where

$$I(k,l_1) = (N(1+(l_1-1)/2^k), N(1+l_1/2^k)]$$

and

$$J(k,l_2) = (N(l_2-1)/2^k, Nl_2/2^k].$$

As before we deduce (using Conjecture 2.2) that

$$T_{l_1,l_2}(k) \sim \frac{1}{4^k} xN \tag{2.6}$$

uniformly in l_1 and l_2 (since k is fixed). Clearly we have

$$\begin{aligned}\sum_{\substack{l_1,l_2 \\ l_1+l_2 \leq 2^k-1}} & T_{l_1,l_2}(k) \\ & \leq \#\left\{(a,b) \in \mathbf{N}^2 \mid N < a < 2N,\ 0 < b \leq 2N-a,\ 2 \mid a+b,\ \|abp\|_q \leq xq/N\right\} \\ & \leq \sum_{\substack{l_1,l_2 \\ l_1+l_2 \leq 2^k}} T_{l_1,l_2}(k).\end{aligned}$$

Recall that $\#\{(l_1,l_2) \in \mathbf{N}^2 \mid l_1 + l_2 \leq m\} = \frac{m(m+1)}{2}$. Using (2.6) we see that

$$\frac{1-2^{-k}}{2} x \leq \liminf_N \frac{\mathfrak{S}}{N} \leq \limsup_N \frac{\mathfrak{S}}{N} \leq \frac{1+2^{-k}}{2} x.$$

Since this holds for any k we must have $\mathfrak{S} \sim xN/2$ as desired. By Proposition 2.1 it remains to prove that there exists $\kappa > 0$ such that for each N sufficiently large we can choose q_n and q_{n+1} such that $N^{3+\kappa} \ll q_n q_{n+1}$. By Conjecture 2.2 we must take $q_n \leq N^{\frac{3(1+\lambda)}{(2+\lambda)}}$. Recall that α is of type $2+\beta$. Choose n such that

$$q_n \leq N^{\frac{3(1+\lambda)}{(2+\lambda)}} \leq q_{n+1}.$$

The condition that α is of type $2+\beta$ implies that

$$q_n q_{n+1} \ll q_n^{2+\beta}$$

and hence

$$q_n q_{n+1} \gg q_{n+1}^{1+\frac{1}{1+\beta}} \geq N^{\frac{3(1+\lambda)}{(2+\lambda)}(1+\frac{1}{1+\beta})}.$$

Thus we can choose $\kappa = \frac{\lambda-\beta}{(1+\beta)(2+\lambda)}$. □

It should be mentioned that there may be some loss in using the rational approximation at an early stage in Proposition 2.1. For this approach to work we must be able to work with $q \geq N^{\frac{3}{2}+\delta}$. In Theorem 1.4 we can say something about values of q that are slightly smaller. In the proof of (1.1) Heath-Brown was able to work with α rather than its convergents and only use the Diophantine approximation at the very end of the proof allowing the use of smaller values of q.

By condition (2.2) we see that the inverse of p_n modulo q_n is $\pm q_{n+1}$. To begin with one could study

$$\sum_{|r| \leq \frac{xq_n}{N}} \sum_{m \equiv r q_{n+1} (q_n)} \tau_N^*(m).$$

Perhaps one can use this information to say more about the pair correlation problem for specific α's such as $\sqrt{2}$ or the golden ratio where the q_n's are known.

3. Preliminary Evidence for Conjecture 1.2

We will now explain why we should expect the asymptotics in Conjecture 1.2. We try the "naive" approach. Define

$$\delta_d(n) = \begin{cases} 1 & \text{if } d \mid n \\ 0 & \text{if } d \nmid n \end{cases}.$$

Assume $M, N \geq 2$. Clearly $\tau_{M,N}(m) = \sum_{d=\lceil m/N \rceil}^{M} \delta_d(m)$. Thus

$$\begin{aligned}
\sum_{r\leq K}\sum_{m\equiv \rho r(q)} \tau_{M,N}(m) &= \sum_{r\leq K}\sum_{\substack{l\leq \frac{MN-[r\rho]_q}{q} \\ (r,l)\neq(0,0)}}\sum_{d=\lceil (ql+[r\rho]_q)/N\rceil}^{M} \delta_d(ql+[r\rho]_q) \\
&= \sum_{\substack{r\leq K\\ r\neq 0}}\sum_{l\leq \frac{MN-[r\rho]_q}{q}}\sum_{(ql+[r\rho]_q)/N\leq d\leq M} \delta_d(ql+[r\rho]_q) + O(MNq^{-1+\varepsilon}) \\
&= \sum_{\substack{r\leq K\\ r\neq 0}}\sum_{[r\rho]_q/N\leq d\leq M}\sum_{0\leq l\leq (Nd-[r\rho]_q)/q} \delta_d(ql+[r\rho]_q) + O(MNq^{-1+\varepsilon})
\end{aligned}$$

where $[\cdot]_q$ denotes the remainder when dividing by q. Now the length of the l-interval can be much smaller than d and this is where the approach fails. We should expect that

$$\sum_{[r\rho]_q/N\leq d\leq M}\sum_{0\leq l\leq (Nd-[r\rho]_q)/q} \delta_d(ql+[r\rho]_q)$$

is roughly

$$\sum_{\substack{[r\rho]_q/N\leq d\leq M\\ (d,q)\mid r}} \frac{Nd-[r\rho]_q}{dq}(d,q) = \frac{N}{q}\sum_{\substack{d\leq M\\ (d,q)\mid r}}(d,q) + O\left(\sum_{\substack{d\leq M\\ (d,q)\mid r}}\frac{(d,q)}{d}\right) + O\left(\frac{N}{q}\sum_{\substack{d\leq q/N\\ (d,q)\mid r}}(d,q)\right),$$

and it is the case if $q \leq N^{1-\delta}$. Using Lemma 3.1 below we see that the "expected" value of $\sum_{r\leq K}\sum_{m\equiv\rho r(q)}\tau_{M,N}(m)$ is

$$\frac{KMN}{q} + O((K+M)Nq^{-1+\varepsilon}).$$

On numerous occasions we will use the fact that

$$\varphi(x,q) = \#\{n\in\mathbf{N} \mid n\leq x,\ (n,q)=1\} = \sum_{d\mid q}\mu(d)\left[\frac{x}{d}\right] = \frac{\varphi(q)x}{q} + O(\tau(q)). \tag{3.1}$$

This implies that

$$\#\{r\in\mathbf{N} \mid r\leq K,\ (r,q)=k\} = \frac{K\varphi(q/k)}{q} + O(q^{\varepsilon}). \tag{3.2}$$

Lemma 3.1. *Let $N, K \geq 1$ and $q\in\mathbf{N}$. Then*

$$\sum_{r\leq K}\sum_{\substack{d\leq N\\ (d,q)\mid r}}(d,r) = KN + O((K+N)q^{\varepsilon}), \tag{3.3}$$

$$\sum_{r\leq K}\sum_{\substack{d\leq N\\ (d,q)\mid r}}\frac{(d,r)}{d} = K\log N + O(Kq^{\varepsilon}) + O(q^{\varepsilon}\log N) \tag{3.4}$$

and

$$\sum_{r\le K} r \sum_{\substack{d\le N\\(d,q)|r}} (d,r) = \frac{1}{2}NK^2 + O(K(K+N)q^{\varepsilon}) \tag{3.5}$$

for any $\varepsilon > 0$.

