Equivariant K-theory and Cohomology of the Free Loop Space of a Projective Space

Ph.d. Thesis

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Contents

Introduction

The free loop space LX of a space X is the space of continuous maps from S^1 to X. The circle group S^1 acts on LX by rotation, and we study the space of homotopy orbits, $LX_{hS^1} = ES^1 \times_{S^1} LX$, sometimes called the Borel construction. The main method for understanding this space will be Morse theory on the energy functional, which to a closed curve associates its energy. This version of Morse theory has been studied by W. Klingenberg in [Klilngenberg1]. As one would expect, the critical points of this functional are the closed geodesics of X, so knowing those will be an important ingredient in understanding LX_{hS^1} via Morse theory.

In this paper we study LX_{hS^1} for a particular space, namely the projective space $X = \mathbb{F}P^r$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$. The goal is to determine the cohomology of $L\mathbb{H}P^r{}_{hS^1}$ and the complex K-theory of $L\mathbb{C}P^r{}_{hS^1}$. This is called S^1-equivariant cohomology (or K-theory) of $L\mathbb{F}P^r$. In general, we get a map

$$ES^1 \times_{S^1} LX \longrightarrow BS^1$$

by projection on the first factor. For a cohomology theory h^*, we therefore get a map $h^*(BS^1) \longrightarrow h^*(LX_{hS^1})$, so $h^*(LX_{hS^1})$ becomes a $h^*(BS^1)$-module. The methods of Morse theory require the use of Thom isomorphism, which destroys the product structure, so we cannot hope to calculate $h^*(L\mathbb{F}P^r{}_{hS^1})$ as a ring. But the $h^*(BS^1)$-module structure is preserved by the Morse theory machinery, so the aim is to calculate $h^*(L\mathbb{F}P^r{}_{hS^1})$ as an $h^*(BS^1)$-module, where h^* is either singular cohomology H^* or complex K-theory K^*.

We will now outline our main results. For $X = \mathbb{H}P^r$, we study the cohomology with $\mathbb{F}_p$-coefficients of LX_{hS^1}, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and obtain a complete description as an $H^*(BS^1;\mathbb{F}_p) = \mathbb{F}_p[u]$-module:

Theorem 1. *As a graded $H^*(BS^1;\mathbb{F}_p) = \mathbb{F}_p[u]$-module, $H^*(L\mathbb{H}P^r{}_{hS^1};\mathbb{F}_p)$ is isomorphic to*

$$\mathbb{F}_p[u] \oplus \bigoplus_{2k\in\mathcal{IF}} \mathbb{F}_p[u]f_{2k} \oplus \bigoplus_{2k\in\mathcal{IF}} \mathbb{F}_p[u]f_{2k-1} \oplus \bigoplus_{2k\in\mathcal{IT}} \left(\mathbb{F}_p[u]/\langle u\rangle\right) t_{2k-1}.$$

Here the lower index denotes the degree of the generator, and the index sets $\mathcal{IF}$ and $\mathcal{IT}$ are known disjoint subsets of $\{(4r+2)i + 4j \mid 0 \le j \le r, i \ge 0\}$. In particular, there is at most one generator in each degree.

For $X = \mathbb{C}P^r$, we study the complex K-theory of $L\mathbb{C}P^r{}_{hS^1}$, and obtain

Theorem 2. *As a $K^*(BS^1) = \mathbb{Z}[[t]]$-module,*

$$K^0(L\mathbb{C}P^r{}_{hS^1}) = K^0(BS^1) = \mathbb{Z}[[t]]\,.$$

As an abelian group, $K^1(L\mathbb{C}P^r{}_{hS^1})$ is torsion-free.

This is one of the first calculations of $K^*(LM_{hS^1})$ for a non-trivial manifold M. The result is quite surprising when compared to $H^*(L\mathbb{C}P^r{}_{hS^1})$, which has a lot of torsion according to [Bökstedt-Ottosen].