Proof. We see that

$$\begin{aligned}
\sum_{r\le K}\sum_{\substack{d\le N\\(d,q)|r}} (d,r) &= \sum_{\substack{k|q\\k\le K}} k\#\{(r,d)\in \mathbf{N}^2 \mid r\le K,\ d\le N,\ (d,q)=k,\ k\mid r\}\\
&= \sum_{\substack{k|q\\k\le K}} k\left[\frac{K}{k}\right]\#\{d\in\mathbf{N}\mid d\le N,\ (d,q)=k\}\\
&= \sum_{\substack{k|q\\k\le K}} k\left[\frac{K}{k}\right]\sum_{a|q/k}\mu(a)\left[\frac{N}{ak}\right]\\
&= \sum_{\substack{k|q\\k\le K}} k\left(\frac{K}{k}+O(1)\right)\sum_{a|q/k}\mu(a)\left(\frac{N}{ak}+O(1)\right)\\
&= KN\sum_{\substack{k|q\\k\le K}}\sum_{a|q/k}\frac{\mu(a)}{ak} + O(Kq^{\varepsilon}) + O(Nq^{\varepsilon})
\end{aligned}$$

for any $\varepsilon > 0$. Now

$$\sum_{\substack{k|q\\k\le K}}\sum_{a|q/k}\frac{\mu(a)}{ak} = q^{-1}\sum_{\substack{k|q\\k\le K}}\varphi(q/k) = 1 - q^{-1}\sum_{\substack{k|q\\k>K}}\varphi(q/k)$$

and

$$\sum_{\substack{k|q\\k>K}}\varphi(q/k) \le q\sum_{\substack{k|q\\k>K}}\frac{1}{k} \le \frac{q}{K}\sum_{\substack{k|q\\k>K}} 1 = O\left(\frac{q^{1+\varepsilon}}{K}\right).$$

Thus

$$\sum_{r\le K}\sum_{\substack{d\le N\\(d,q)|r}} (d,r) = KN\left(1+O\left(\frac{q^{\varepsilon}}{K}\right)\right) + O(Kq^{\varepsilon}) + O(Nq^{\varepsilon}) = KN + O((K+N)q^{\varepsilon}).$$

Using partial summation we see that

$$\begin{aligned}
\sum_{r\le K}\sum_{\substack{d\le N\\(d,q)|r}}\frac{(d,r)}{d} &= N^{-1}\sum_{r\le K}\sum_{\substack{d\le N\\(d,q)|r}}(d,r) + \int_1^N \frac{1}{t^2}\sum_{r\le K}\sum_{\substack{d\le t\\(d,q)|r}}(d,r)dt\\
&= K + O((K/N+1)q^{\varepsilon}) + K\log N + O(Kq^{\varepsilon}) + O(q^{\varepsilon}\log N)\\
&= K\log N + O((K+1)q^{\varepsilon}) + O(q^{\varepsilon}\log N).
\end{aligned}$$

The last part of the proposition is also proved using partial summation. We omit the details. □

The difficulty in proving Conjecture 1.2 obviously lies in dealing with the fact that we only consider a very small number of all the residue classes (K can be much smaller than q) and at the same time M and N can be much smaller than q (as pointed out earlier). Indeed the above shows:

Proposition 3.2. *Let $K, M, N \geq 1$ with $M \asymp N$ and $K \geq N^\eta$ for some fixed $\eta > 0$. Assume also that $q \leq N^{1-\delta}$ for some $\delta > 0$ and $(q, \rho) = 1$. Then*

$$\sum_{r \leq K} \sum_{m \equiv \rho r(q)} \tau_{M,N}(m) \sim \frac{KMN}{q}$$

as $N \to \infty$ uniformly in M, K, q and ρ.

Conjecture 1.2 says that the asymptotic formula above still holds if we extend the range of q to $q \leq N^{2-\delta}$. In the same way we see that Conjecture 1.5 holds for small values of q. More precisely we have

$$\sum_{m \equiv r(q)} \tau_N^*(m) \ll \frac{N^2 \varphi(q)}{q^2}$$

for $q \leq N^{1-\delta}$.

The following lemma will be useful in the next section. It shows that (1.4) would imply the asymptotics in Conjecture 2.2.

Lemma 3.3. *Let $q \in \mathbf{N}$. Then*

$$\sum_{r=1}^{q} \sum_{d|(q,r)} \sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) = q^2.$$

If $K > 0$ then for any $\varepsilon > 0$

$$\sum_{r \leq K} \sum_{d|(q,r)} \sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) = Kq + O((K+q)q^\varepsilon).$$

Proof. We see that

$$\sum_{r=1}^{q} \sum_{d|(q,r)} \sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) = \sum_{k|q} \varphi(q/k) \sum_{d|k} \sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right).$$

Since the left hand side is a multiplicative function of q it suffices to prove that

$$\sum_{n=0}^{l} \varphi(p^{l-n}) \sum_{m=0}^{n} \sum_{j=0}^{l-m} p^{m+j} \mu(p^{l-m-j}) = p^{2l}. \tag{3.6}$$

for a prime p. One easily checks that

$$\begin{aligned}
\sum_{n=0}^{l} \varphi(p^{l-n}) \sum_{m=0}^{n} \sum_{j=0}^{l-m} p^{m+j} \mu(p^{l-m-j}) &= p^{l-1} + p^l(1-p^{-1}) \sum_{n=0}^{l} \varphi(p^{l-n})(n+1) \\
&= p^{l-1} + p^l(1-p^{-1}) \left(l + 1 + p^l(1-p^{-1}) \sum_{n=0}^{l-1} \frac{n+1}{p^n} \right).
\end{aligned}$$

The identity (3.6) now follows since

$$\sum_{n=0}^{l-1} \frac{n+1}{p^n} = \frac{1-p^{-l-1}}{(1-p^{-1})^2} - \frac{l+1}{p^l(1-p^{-1})}.$$

From the first part we deduce using (3.2) that

$$\begin{aligned}\sum_{r\le K}\sum_{d|(q,r)}\sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) &= \frac{K}{q}\sum_{\substack{k|q\\ k\le K}}\varphi\left(\frac{q}{k}\right)\sum_{d|k}\sum_{c|\frac{q}{d}} dc\mu\left(\frac{q}{dc}\right) + O(Kq^{\varepsilon})\\ &= Kq + \frac{K}{q}\sum_{\substack{k|q\\ k>K}}\varphi\left(\frac{q}{k}\right)\sum_{d|k} d\varphi\left(\frac{q}{d}\right) + O(Kq^{\varepsilon})\\ &= Kq + O((q+K)q^{\varepsilon}).\end{aligned}$$

□

4. Average Results

From [4] we know how to count elements of an arithmetic progression in a given interval:

Lemma 4.1. *Let $a<b$, $r\in\mathbf{Z}$ and $H\in\mathbf{N}$ with $H<q$. Then*

$$\sum_{\substack{a<m\le b\\ m\equiv r(q)}} 1 = \frac{b-a}{q} + \sum_{1\le|h|\le H} C_{a,b}(h)e(-hr/q) + O\left(\theta_H((a-r)/q)+\theta_H((b-r)/q)\right),$$

where

$$\theta_H(s) = \begin{cases}\min(1,1/(H\|s\|)) & \text{if } s\notin\mathbf{Z}\\ 1 & \text{if } s\in\mathbf{Z}\end{cases}$$

and

$$C_{a,b}(n) = \frac{e(bn/q)-e(an/q)}{2\pi i n}.$$

Note that $H_1\le H_2$ implies that $\theta_{H_1}(s)\ge\theta_{H_2}(s)$. It will be convenient to set

$$E_H(a,b,r,q) = \theta_H((a-r)/q)+\theta_H((b-r)/q).$$

In [1] the following lemma was proved:

Lemma 4.2. *Let $(r,q)=1$, H as in Lemma 4.1 and $t\in\mathbf{R}$. Then*

$$\sum_{c\in\mathbf{Z}_q}\theta_H((t-rc)/q) \ll \frac{q^{1+\varepsilon}}{H}$$

for any $\varepsilon>0$, where the constant implied depends at most on ε.

We will use these results to prove a technical lemma. Define

$$I(m,M,\alpha,\beta,b,t) = \{x\in\mathbf{N}\mid \max(m,(\alpha+t)/b)\le x\le\min(M,(\beta+t)/b)\}$$

and

$$J(m,M,\alpha,\beta,b,t) = \{x\in\mathbf{N}\mid \max(m,(\alpha+t)/b)< x\le\min(M,(\beta+t)/b)\}.$$

Lemma 4.3. *Let $N\ge 1$. Then*

$$\begin{aligned}&\sum_{\substack{a\le N_1\\ b\le N_2}}\sum_{c\in\mathbf{Z}}\#\{x\in I(1,N_3,a,N_4a,b,r+cq)\mid xb\equiv r+cq(a)\}\\ &\quad=\sum_{\substack{a\le N_1\\ b\le N_2}}\sum_{\substack{c\in\mathbf{Z}\\ (a,b)|cq+r}}\frac{(a,b)}{a}\int_0^{N_3}1_{\{t\in\mathbf{R}|(r+cq)/b\le t\le(N_4a+r+cq)/b\}}(t)dt + O\left(\frac{N^{\frac{7}{2}+\varepsilon}}{q}\right)\end{aligned}$$

for $N_i\asymp N$, $q\le N^{2-\delta}$ and $|r|\ll N^2$. In the first sum $a,b\in\mathbf{N}$.

Proof. We see that

$$\sum_{\substack{a\le N_1\\ b\le N_2}}\sum_{c\in\mathbf{Z}}\#\left\{x\in I(1,N_3,a,N_4a,b,r+cq)\mid xb\equiv r+cq(a)\right\}$$
$$=\sum_{\substack{a\le N_1\\ b\le N_2}}\sum_{c\in\mathbf{Z}}\#\left\{x\in J(0,N_3,0,N_4a,b,r+cq)\mid xb\equiv r+cq(a)\right\}+O\left(\frac{N^{3+\varepsilon}}{q}\right).$$

We apply Lemma 4.1 and get

$$\sum_{\substack{a\le N_1\\ b\le N_2}}\sum_{c\in\mathbf{Z}}\#\left\{x\in J(0,N_3,0,N_4a,b,r+cq)\mid xb\equiv r+cq(a)\right\}$$
$$=\sum_{\substack{a\le N_1\\ b\le N_2}}\sum_{\substack{c\in\mathbf{Z}\\ (a,b)\mid cq+r}}\frac{(a,b)}{a}\int_0^{N_3}1_{\{t\in\mathbf{R}\mid (r+cq)/b\le t\le (N_4a+r+cq)/b\}}(t)dt+\mathcal{F}+\mathcal{E}_1+\mathcal{E}_2$$

where the error term $\mathcal{E}_1$ accounts for the contribution from E_H (we choose $H=[N^\lambda]$) for a and b such that

$$\min\left(\frac{a}{(a,r+cq)},\frac{b}{(b,r+cq)}\right)>N^\lambda, \tag{4.1}$$

$\mathcal{F}$ is the exponential sum (we set $M=\max N_i$)

$$\sum_{|c|\le\frac{M^2}{q}}\sum_{k\mid cq+r}\left|\sum_{N^\lambda\le\alpha\le\frac{N_1}{k}}\sum_{\substack{N^\lambda\le\beta\le\frac{N_2}{k}\\ (\alpha,\beta)=1}}\sum_{1\le|h|\le H}C_{\max\left(0,\frac{r+cq}{\beta k}\right),\min\left(N,\frac{N\alpha k+r+cq}{\beta k}\right)}(h)e\left(\frac{-h(r+cq)\overline{\beta}}{\alpha k}\right)\right|,$$

and $\mathcal{E}_2$ accounts for the the entire error coming from a and b not satisfying (4.1). We estimate $\mathcal{E}_2$ trivially by

$$\mathcal{E}_2\ll\sum_{|c|\le\frac{M^2}{q}}\sum_{k\mid cq+r}(N_1+N_2)N^\lambda\ll\frac{N^{3+\lambda+\varepsilon}}{q}.$$

We see that the error term $\mathcal{E}_1$ is at most of order

$$\sum_{|c|\le\frac{M^2}{q}}\sum_{k\mid cq+r}\sum_{\delta_1,\delta_2\mid\frac{r+cq}{k}}\sum_{(\alpha,\beta)\in\mathcal{A}}E_H\left(\max\left(0,\frac{r+cq}{\beta k}\right),\min\left(N,\frac{N\alpha k+r+cq}{\beta k}\right),\frac{\overline{\beta}(r+cq)}{k},\alpha\right),$$

where

$$\mathcal{A}=\left\{(\alpha,\beta)\in\mathbf{N}^2\,\middle|\,\alpha\in I_1,\ \beta\in I_2,\ (\alpha,\beta)=1,\ \left(\alpha,\frac{r+cq}{k}\right)=\delta_1,\ \left(\beta,\frac{r+cq}{k}\right)=\delta_2\right\},$$

$I_1=[\delta_1N^\lambda,N_1/k]$ and $I_2=[\delta_2N^\lambda,N_2/k]$. First we consider

$$\sum_{(\alpha,\beta)\in\mathcal{A}}\theta_H\left(\frac{\overline{\beta}(r+cq)}{\alpha k}\right).$$

From Lemma 4.2 it follows that

$$\sum_{\substack{\beta\in I_2\\ (\alpha,\beta)\in\mathcal{A}}}\theta_H\left(\frac{\overline{\beta}(r+cq)}{\alpha k}\right)\ll N^{-\lambda}\left(\frac{N_2\left(\alpha,\frac{r+cq}{k}\right)}{\alpha k}+1\right)\left(\frac{\alpha}{\left(\alpha,\frac{r+cq}{k}\right)}\right)^{1+\varepsilon},$$

and

$$\sum_{\alpha\le\frac{N_1}{k}}\left(\frac{N_2\left(\alpha,\frac{r+cq}{k}\right)}{\alpha k}+1\right)\left(\frac{\alpha}{\left(\alpha,\frac{r+cq}{k}\right)}\right)^{1+\varepsilon}\ll N^{2+2\varepsilon}.$$

Thus it follows that

$$\sum_{|c|\le\frac{M^2}{q}}\sum_{k|cq+r}\sum_{(\alpha,\beta)\in\mathcal{A}}\theta_H\left(\frac{\overline{\beta}(r+cq)}{\alpha k}\right)\ll\frac{N^{4-\lambda+3\varepsilon}}{q}.$$

We now consider

$$\sum_{(\alpha,\beta)\in\mathcal{A}}\theta_H\left(\frac{(\overline{\beta}-1/\beta)(r+cq)}{\alpha k}\right)=\sum_{(\alpha,\beta)\in\mathcal{A}}\theta_H\left(\frac{\overline{\alpha}(r+cq)}{\beta k}\right),$$

where $\overline{\alpha}$ is the inverse of α modulo β. Again we apply Lemma 4.2 and obtain

$$\sum_{|c|\le\frac{M^2}{q}}\sum_{k|cq+r}\sum_{(\alpha,\beta)\in\mathcal{A}}\theta_H\left(\frac{(\overline{\beta}-1/\beta)(r+cq)}{\alpha k}\right)\ll\frac{N^{4-\lambda+\varepsilon}}{q}.$$

The remaining terms of similar type are estimated in the same way.

We now consider the exponential sum

$$\sum_{N^\lambda<\alpha\le\frac{N_1}{k}}\sum_{\substack{N^\lambda<\beta\le\frac{N_2}{k}\\(\alpha,\beta)=1}}\sum_{1\le|h|\le H}C_{\max\left(0,\frac{r+cq}{\beta k}\right),\min\left(N,\frac{N\alpha k+r+cq}{\beta k}\right)}(h)e\left(\frac{-h(r+cq)\overline{\beta}}{\alpha k}\right).$$

First we look at

$$\sum_{1\le|h|\le H}\frac{1}{2\pi ih}\sum_{N^\lambda<\alpha\le\frac{N_1}{k}}\sum_{\substack{N^\lambda<\beta\le\frac{N_2}{k}\\(\alpha,\beta)=1}}e\left(\frac{-h(r+cq)\overline{\beta}}{\alpha k}\right).$$