Unfortunately, we have not been able to determine $K^1(L\mathbb{C}P^r{}_{hS^1})$ as a $K^*(BS^1)$-module. As a partial result in this direction, we have

Theorem 3. *There is a spectral sequence of $K^*(BS^1) = \mathbb{Z}[[t]]$-modules converging strongly to $K^*(L\mathbb{C}P^r{}_{hS^1})$, which has E_1 page,*

$$\begin{aligned} E_1^{0,j} &= \begin{cases} \mathbb{Z}[[t]] \otimes_{\mathbb{Z}} \mathbb{Z}[h]/\langle h^r \rangle, & j \text{ even;} \\ 0, & j \text{ odd.} \end{cases} \\ E_1^{n,j} &= \begin{cases} \mathbb{Z}[[t]]^{(n)} \otimes_{R(S^1)} \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1} \rangle, & j \text{ odd;} \\ 0, & j \text{ even.} \end{cases} \end{aligned}$$

The first differential d_1 is given by $d_1(p(t) \otimes h^j) = p(t) \otimes (x^j - y^j)$, where $p(t) \in \mathbb{Z}[[t]]$.

Theorem 2 states that $K^0(L\mathbb{C}P^r{}_{hS^1})$ is (almost) trivial, while $K^1(L\mathbb{C}P^r{}_{hS^1})$ is free abelian. This is rather similar to the well-known case of $K^0(BG)$ as the completion of the representation ring $R(G)$ for a compact Lie group G, while $K^1(BG) = 0$. This is a classical result of M. Atiyah. One can also compare to e.g. [Freed-Hopkins-Teleman], who find $K^*_\tau(LBG)$ as the completion of certain representations of the loop group LG, although it should be remarked that they consider K-theory twisted by a cohomology class τ, and not S^1-equivariant K-theory as we do. Still, this prompts the following

Conjecture. The exists a "representation theory" type group, such that $K^1(L\mathbb{C}P^r{}_{hS^1})$ is a completion of this group.

The outline of this paper is as follows: The paper consists of two main parts, each divided in three sections. The first section of each part treats the general theory needed and investigates the relevant spaces and structures, while the next two sections are more computational and deal, respectively, with the cohomology for $\mathbb{F} = \mathbb{H}$, and the K-theory for $\mathbb{F} = \mathbb{C}$.

Section 1 investigates $\mathbb{F}P^r$ and its geodesics, obtaining some useful fibrations. We consider both the space of parametrized and unparametrized geodesics; the latter being the quotient of the former under the action of S^1 by rotation.

Section 2 calculates the cohomology of the above spaces using Serre's spectral sequence for the fibrations found in section 1. We then turn to S^1-equivariant cohomology of the space of parametrized geodesics, via two fibrations and the non-equivariant cohomology results from the previous section.

Section 3 obtains similar results for K-theory. We use the Atiyah-Hirzebruch spectral sequence along with the known cohomology results for $\mathbb{C}P^r$ to determine the K-theory of the space of unparametrized geodesics. The S^1-equivariant K-theory is determined using the same fibrations as for cohomology, but the method is different, employing the result of Atiyah about K-theory of classifying spaces.

Section 4 studies of the free loop space, $L\mathbb{F}P^r{}_{hS^1}$. First we explain the workings of Morse theory in this setting, then we apply this to $L\mathbb{F}P^r$ and $L\mathbb{F}P^r{}_{hS^1}$ to get the so-called Morse spectral sequence.

Section 5 is dedicated to proving Theorem 1. The method is closely based upon a similar calculation by M. Bökstedt and I. Ottosen in their paper *String Cohomology Groups of Complex Projective Spaces*, [Bökstedt-Ottosen]. We extract a lot of information about the Morse spectral sequence, its size, its differentials, and the relation between the equivariant and non-equivariant case. All this information is brought together to prove the Main Theorem for cohomology, Theorem 1 above. But even then, it is necessary to turn to other sources of information to complete the proof. One is localization, the other is comparison with the Serre spectral sequence also converging to $H^*(L\mathbb{H}P^r{}_{hS^1})$.

Section 6 is dedicated to proving Theorem 2. The methods here are quite different, relying on the fact that the Morse spectral sequence in Theorem 3 has a rather special configuration, which implies that all its non-trivial differentials start from the zeroth column. A very important point is the calculation of the first differential d_1. The central idea is then to twist the rotation action of S^1 with a positive integer, which gives new Morse spectral sequences related to the standard one. This gives enough information to prove Theorem 2.

For the reader's convenience, we have assembled a table of notation at the end of this document.

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1 Projective space and geodesics

1.1 The quaternions

I start by introducing the quaternions, $\mathbb{H}$, as an associative algebra of real dimension 4, generated by $1, i, j, k$ with the following multiplication rules:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

It should be stressed, even though it is obvious from the above relations, that $\mathbb{H}$ is *not* commutative. If one wants to be concrete, one can realize $\mathbb{H}$ as a subalgebra of $M_2(\mathbb{C})$ generated over $\mathbb{R}$ by (in the matrix entries, $i = \sqrt{-1}$):

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

It is straightforward to check the above multiplication rules. Similar to complex conjugation, there is an $\mathbb{R}$-linear map, also called conjugation,

$$\begin{array}{rcl} \mathbb{H} & \xrightarrow{*} & \mathbb{H} \\ z = x_0 + x_1 i + x_2 j + x_3 k & \mapsto & z^* = x_0 - x_1 i - x_2 j - x_3 k, \end{array}$$

satisfying the usual rule $(zw)^* = w^* z^*$. In the matrix description, this is precisely the usual $*$-operation of taking the conjugate transpose. This can be used to define an inner product $\langle z, w \rangle_{\mathbb{H}} = w^* z$, whose real part is the usual inner product on $\mathbb{R}^4$. Noting that $\langle z, z \rangle_{\mathbb{H}} \in \mathbb{R}$ we can then define a norm $|z| = \sqrt{\langle z, z \rangle_{\mathbb{H}}}$. This satisfies $|zw| = |z|\,|w|$ and $|z^*| = |z|$. The unit sphere in $\mathbb{H}$ is usually denoted $Sp(1) = \{z \in \mathbb{H} \mid |z| = 1\}$, and this is canonically identified with S^3. Finally we note that if $z \neq 0$ then z is invertible – this is most easily seen by using the matrix description, which gives an explicit inverse, and checking that this belongs to $\mathbb{H}$.

We can take the direct product of $\mathbb{H}$ with itself to form $\mathbb{H}^r$. The operations $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and $|\cdot|$ from $\mathbb{H}$ are extended to $\mathbb{H}^r$ in the usual way: For $z = (z_1, \ldots, z_r)$ and $w = (w_1, \ldots, w_r)$, we set

$$\langle z, w \rangle_{\mathbb{H}} = \sum_{j=1}^{r} \langle z_j, w_j \rangle_{\mathbb{H}}, \quad |z| = \sqrt{|z_1|^2 + \ldots + |z_r|^2}.$$

1.2 Spaces of geodesics

Let $\mathbb{F}$ denote either $\mathbb{C}$ or $\mathbb{H}$. To ease the notation we denote the unit sphere in $\mathbb{F}$ by $S(\mathbb{F})$. We define the projective space $\mathbb{F}P^r$ as the set of all 1-dimensional

$\mathbb{F}$-subspaces $z\mathbb{F}$ of $\mathbb{F}^{r+1}$, for $z \in \mathbb{F}^{r+1}$. We define the projection map

$$\begin{aligned} \pi : \mathbb{F}^{r+1} \setminus \{0\} &\longrightarrow \mathbb{F}P^r \\ z = (z_0, \ldots, z_r) &\mapsto [z_0, \ldots, z_r] = z\mathbb{F}, \end{aligned} \tag{1}$$

so $\pi(z) = z\mathbb{F}$ is the subspace spanned by z. Note that for $\mathbb{F} = \mathbb{H}$ it is important that we specify which side we multiply on; I have chosen to multiply from the right. We give $\mathbb{F}P^r$ the quotient topology from π. To show that $\mathbb{F}P^r$ is a smooth manifold of real dimension $2r$ (resp. $4r$) for $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{F} = \mathbb{H}$), we display the explicit charts