Using standard exponential sum techniques we rewrite the inner sum (see e.g. [1])

$$\begin{aligned}\sum_{\substack{N^\lambda<\beta\le\frac{N_2}{k}\\(\alpha,\beta)=1}}e\left(\frac{-h(r+cq)\overline{\beta}}{\alpha k}\right)&=\sum_{\substack{\beta=0\\(\alpha,\beta)=1}}^{\alpha-1}e\left(\frac{-h(r+cq)\overline{\beta}}{\alpha k}\right)\sum_{N^\lambda<\zeta\le\frac{N_2}{k}}\frac{1}{\alpha}\sum_{\xi=0}^{\alpha-1}e\left(\frac{\xi(\beta-\zeta)}{\alpha}\right)\\&=\frac{1}{\alpha}\sum_{\xi=0}^{\alpha-1}\sum_{\substack{\beta=0\\(\alpha,\beta)=1}}^{\alpha-1}e\left(\frac{\xi\beta-h\overline{\beta}(r+cq)/k}{\alpha}\right)\sum_{N^\lambda<\zeta\le\frac{N_2}{k}}e\left(-\frac{\xi\zeta}{\alpha}\right).\end{aligned}$$

The last sum is just a Weyl sum and is easily estimated (see e.g. [7] Section 8.2) by

$$\left|\sum_{N^\lambda<\zeta\le\frac{N_2}{k}}e\left(-\frac{\xi\zeta}{\alpha}\right)\right|\le\min\left(N_2,\frac{\alpha}{\xi}\right).$$

Thus using the Weil bound for Kloosterman sums we obtain

$$\begin{aligned}
\Bigg|\sum_{\substack{N^\lambda<\beta\le\frac{N_2}{k}\\(\alpha,\beta)=1}} e\left(\frac{-h(r+cq)\overline{\beta}}{\alpha k}\right)\Bigg| &\le \frac{1}{\alpha}\sum_{\xi=0}^{\alpha-1}\Bigg|\sum_{\substack{\beta=0\\(\alpha,\beta)=1}}^{\alpha-1} e\left(\frac{\xi\beta-h\overline{\beta}(r+cq)/k}{\alpha}\right)\Bigg|\Bigg|\sum_{N^\lambda<\zeta\le\frac{N_2}{k}} e\left(-\frac{\xi\zeta}{\alpha}\right)\Bigg| \\
&\ll \alpha^{-\frac{1}{2}+\varepsilon}\left(\alpha, h\frac{r+cq}{k}\right)^{\frac{1}{2}}\left(N+\sum_{\xi=1}^{\alpha-1}\frac{\alpha}{\xi}\right) \\
&\ll N^{\frac{1}{2}+2\varepsilon}\left(\alpha, h\frac{r+cq}{k}\right)^{\frac{1}{2}}
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{\alpha\le N_1}\left(\alpha, h\frac{r+cq}{k}\right)^{\frac{1}{2}} &= \sum_{d|h\frac{r+cq}{k}}\sqrt{d}\#\left\{\alpha\in\mathbf{N}\,\Big|\,\alpha\le N_1,\ \left(\alpha,h\frac{r+cq}{k}\right)=d\right\} \\
&\ll \sum_{\substack{d|h\frac{r+cq}{k}\\ d\le N_1}}\frac{\sqrt{d}N_1}{d} \\
&\ll N^{1+\varepsilon}.
\end{aligned}$$

The contribution to $\mathcal{F}$ is

$$\sum_{|c|\le\frac{M^2}{q}}\sum_{k|cq+r}\sum_{1\le h\le N^\lambda}\frac{N^{\frac{3}{2}+3\varepsilon}}{h}\ll\frac{N^{\frac{7}{2}+5\varepsilon}}{q}.$$

The remaining terms are handled in a similar way. Choosing $\lambda = 1/2$ yields the desired result. □

We need a slightly different version of the previous lemma as well

Lemma 4.4. *Let $N\ge 1$, $k \mid q$ and $d_1, d_2 \mid k$. Then*

$$\begin{aligned}
&\sum_{\substack{a_i\le M/d_i\\(a_i,q/d_i)=1}}\sum_{c\in\mathbf{Z}}\#\left\{x\in I(1,Nd_1/k,a_2,Na_2d_2,ka_1,cq)\mid xa_1\equiv cq/k(a_2),\ (x,q/k)=1\right\} \\
&\quad=\frac{k\varphi(q/k)}{q}\sum_{\substack{a_i\le M/d_i\\(a_i,q/d_i)=1}}\sum_{\substack{c\in\mathbf{Z}\\(a_1,a_2)|c}}\frac{(a_1,a_2)}{a_2}\int_0^{Nd_1/k}1_{\{t\in\mathbf{R}|cq/(ka_1)\le t\le(Na_2d_2+cq)/(ka_1)\}}(t)dt+ \\
&\quad\quad O\left(\frac{N^{\frac{7}{2}+\varepsilon}}{q}\right)
\end{aligned}$$

for $M\asymp N$ and $q\le N^{2-\delta}$.

Proof. As in the proof of Lemma 4.3 we see that

$$\sum_{\substack{a_i\le M/d_i\\(a_i,q/d_i)=1}}\sum_{c\in\mathbf{Z}}\#\left\{x\in I(1,Nd_1/k,a_2,Na_2d_2,ka_1,cq)\mid xa_1\equiv cq/k(a_2),\ (x,q/k)=1\right\}$$
$$=\sum_{\substack{a_i\le M/d_i\\(a_i,q/d_i)=1}}\sum_{c\in\mathbf{Z}}\#\left\{x\in J(0,Nd_1/k,0,Na_2d_2,ka_1,cq)\mid xa_1\equiv cq/k(a_2),\ (x,q/k)=1\right\}+O\left(\frac{N^{3+\varepsilon}}{q}\right).$$

Assume $(q,l)=1$ and $l\mid r$. Then

$$\begin{aligned}\#\{x\in\mathbf{Z}\mid a<x\le b,\ x\equiv r(q),\ (x,l)=1\}&=\#\{c\in\mathbf{Z}\mid (a-r)/q<c\le(b-r)/q,\ (c,l)=1\}\\&=\sum_{d\mid l}\mu(d)\left(\left[\frac{b-r}{qd}\right]-\left[\frac{a-r}{qd}\right]\right)\\&=\sum_{d\mid l}\mu(d)\#\{x\in\mathbf{Z}\mid a<x\le b,\ x\equiv r(qd)\}.\end{aligned}$$

Thus

$$\#\left\{x\in J(0,Nd_1/k,0,Na_2d_2,ka_1,cq)\mid xa_1\equiv cq/k(a_2),\ (x,q/k)=1\right\}=\sum_{d\mid\frac{q}{k}}\mu(d)\#\left\{x\in J(0,Nd_1/k,0,Na_2d_2,ka_1,cq)\mid xa_1\equiv cq/k(a_2)\right\}.$$

We can now proceed as in the proof of Lemma 4.4. The idea is the same so we only sketch the rest of the proof.

For the θ_H sums we are lead to consider (essentially) sums of the form

$$\sum_{|c|\le\frac{N^2}{q}}\sum_{\delta\mid\frac{q}{k}}\sum_{\substack{\alpha_i\le M/d_i\\(\alpha_i,q/d_i)=1}}\theta_H\left(\frac{cq\overline{\alpha}_1}{k\delta\alpha_2}\right)$$

and these can be estimated just as in Lemma 4.4 since (roughly speaking) the sum above just has fewer terms than the ones considered in the previous lemma (we exclude the terms that do not meet a certain coprimality condition) and all terms are non-negative. We have to be more careful with the exponential sum terms. We consider sums of the form

$$\sum_{|c|\le\frac{N^2}{q}}\sum_{l\mid\frac{q}{k}}\sum_{\delta\mid c}\left|\sum_{\substack{H<\alpha_i\le\frac{M}{\delta d_i}\\(\alpha_1,\alpha_2)=(\alpha_i,q/d_i)=1}}\sum_{1\le|h|\le H}\frac{1}{h}e\left(-\frac{hcq\overline{\alpha}_1}{kl\delta\alpha_2}\right)\right|.\tag{4.2}$$

In the inner sum we replace the coprimality condition $(\alpha_1,q/d_1)=1$ with a Möbius sum (setting $\alpha_1=\gamma\lambda$) and get

$$\sum_{\gamma\mid\frac{q}{d_1}}\mu(\gamma)\sum_{1\le|h|\le H}\frac{1}{h}\sum_{\substack{H<\alpha_2\le\frac{M}{\delta d_2}\\(\alpha_2,\gamma q/d_2)=1}}\sum_{\substack{\frac{H}{\gamma}<\lambda\le\frac{M}{\gamma\delta d_1}\\(\lambda,\alpha_2)=1}}e\left(-\frac{\overline{\gamma}hcq\overline{\lambda}}{kl\delta\alpha_2}\right).$$

The inner sum is estimated as in Lemma 4.3 by

$$\sum_{\substack{H<\alpha_2\le\frac{M}{\delta d_2}\\(\alpha_2,\gamma q/d_2)=1}}\left|\sum_{\substack{\frac{H}{\gamma}<\lambda\le\frac{M}{\gamma\delta d_1}\\(\lambda,\alpha_2)=1}}e\left(-\frac{\overline{\gamma}hcq\overline{\lambda}}{kl\delta\alpha_2}\right)\right|\ll N^{\frac12+\varepsilon}\sum_{\substack{\alpha_2\le M\\(\alpha_2,\gamma q/d_2)=1}}\left(\alpha_2,\frac{hcq}{lk\delta}\right)^{\frac12}\ll N^{\frac32+2\varepsilon}.$$

Thus we can estimate (4.2) by $\frac{N^{\frac{7}{2}+3\varepsilon}}{q}$. $\square$

Using Lemma 4.3 we can now prove the following:

Theorem 4.5. *Let $K, M, N \geq 1$, $q \in \mathbf{N}$ and $N^\eta \leq K$ for some $\eta > 0$. Assume also that $q \leq N^{2-\delta}$ for some $\delta > 0$ and $(\rho, q) = 1$. Then*

$$\sum_{s=1}^{q}\left(\sum_{r\leq K}\sum_{m\equiv\rho r+s(q)}\tau_{M,N}(m)-\frac{KMN}{q}\right)^2 \ll \frac{K^2N^{\max(\frac{7}{2}+\varepsilon,4-\delta,4-\eta+\varepsilon)}}{q}$$

for any $\varepsilon > 0$.

Proof. We see that

$$\begin{aligned}\sum_{s=1}^{q}\left(\sum_{r\leq K}\sum_{m\equiv\rho r+s(q)}\tau_{M,N}(m)\right)^2 &= \sum_{\substack{r\leq K\\ r'\leq K}}\sum_{m\equiv m'+\rho(r-r')(q)}\tau_{M,N}(m)\tau_{M,N}(m')\\ &= \sum_{|l|\leq K}R_l\sum_{m\equiv m'+l\rho(q)}\tau_{M,N}(m)\tau_{M,N}(m')\end{aligned}$$

where $R_l = [K] + 1 - |l|$. We consider the innermost sum

$$\begin{aligned}\sum_{m\equiv m'+l\rho(q)}&\tau_{M,N}(m)\tau_{M,N}(m') = \\ &\#\{(a_1,b_1,a_2,b_2)\in\mathbf{N}^4 \mid a_1,b_1\leq N,\ a_2,b_2\leq M,\ a_1b_1\equiv a_2b_2+\rho l(q)\}\end{aligned}\tag{4.3}$$

We want to "switch" the roles of q and a_2 and apply Lemma 4.3. Note that

$$a_1b_1 \equiv a_2b_2 + \rho l(q)$$

exactly if there exists $c \in \mathbf{Z}$ such that

$$a_1b_1 - \rho l - cq = a_2b_2.$$

Now fix a_1 and a_2. We see that (4.3) is

$$\begin{aligned}&\sum_{\substack{a_1,a_2\leq N\\ c\in\mathbf{Z}}}\#\left\{x\in I(1,M,a_2,Ma_2,a_1,\rho l+cq)\;\middle|\; xa_1\equiv\rho l+cq(a_2)\right\}\\ &=\sum_{\substack{a_1,a_2\leq N\\ c\in\mathbf{Z}\\ (a_1,a_2)|cq+\rho l}}\frac{(a_1,a_2)}{a_2}\int_0^M 1_{\{t\in\mathbf{R}|(\rho l+cq)/a_1\leq t\leq(\rho l+cq)/a_1+Ma_2/a_1\}}(t)dt+O\left(\frac{N^{\frac{7}{2}+\varepsilon}}{q}\right)\\ &=\sum_{\substack{a_2,a_1\leq N\\ (a_2,a_1,q)|l}}\frac{(a_2,a_1)}{a_2}\int_0^M\sum_{\substack{(ta_1-l\rho)/q-Ma_2/q\leq c\leq(ta_1-l\rho)/q\\ (a_2,a_1)|cq+\rho l}}1dt+O\left(\frac{N^{\frac{7}{2}+\varepsilon}}{q}\right)\\ &=M\sum_{\substack{a_2,a_1\leq N\\ (a_2,a_1,q)|l}}\frac{(a_2,a_1)}{a_2}\frac{Ma_2(a_2,a_1,q)}{q(a_2,a_1)}+O\left(N^2+\frac{N^{\frac{7}{2}+\varepsilon}}{q}\right)\\ &=\frac{M^2}{q}\sum_{\substack{a_2,a_1\leq N\\ (a_2,a_1,q)|l}}(a_2,a_1,q)+O\left(\frac{N^{\max(\frac{7}{2}+\varepsilon,4-\delta)}}{q}\right).\end{aligned}$$

Again we consider the entire sum and see (using Lemma 3.1)

$$\begin{aligned}\sum_{|l|\le K} R_l \sum_{\substack{a_2,a_1\le N\\(a_2,a_1,q)|l}} (a_2,a_1,q) &= \sum_{a_1=1}^{N}\sum_{|l|\le K} R_l \sum_{\substack{a_2\le N\\(a_2,a_1,q)|l}} (a_2,a_1,q)\\ &= 2([K]+1)\sum_{a_1=1}^{N}(KN+O((K+N)q^{\varepsilon}))+O(N^2q^{\varepsilon})-\\ &\quad \sum_{a_1=1}^{N}\left(K^2N+O(K(K+N)q^{\varepsilon})\right)\\ &= K^2N^2+O(NK(N+K)q^{\varepsilon}).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{s=1}^{q}\left(\sum_{r\le K}\sum_{m\equiv\rho r(q)}\tau_{M,N}(m)-\frac{KMN}{q}\right)^2 &= \sum_{\substack{r\le K\\r'\le K}}\sum_{m\equiv m'+\rho(r-r')(q)}\tau_{M,N}(m)\tau_{M,N}(m')-\\ &\quad \frac{K^2M^2N^2}{q}\\ &= O\left(\frac{K^2N^{\max(\frac{7}{2}+\varepsilon,4-\delta,4-\eta+\varepsilon)}}{q}\right).\end{aligned}$$

□

This also shows that for the individual terms the asymptotics one should expect from Conjecture 2.2 holds for a subset of $\{s\in\mathbf{N}\mid 1\le s\le q\}$ of full density in the following sense:

Corollary 4.6. *Let $\nu>0$ and assumptions be as in Theorem 4.5. Then*

$$\#\left\{s\in\{1,\dots,q\}\;\Big|\;\Big|1-\frac{q}{KMN}\sum_{r\le K}\sum_{m\equiv\rho r+s(q)}\tau_{M,N}(m)\Big|>\nu\right\}\ll qN^{-\min(\delta,\frac{1}{2},\eta)+\varepsilon}$$

uniformly in K, q and ρ for any $\varepsilon>0$.

In this connection it should be mentioned that if s is "bad" then so is its neighbors. In fact we see that just one bad s (in the sense of Corollary 4.6) will imply that there are at least $K^{1-\varepsilon}$ bad values of s.

We now proceed to the proof of Theorem 1.8.