$$h_j : U_j = \{[z_0, \ldots, z_r] \in \mathbb{F}P^r \mid z_j \neq 0\} \longrightarrow \mathbb{F}^r,$$
$$h_j([z_0, \ldots, z_r]) = (z_0 {z_j}^{-1}, \ldots, \widehat{z_j {z_j}^{-1}}, \ldots, z_r {z_j}^{-1}),$$

where the hat denotes omission; the charts have inverses

$$h_j^{-1}(w_1, \ldots, w_r) = [w_1, \ldots, 1, \ldots, w_r],$$

with the 1 at the jth place.

Example 1.1. We will show $\mathbb{H}P^1$ is diffeomorphic to S^4. This can be seen by stereographic projection. Think of $S^4 \subseteq R^5 = \mathbb{R} \times \mathbb{H}$ with north pole $p_+ = (1, 0)$ and south pole $p_- = (-1, 0)$. Stereographic projection are the maps

$$\psi_\pm : S^4 \setminus \{p_\pm\} \longrightarrow \mathbb{H},$$

which takes a point (t, z) in S^4 to the intersection of the line through (t, z) and $p_\pm$ with $0 \times \mathbb{H}$. This is easily computed:

$$\psi_+(t, z) = \frac{z}{1-t}, \quad \psi_-(t, z) = \frac{z}{1+t},$$

and are clearly smooth maps. Now we want to compose ψ_+ and ψ_- with the h_j^{-1} to get two maps to $\mathbb{H}P^1$. When we do this, we would like the two maps to agree when $t \in]-1, 1[$. To achieve this, we replace ψ_+ with its conjugate $\psi_+^*(t, z) = \frac{z^*}{1-t}$. Doing this, we get maps,

$$S^4 \setminus \{p_+\} \xrightarrow{\psi_+^*} \mathbb{H} \xrightarrow{h_0^{-1}} \mathbb{H}P^1, \quad S^4 \setminus \{p_-\} \xrightarrow{\psi_-} \mathbb{H} \xrightarrow{h_1^{-1}} \mathbb{H}P^1,$$

given by

$$(t, z) \mapsto \left[1, \frac{z^*}{1+t}\right], \quad (t, z) \mapsto \left[\frac{z}{1-t}, 1\right].$$

By multiplying the first expression from the right by $\frac{z}{1-t}$ and using that $1 = |(t, z)| = t^2 + |z|^2 = t^2 + z^* z$, we see that these two maps agree when $t \in]-1, 1[$, so they combine to a diffeomorphism $S^4 \longrightarrow \mathbb{H}P^1$. $\square$

We can modify the projection map π in (1) to a map

$$\pi : S(\mathbb{F}^{r+1}) \longrightarrow \mathbb{F}P^r$$

where $S(\mathbb{F}^{r+1}) \subseteq \mathbb{F}^{r+1}$ is the unit sphere. This can be used to describe the tangent bundle of $\mathbb{F}P^r$. Specifically for $z \in S(\mathbb{F}^{r+1})$ there is an $\mathbb{F}$-linear isometry,

$$t_z : (z\mathbb{F})^\perp \subseteq T_z S(\mathbb{F}^{r+1}) \xrightarrow{\pi_*} T_{\pi(z)}\mathbb{F}P^r,$$

where $(z\mathbb{F})^\perp = \{w \in \mathbb{F}^{r+1} \mid \langle w, z\rangle_{\mathbb{F}} = 0\}$. This map satisfies

$$t_{z\lambda}(w\lambda) = t_z(w) \quad \text{for } \lambda \in S(\mathbb{F}). \tag{2}$$

The above properties of $\mathbb{F}P^r$ are rather elementary, and the reader can see e.g. [Madsen-Tornehave] Chapter 14 for proofs of the results in the case of $\mathbb{C}P^r$.