Proof of Theorem 1.8. Recall that $\varphi(q)/q\gg q^{1-\varepsilon}$ for any $\varepsilon>0$. Applying (3.2) we see that

$$\begin{aligned}\sum_{(r,q)=k}\sum_{m\equiv r(q)}\tau_{M,N}(m) &= \sum_{(m,q)=k}\tau_{M,N}(m)\\ &= \sum_{c|k}\varphi(M/c,q/c)\varphi(Nc/k,q/k)\\ &= \sum_{c|k}\frac{\varphi(q/c)\varphi(q/k)cMN}{q^2}+O(q^{\varepsilon}N).\end{aligned}$$

We proceed as in the proof of Theorem 4.5 and obtain using Lemma 4.4

$$\sum_{(q,r)=k}\sum_{\substack{m\equiv r(q)\\ m'\equiv r(q)}}\tau_{M,N}(m)\tau_{M,N}(m')$$
$$=\sum_{d_i|k}\sum_{\substack{\alpha_i\le\frac{M}{d_i}\\ (\alpha,q/d_i)=1}}\sum_{\substack{c\in\mathbf{Z}\\ (\alpha_1,\alpha_2)|c}}$$
$$\#\left\{x\in I\left(1,\frac{Nd_1}{k},\alpha_2,N\alpha_2d_2,k\alpha_1,cq\right)\middle| x\alpha_1\equiv\frac{cq}{k}(\alpha_2),\ \left(x,\frac{q}{k}\right)=1\right\}$$
$$=\sum_{d_i|k}\sum_{\substack{\alpha_i\le\frac{M}{d_i}\\ (\alpha,q/d_i)=1}}\sum_{\substack{c\in\mathbf{Z}\\ (\alpha_1,\alpha_2)|c}}\frac{(\alpha_1,\alpha_2)\varphi(q/k)k}{\alpha_2 q}\times$$
$$\int_0^{\frac{Nd_1}{k}}1_{\{t\in\mathbf{R}|cq/(k\alpha_1)<t\le(cq+N\alpha_2d_2)/(k\alpha_1)\}}(t)dt+O\left(\frac{N^{\frac{7}{2}+\varepsilon}}{q}\right)$$
$$=\frac{N^2\varphi(q/k)}{q^2}\sum_{d_i|k}d_1d_2\sum_{\substack{\alpha_i\le\frac{M}{d_i}\\ (\alpha,q/d_i)=1}}1+O\left(\frac{N^{\max(4-\delta,\frac{7}{2}+\varepsilon)}}{q}\right)$$
$$=\frac{M^2N^2\varphi(q/k)}{q^2}\left(\sum_{d|k}d\varphi(q/d)\right)^2+O\left(\frac{N^{\max(4-\delta,\frac{7}{2}+\varepsilon)}}{q}\right).$$

This proves the theorem. □

Finally we prove Theorem 1.9.

Proof of Theorem 1.9. We see that

$$\sum_{(\rho,q)=1}\sum_{r\le K}\sum_{m\equiv\rho r(q)}\tau_{M,N}(m)=\sum_{r\le K}\frac{\varphi(q)}{\varphi(q/(q,r))}\sum_{(m,q)=(r,q)}\tau_{M,N}(m)$$
$$=\sum_{\substack{k\le K\\ k|q}}\frac{\varphi(q)}{\varphi(q/k)}\#\{r\in\mathbf{N}\mid r\le K,\ (r,q)=k\}\times$$
$$\#\{(a,b)\in\mathbf{N}^2\mid a\le M,\ b\le N,\ (ab,q)=k\}.$$

In the same way we see that

(4.4)
$$\#\{(a,b)\in\mathbf{N}^2\mid a\le M,\ b\le N,\ (ab,q)=k\}=\sum_{l|k}\#\{b\in\mathbf{N}\mid b\le N,\ (b,q)=l\}\times$$
$$\#\{a\in\mathbf{N}\mid a\le M,\ (a,q/l)=k/l\}$$
$$=\sum_{l|k}\frac{N}{q}(\varphi(q/l)+O(q^{\varepsilon}))\frac{Ml}{q}(\varphi(q/k)+O(q^{\varepsilon}))$$
$$=\frac{\varphi(q/k)N^2}{q^2}\sum_{l|k}l\varphi(q/l)+O\left(\frac{MN}{q^{1-\varepsilon}}\right)$$
$$\le\frac{MNd(k)}{k}+O\left(\frac{N^2}{q^{1-\varepsilon}}\right).$$

Thus

$$\begin{aligned}
&\sum_{\substack{k\le K\\ k|q}} \#\{(a,b)\in\mathbf{N}^2 \mid a\le M,\ b\le N,\ (ab,q)=k\}\\
&\qquad = MN - \sum_{\substack{k> K\\ k|q}} \#\{(a,b)\in\mathbf{N}^2 \mid a\le M,\ b\le N,\ (ab,q)=k\}\\
&\qquad = MN + O\left(\frac{N^2 q^{\varepsilon}}{K}\right).
\end{aligned}\tag{4.5}$$

Combining (3.2) and (4.5) it follows that

$$\sum_{(\rho,q)=1}\sum_{r\le K}\sum_{m\equiv\rho r(q)} \tau_{M,N}(m) = \frac{\varphi(q)KMN}{q} + O\left(\frac{\varphi(q)N^2q^{\varepsilon}}{q}\right).\tag{4.6}$$

Now we look at

$$\left(\sum_{r\le K}\sum_{m\equiv r\rho(q)} \tau_{M,N}(m) - \frac{KMN}{q}\right)^2$$

and rewrite it as

$$\begin{aligned}
&\left(\sum_{r\le K}\left(\sum_{m\equiv r\rho(q)}\tau_{M,N}(m) - \frac{MN}{q^2}\sum_{d|(r,q)} d\varphi\left(\frac{q}{d}\right)\right)\right)^2 + \left(\frac{MN}{q^2}\left(Kq - \sum_{r\le K}\sum_{d|(r,q)} d\varphi\left(\frac{q}{d}\right)\right)\right)^2 +\\
&2\left(\sum_{r\le K}\left(\sum_{m\equiv r\rho(q)}\tau_{M,N}(m) - \frac{MN}{q^2}\sum_{d|(r,q)} d\varphi\left(\frac{q}{d}\right)\right)\right)\left(\frac{MN}{q^2}\left(Kq - \sum_{r\le K}\sum_{d|(r,q)} d\varphi\left(\frac{q}{d}\right)\right)\right).
\end{aligned}$$

Lemma 3.3 implies that

$$\left|\frac{MN}{q^2}\left(Kq - \sum_{r\le K}\sum_{d|(r,q)} d\varphi\left(\frac{q}{d}\right)\right)\right| \ll \frac{N^2(K+q)q^{\varepsilon}}{q^2}.\tag{4.7}$$

We also see that Lemma 3.3 and (4.6) implies

$$\left|\sum_{(\rho,q)=1}\sum_{r\le K}\left(\sum_{m\equiv r\rho(q)}\tau_{M,N}(m) - \frac{MN}{q^2}\sum_{d|(r,q)} d\varphi\left(\frac{q}{d}\right)\right)\right| \ll \frac{K\varphi(q)N^2q^{\varepsilon}}{q}\left(\frac{1}{K}+\frac{1}{q}\right).\tag{4.8}$$

Thus it remains to look at

$$\sum_{(\rho,q)=1}\left(\sum_{r\le K}\left(\sum_{m\equiv r\rho(q)}\tau_{M,N}(m) - \frac{MN}{q^2}\sum_{d|(r,q)} d\varphi\left(\frac{q}{d}\right)\right)\right)^2.\tag{4.9}$$

Using Cauchy-Schwarz inequality and Theorem 1.8 we estimate (4.9) by

$$
\begin{aligned}
K\sum_{r\le K}\sum_{(\rho,q)=1}&\left(\sum_{m\equiv r\rho(q)}\tau_{M,N}(m)-\frac{MN}{q^2}\sum_{d|(r,q)}d\varphi\left(\frac{q}{d}\right)\right)^2\\
&=K\sum_{r\le K}\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,r)}\right)}\sum_{(s,q)=(r,q)}\left(\sum_{m\equiv s(q)}\tau_{M,N}(m)-\frac{MN}{q^2}\sum_{d|(r,q)}d\varphi\left(\frac{q}{d}\right)\right)^2\\
&\ll\frac{KN^{\max(\frac{7}{2}+\varepsilon,4-\delta)}}{q}\sum_{r\le K}\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,r)}\right)}\\
&\ll\frac{KN^{\max(\frac{7}{2}+\varepsilon,4-\delta)}}{q}\sum_{r\le K}(q,r)\\
&\ll\frac{K^2N^{\max(\frac{7}{2}+\varepsilon,4-\delta)}}{q}
\end{aligned}
$$

Together with the estimates (4.7) and (4.8) this proves the theorem. □

5. Estimates for τ_M^* and τ_M

Through out this section we will restrict our discussion to τ_M^* though the results (with suitable modifications) clearly can be extended to cover $\tau_{M,N}$ as well.

We see that

$$
\sum_{m\le x}\tau_M(m)=\sum_{m\le x}\sum_{d=1}^{[M]}\delta_d(m)=\sum_{d=1}^{[M]}\frac{x}{d}+O(M)=x\log M+O(M+x). \tag{5.1}
$$

Let τ denote the usual divisor function and note that

$$
\tau_M^*(m)=\begin{cases}\tau(m) & \text{if } m\le M\\ 2\tau_M(m)-\tau(m) & \text{if } M<m<M^2\\ 0 & \text{if } m\ge M^2\end{cases}.
$$

It is well known that

$$
\sum_{m\le x}\tau(m)=x\log x+(2\gamma-1)x+O(\sqrt{x}). \tag{5.2}
$$

Thus

$$
\sum_{m\le x}\tau_M^*(m)=\begin{cases}x\log x+(2\gamma-1)x+O(\sqrt{x}) & \text{if } x\le M\\ x\log\frac{M^2}{x}+O(x) & \text{if } M<x<M^2\\ [M]^2 & \text{if } x\ge M^2\end{cases},
$$

where the constants implied are absolute.

Conjecture 1.5 is probably hard to prove. We can however, give an estimate for the sum using a result due to Nair and Tenenbaum [11]. Before we state the result we need to introduce some notation. Let $F:\mathbf{N}\to\mathbf{R}_+$. We say that $F\in\mathcal{M}(A,B,\varepsilon)$ if F satisfies (for $(m,n)=1$)

$$
F(mn)\le\min(A^{\Omega(m)},Bm^{\varepsilon})F(n)
$$

for some $A,B\ge 0$, where $\Omega(m)$ denotes the total number of prime factors of m, counted with multiplicity. In [11] the following was proved:

Theorem 5.1. *Let* $F \in \mathcal{M}(A, B, \frac{\varepsilon\delta}{3})$, $0 < \delta \le 1$, $0 < \varepsilon < \frac{1}{8}$. *Then*

$$\sum_{\substack{x<n\le x+y \\ n\equiv r(q)}} F(n) \ll \frac{y}{\varphi(q)\log x} \sum_{\substack{n\le \frac{x}{q} \\ (n,q)=1}} \frac{F(n)}{n}$$

uniformly for $(r, q) = 1$, $x^{\frac{1+4\varepsilon\delta}{1+\delta}} \le y \le x$, $x \ge c_0 q^{1+\delta}$, *where* c_0 *and the constant implied depends at most on* A, B, δ *and* ε.

Proposition 1.6 can be proved quite easily, since τ_N^* is closely related to the Hooley Δ-function defined by

$$\Delta(n) = \max_{u\in\mathbf{R}} \#\{d \in \mathbf{N} \mid e^u < d \le e^{u+1},\ d \mid n\}.$$

One easily checks that for all $N \ge 1$, $k \in \mathbf{N}$ we have

$$\tau_N^*(m) \le 2k\Delta(m) \tag{5.3}$$

whenever $\frac{N^2}{2^k} < m \le \frac{N^2}{2^{k-1}}$, and this implies that

$$\sum_{m\equiv r(q)} \tau_N^*(m) \ll \sum_{\substack{m\le N^2 \\ m\equiv r(q)}} \Delta(m).$$

One easily checks that $\Delta \in \mathcal{M}(2, B, \varepsilon)$ for any $\varepsilon > 0$ and suitable B (chosen according to ε). Proposition 1.6 now follows from Theorem 5.1 since

$$\sum_{n\le x} \frac{\Delta(n)}{n} \ll e^{\sqrt{(2+\varepsilon)(\log\log x)(\log\log\log x)}} \log x.$$

This was proved in [15].

Note that for $x \gg 1$

$$e^{\sqrt{(2+\varepsilon)(\log\log x)(\log\log\log x)}} \ll (\log x)^{\varepsilon'} \tag{5.4}$$

for any $\varepsilon' > 0$. To see this note that

$$\frac{\log\log\log x}{\log\log x} \to 0$$

as $x \to \infty$. In particular

$$\frac{\log\log\log x}{\log\log x} \le \frac{\varepsilon'^2}{2+\varepsilon}$$

for any $\varepsilon' > 0$ for x large. Hence

$$(2+\varepsilon)(\log\log x)(\log\log\log x) \le (\varepsilon' \log\log x)^2.$$

From this (5.4) follows easily.

To prove Theorem 1.7 we need the following lemma.

Lemma 5.2. *Let* $\delta, \varepsilon > 0$. *Let* $q \le x^{1-\delta}$ *and* $2 \le N$. *Assume also that* $N \ge q^{\varepsilon}$. *Then*

$$\sum_{\substack{n\le x \\ (n,q)=1}} \frac{\tau_N(n)}{n} \ll \frac{\varphi(q)^2 \log x \log N}{q^2}.$$

Proof. We see that

$$\sum_{\substack{n\le x \\ (n,q)=1}} \frac{\tau_N(n)}{n} = \sum_{\substack{ab\le x,\ a\le N \\ (a,q)=(b,q)=1}} \frac{1}{ab} = \sum_{\substack{a\le N \\ (a,q)=1}} \frac{1}{a} \sum_{\substack{b\le x/a \\ (b,q)=1}} \frac{1}{b}. \tag{5.5}$$

Thus we must consider

$$\begin{aligned}\sum_{\substack{a\le Y\\(a,q)=1}}\frac{1}{a}&=\sum_{a\le Y}\frac{1}{a}\sum_{\substack{d|q\\d|a}}\mu(d)\\&=\sum_{d|q}\frac{\mu(d)}{d}\sum_{\alpha\le Y/d}\frac{1}{\alpha}\\&=\sum_{d|q}\frac{\mu(d)}{d}\left(\log\frac{Y}{d}+\gamma+O\left(\frac{d}{Y}\right)\right)\\&=(\log Y+\gamma)\frac{\varphi(q)}{q}+O\left(\frac{\tau(q)}{Y}\right)-\sum_{d|q}\frac{\mu(d)\log d}{d}.\end{aligned}$$

Let q' denote the square free part of q and p be a prime number. We see that

$$\begin{aligned}\sum_{d|q}\frac{\mu(d)\log d}{d}&=\sum_{d|q}\frac{\mu(d)}{d}\sum_{c|d}\Lambda(c)\\&=\sum_{c|q'}\Lambda(c)\sum_{\substack{c|d\\d|q'}}\frac{\mu(d)}{d}\\&=\sum_{p|q'}\Lambda(p)\sum_{\delta|\frac{q'}{p}}\frac{\mu(\delta p)}{\delta p}\\&=-\sum_{p|q'}\frac{\Lambda(p)}{p}\sum_{\delta|\frac{q'}{p}}\frac{\mu(\delta)}{\delta}.\end{aligned}$$

Thus

$$\sum_{d|q}\frac{\mu(d)\log d}{d}=O\left(\sum_{p|q'}\frac{\Lambda(p)}{p}\right)=O\left(\sum_{p|q}\frac{\log p}{p}\right).$$

We split the last sum in two parts:

$$\sum_{p|q}\frac{\log p}{p}=\sum_{\substack{p|q\\p\le(\log q)^2}}\frac{\log p}{p}+\sum_{\substack{p|q\\p>(\log q)^2}}\frac{\log p}{p}.$$

Clearly

$$\sum_{\substack{p|q\\p>(\log q)^2}}\frac{\log p}{p}\le(\log q)^{-2}\sum_{p|q}\log p\le 1.$$

We know that

$$\sum_{p\le x}\frac{1}{p}=O(\log\log x)$$

Hence

$$\sum_{\substack{p|q\\p\le(\log q)^2}}\frac{\log p}{p}\le 2\log\log q\sum_{p\le q}p^{-1}=O((\log\log q)^2).$$

From this it follows that

$$\sum_{\substack{a\le Y\\(a,q)=1}} \frac{1}{a} = (\log Y+\gamma)\frac{\varphi(q)}{q} + O\left(\frac{\tau(q)}{Y}\right) + O((\log\log q)^2).$$

Recall that $\varphi(q) \gg \frac{q}{\log\log q}$. Thus it follows that

$$\sum_{\substack{a\le N\\(a,q)=1}} \frac{1}{a} \ll \frac{\varphi(q)\log N}{q}$$

and

$$\sum_{\substack{b\le x\\(b,q)=1}} \frac{1}{b} \ll \frac{\varphi(q)\log x}{q}.$$

The result now follows from (5.5). □

Theorem 1.7 follows immediately from Theorem 5.1 and Lemma 5.2 since $\tau_N \in \mathcal{M}(2,B,\varepsilon)$.