Consider the Riemannian metric on $\mathbb{F}P^r$ given by the real part of the inner product on $\mathbb{F}^{r+1}$. This is the standard metric on $\mathbb{F}P^r$, and we will use a metric g which is a scalar multiple of this metric. Take the unique connection on $T(\mathbb{F}P^r)$ compatible with this metric, called the Levi-Civita connection. We now define $G(r) = G(\mathbb{F}P^r)$ as the space of parametrized, simple, closed geodesics $f : [0,1] \longrightarrow \mathbb{F}P^r$ with respect to this connection. The scalar determining g is specified by requiring that such a geodesic has length 1 with respect to g. Note that every geodesic in $\mathbb{F}P^r$ is closed: The group of $\mathbb{F}$-orthogonal matrices ($U(r+1)$ or $Sp(r+1)$, respectively) acts transitively on $\mathbb{H}P^r$, so it is only necessary to check it for one geodesic, e.g. on $\mathbb{F}P^1 \subseteq \mathbb{F}P^r$, and since $\mathbb{C}P^1 \cong S^2$ and $\mathbb{H}P^1 \cong S^4$, all geodesics on $\mathbb{F}P^1$ are known to be closed.

We also consider the set of n times iterated geodesics $G_n(r)$ for every integer $n \geq 1$, whose elements $\gamma : [0,1] \longrightarrow \mathbb{F}P^r$ are given by $\gamma(t) = f(nt)$ for some $f \in G(r)$, where we make the obvious identification of the intervals $[j-1, j]$ with $[0,1]$ for $j = 2, \ldots, n$. There is an action on $G_n(r)$ by S^1 given by rotation; explicitly,

$$\begin{array}{ccc} S^1 \times G_n(r) & \longrightarrow & G_n(r) \\ (e^{2\pi i\theta}, f(t)) & \mapsto & f(t-\theta). \end{array}$$

We can twist the rotation action on $G(r)$ by an integer $n \geq 1$, and we denote the resulting S^1-space $G(r)^{(n)}$:

$$\begin{array}{ccc} S^1 \times G(r)^{(n)} & \longrightarrow & G(r)^{(n)} \\ (e^{2\pi i\theta}, f(t)) & \mapsto & f(t-n\theta). \end{array} \tag{3}$$

This action is the rotation action precomposed with the nth power map $\mathcal{P}_n : S^1 \longrightarrow S^1$, $\mathcal{P}_n(z) = z^n$ in complex notation. Then $G_n(r)$ and $G(r)^{(n)}$ are isomorphic as S^1-spaces via the obvious map $G(r)^{(n)} \longrightarrow G_n(r)$ given by $f(t) \mapsto f(nt)$, so from now on, we will chiefly use $G(r)^{(n)}$ instead of $G_n(r)$. We also consider the quotient $\Delta(r) = S^1 \setminus G(r)$ under the rotation action, which is the space of oriented, unparametrized, simple, closed geodesics on $\mathbb{F}P^r$.

We now want to get a more concrete description of $G(r)$ and $\Delta(r)$, following [Bökstedt-Ottosen], §2. Let $V_2 = V_2(\mathbb{F}^{r+1})$ be the Stiefel manifold of $\mathbb{F}$-orthonormal 2-frames in $\mathbb{F}^{r+1}$, so

$$V_2 = \left\{(v, w) \in \mathbb{F}^{r+1} \times \mathbb{F}^{r+1} \mid \|v\| = \|w\| = 1, \langle v, w\rangle_{\mathbb{F}} = 0\right\},$$

and let PV_2 be the quotient manifold by the right diagonal $S(\mathbb{F})$ action, $(v, w) * z = (vz, wz)$. On V_2 we have a left action of S^1 by rotation by an angle θ: For $\theta \in \mathbb{R}$, the action is $\binom{v}{w} \mapsto R(\theta)\binom{v}{w}$, where