6. Proof of Theorem 1.4

We follow Section 5 in [5]. For $\delta \in (0,1)$ define

$$\mathcal{R}(M,\alpha,\delta) = \#\{(x,y)\in \mathbf{Z}^2 \mid |x\alpha-y|\le\delta,\ |x|\le M\}.$$

The proof of Theorem 1.4 is based on the following identity

$$\mathcal{S}(M,N,K,p,q) = \sum_{a\le N}\mathcal{R}(M,ap/q,K/q) = N + 2\sum_{|r|\le K}\sum_{pm\equiv r(q)}\tau_{M,N}(m).$$

We can transform it into a lattice point problem since

$$\{(x,y)\in\mathbf{Z}^2 \mid |x\alpha-y|\le\delta,\ |x|\le M\} = \{(x,y)\in\mathbf{Z}^2 \mid x\mathbf{u}+y\mathbf{v}\in[-\sqrt{M\delta},\sqrt{M\delta}]^2\}$$

where $\mathbf{u} = (\sqrt{\delta/M}, \alpha\sqrt{M/\delta})$ and $\mathbf{v} = (0,-\sqrt{M/\delta})$. Since $\mathbf{u}$ and $\mathbf{v}$ generate a lattice of determinant 1 we obtain

$$\mathcal{R}(M,\alpha,\delta) = 4M\delta + O(\sqrt{M\delta/\lambda_1}) + O(1),$$

where λ_1 is the length of the shortest non-zero vector in the lattice. We have $\delta = K/q$ and $\alpha = ap/q$ in our case. Thus we expect that the main term in $\mathcal{S}(M,N,K,p,q)$ is KMN/q. It is the error term $O(\sqrt{M\delta/\lambda_1})$ that needs attention. In particular one is concerned with the case where λ_1 is small. The idea is to consider $\sqrt{M\delta/\lambda_1}$ in dyadic intervals

$$E < \sqrt{M\delta/\lambda_1} \le 2E.$$

Note that E can be at most M since $\lambda_1 \ge \sqrt{\delta/M}$. The a's for which $E \le \sqrt{MK/q}$ contribute $N\sqrt{KM/q}$ to the error term for $\mathcal{S}(M,N,K,p,q)$. Following Lemma 4 in [5] the contribution from values of a for which $E \ge \sqrt{MK/q}$ can be estimated by

$$\sum_{\sqrt{MK/q}\le E=2^k\le M}\ \sum_{1\le F=2^h\le N} EFV(M/E,N/F,K/(qEF),p/q),$$

where

$$V(A,B,D,\beta) = \#\left\{(a,b,z)\in\mathbf{N}^2\times\mathbf{Z} \mid a\le A,\ b\le B,\ (ab,z)=1,\ |ab\beta-z|\le D\right\}.$$

Using Lemma 6 and Lemma 7 in [5] we obtain the estimate

$$\sum_{\sqrt{MK/q}\le E=2^k\le M}\ \sum_{\substack{1\le F=2^h\le N\\ EF\le(\log N)^{\frac{5}{4}}}} EFV(M/E,N/F,K/(qEF),p/q)\ll N(KN/q)^{\frac{7}{8}}.$$

Clearly we have

$$V(A,B,D,p/q)\le \sum_{|r|\le qD}\ \sum_{\substack{m\le AB\\ pm\equiv r(q)}}\tau_{A,B}(m)\le \sum_{|r|\le qD}\ \sum_{\substack{m\le AB\\ pm\equiv r(q)}}\tau(m).$$

The previous results in the present paper suggest that there is a loss of roughly a factor $\log(AB)$ in the last inequality, but the last estimate will be sufficient for our purpose.

We now need the fact that p/q is of type $(2+\gamma,\mathcal{K})$. This implies

$$\left|\frac{p}{q}-\frac{z}{xy}\right|\ge\frac{1}{\mathcal{K}(xy)^{2+\gamma}}$$

unless $xy\mid q$ (remember that $(xy,z)=1$). Thus if

$$\left|\frac{xyp}{q}-z\right|\le\frac{K}{qEF}$$

we conclude that

$$(xy)^{1+\gamma}\ge\frac{qEF\mathcal{K}}{K}$$

unless $xy\mid q$. Thus for such xy we can assume that

$$EF\le\left(\frac{(MN)^{1+\gamma}K}{q\mathcal{K}}\right)^{\frac{1}{2+\gamma}}.$$

Using the assumption (1.2) we see that

$$\frac{MN}{EF}\ge\frac{MN}{\left(\frac{(MN)^{1+\gamma}K}{q\mathcal{K}}\right)^{\frac{1}{2+\gamma}}}\gg q^{1+\frac{\delta(1+\gamma)}{2+\gamma}}. \tag{6.1}$$

It follows that

$$V(M/E,N/F,K/(qEF),p/q)\le\sum_{|r|\le\frac{K}{EF}}\ \sum_{\substack{m\le\frac{MN}{EF}\\ pm\equiv r(q)}}\tau(m)+\tau(q)^2.$$

Using the Linnik-Vinogradov estimate (1.6) one easily deduces that for $q\ll x^{1-\delta}$

$$\sum_{|r|\le S}\ \sum_{\substack{m\le x\\ m\equiv s(q)}}\tau(m)\ll\left(\tau(q)^2+S(\log\log q)^2\right)\frac{x\log x}{q},$$

where the constant implied depends on δ only. Since (6.1) holds we can use this result and we obtain

$$V(M/E,N/F,K/(qEF),p/q)\ll\left(\tau(q)^2+\frac{K}{EF}(\log\log q)^2\right)\frac{N^2\log N}{EFq}.$$

Clearly

$$\sum_{\sqrt{MK/q}\le E=2^k\le M}\ \sum_{\substack{1\le F=2^h\le N\\ EF\ge(\log N)^{\frac{5}{4}}}}1\ll(\log N)^2.$$

For $t \in (0,1)$ we recall the formula

$$\sum_{j=0}^{n}(j+1)t^j = \frac{(t-1)(n+1)t^n + 1 - t^{n+1}}{(1-t)^2}.$$

Using this we see that

$$\sum_{1 \le E = 2^k \le M} \sum_{\substack{1 \le F = 2^h \le N \\ EF \ge (\log N)^{\frac{5}{4}}}} \frac{1}{EF} \ll \sum_{\frac{5}{4}\log\log N \le l \le 3\log N} \frac{l+1}{2^l} \ll \frac{\log\log N}{(\log N)^{\frac{5}{4}}}.$$

This implies

$$\sum_{\sqrt{MK/q} \le E = 2^k \le M} \sum_{\substack{1 \le F = 2^h \le N \\ EF \ge (\log N)^{\frac{5}{4}}}} EFV(M/E, N/F, K/(qEF), p/q)$$

$$\ll \frac{N^2}{q}\left(\tau(q)^2(\log N)^3 + \frac{K(\log\log N)^2}{(\log N)^{\frac{1}{4}}}\right).$$

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