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

For each $n \in \mathbb{N}$, we can define an action of S^1 on PV_2, and we denote the resulting S^1-space by $PV_2^{(n)}$:

$$S^1 \times PV_2^{(n)} \longrightarrow PV_2^{(n)}; \quad e^{2\pi i\theta} * [x, y] = [R(n\pi\theta)(x, y)].$$

This gives a well-defined S^1-action on PV_2, because we multiply the matrix R on the left, while $PV_2 = V_2/\mathrm{diag}S(\mathbb{F})$, where we multiply on the right. We can now make an S^1-equivariant diffeomorphism

$$\begin{array}{rcl} \varphi_1 : PV_2^{(n)} & \longrightarrow & G(r)^{(n)} \\ {[x, y]} & \mapsto & \pi \circ c(x, y) \end{array} \tag{4}$$

where $\pi : S(\mathbb{F}^{r+1}) \longrightarrow \mathbb{F}P^r$ is the projection, and $c(x, y)$ is the simple closed geodesic starting at x in direction y; explicitly,

$$c(x, y)(t) = \cos(\pi t)x + \sin(\pi t)y, \quad \text{for } t \in [0, 1].$$

This is well-defined, and a bijection because every geodesic on $\mathbb{F}P^r$ is closed. Clearly, φ_1 is a diffeomorphism, and it is straightforward to check that it is S^1-equivariant, using the trigonometric formulas.

Another very useful model for $G(r)$ is $S(\tau) = S(T(\mathbb{H}P^r))$, the sphere bundle of the tangent bundle τ of $\mathbb{F}P^r$. There is a diffeomorphism

$$\begin{array}{rcl} \psi : PV_2 & \longrightarrow & S(\tau) \\ {[x, y]} & \mapsto & t_x(y) \in T_{\pi(x)}\mathbb{F}P^r \end{array}$$

This is well-defined because of (2), and we can give an explicit inverse: Given $y \in T_{\pi(x)}\mathbb{F}P^r$, $\psi^{-1}(y) = [x, t_x^{-1}(y)]$. Thus we can give $S(T(\mathbb{F}P^r))$ a rotation action of S^1, namely the action that makes this diffeomorphism S^1-equivariant. Combining this with (4), we have an S^1-equivariant diffeomorphism

$$\psi^{-1} \circ \varphi_1 : S(\tau) \longrightarrow G(r). \tag{5}$$

The last description only works for $\mathbb{C}P^r$. Going back to $PV_2(\mathbb{C}^{r+1})$, we first change coordinates as follows

$$\varphi_2 : PV_2(\mathbb{C}^{r+1}) \longrightarrow \widetilde{PV}_2(\mathbb{C}^{r+1}), \quad [x, v] \mapsto \left[\frac{x+iv}{\sqrt{2}}, \frac{x-iv}{\sqrt{2}}\right].$$

Here $\widetilde{PV}_2$ is PV_2 equipped the S^1-action induced from this change of coordinates. It is easily computed that the action of $\theta \in [0,1]$ is $\theta * [a,b] = [za, zb]$ where $z = e^{\pi i \theta} \in S^1$.

We are interested in $\Delta(\mathbb{C}P^r)$, i.e. we divide out the rotation action. Therefore we now consider the following space: Let γ_2 be the standard 2-dimensional bundle over the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{r+1})$ of 2-planes in $\mathbb{C}^{r+1}$, and let $p : \mathbb{P}(\gamma_2) \longrightarrow \mathrm{Gr}_2(\mathbb{C}^{r+1})$ be the associated projective bundle. Then $\mathbb{P}(\gamma_2) = \{V_1 \subseteq V_2 \subseteq \mathbb{C}^{r+1} \mid \dim_{\mathbb{C}}(V_j) = j\}$. We can make a diffeomorphism,

$$\varphi_3 : S^1 \setminus \widetilde{PV}_2(\mathbb{C}^{r+1}) \longrightarrow \mathbb{P}(\gamma_2), \quad [a,b] \mapsto \mathrm{span}_{\mathbb{C}}\{a\} \subseteq \mathrm{span}_{\mathbb{C}}\{a,b\}.$$

This is well-defined, but only for $\mathbb{F} = \mathbb{C}$. In conclusion we get a composite S^1-equivariant diffeomorphism

$$\varphi : \Delta(\mathbb{C}P^r) \xrightarrow{\varphi_1^{-1}} S^1 \setminus PV_2(\mathbb{C}^{r+1}) \xrightarrow{\varphi_2} S^1 \setminus \widetilde{PV}_2(\mathbb{C}^{r+1}) \xrightarrow{\varphi_3} \mathbb{P}(\gamma_2). \tag{6}$$

1.3 Fibrations involving spaces of geodesics

We are going to compute the cohomology and K-theory of the spaces $G(r)$ and $\Delta(r)$. In cohomology, our most important tool will be Serre's spectral sequence. I will write down the most important part; for the complete formulation and proof, see e.g [Hatcher2] Thm 1.14 pp.

Theorem 1.2 (Serre's Spectral Sequence)**.** *Let $F \longrightarrow X \longrightarrow B$ be a fibration, with B a path-connected CW complex, and $\pi_1(B)$ acting trivially on $H^*(F;G)$. Then there is a spectral sequence $\{E_r^{p,q}, d_r\}$ converging to $H^*(X;G)$ with*

$$E_2^{p,q} \cong H^p(B; H^q(F;G)).$$

If $G = R$ is a ring, then there is a product $E_r^{p,q} \times E_r^{s,t} \longrightarrow E_r^{p+s,r+t}$, and the differentials are derivations, i.e. $d(xy) = (dx)y + (-1)^{p+q}x(dy)$. For $r = 2$ the product is $(-1)^{qs}$ times the standard cup product. The product structure on E_∞ coincide with that induced by the cup product on $H^*(X; R)$.

For the definition of a fibration, and the useful fact that fiber bundles are fibrations, see [Hatcher1], p. 375 and Prop. 4.48.

There is a similar result for a fibration in K-theory, but I am chiefly going to use the important special case where the fibration is $* \longrightarrow X \longrightarrow X$, called the Atiyah-Hirzebruch spectral sequence, see [Atiyah-Hirzebruch]:

Theorem 1.3 (Atiyah-Hirzebruch Spectral Sequence). *Let X be a finite CW complex. Then there is a spectral sequence $\{E_r^{p,q}, d_r\}$ converging to $K^*(X)$ with*

$$E_2^{p,q} \cong H^p(X; K^q(*)).$$

We will need a way to build fibrations from other fibrations, and this is provided by the following theorem.

Theorem 1.4. *Let $F \longrightarrow X \longrightarrow B$ be a fibration, and assume that the group G acts freely on X. Then,*

(i) If the G-action preserves the fibres, $F/G \longrightarrow X/G \longrightarrow B$ is a fibration.

(ii) If G acts freely on B, then $F \longrightarrow X/G \longrightarrow B/G$ is a fibration.

Proof. This follows from the fact that $G \longrightarrow X \longrightarrow X/G$ is a fibration, which is a consequence of the "slice theorem", [Bredon] Thm. 5.4. □

To apply the spectral sequences, we must know some fibrations involving the spaces of geodesics. First by definition we have the fibration

$$S^1 \longrightarrow G(r) \longrightarrow \Delta(r). \tag{7}$$

For the application of Serre's spectral sequence, note that the base is 1-connected. This can be seen from the long exact sequence of homotopy groups, using that $G(r) \cong S(\tau)$ is 1-connected.

Then there is the map

$$PV_2(\mathbb{F}^{r+1}) \longrightarrow \mathrm{Gr}_2(\mathbb{F}^{r+1})$$

induced by the map $V_2(\mathbb{F}^{r+1}) \longrightarrow \mathrm{Gr}_2(\mathbb{F}^{r+1})$, $(x, y) \mapsto \{x\lambda + y\mu \mid \lambda, \mu \in \mathbb{F}\}$, which is well-defined on PV_2. The fibre is $PV_2(\mathbb{F}^2)$. By the diffeomorphism (4), this means we have the fibration

$$G(1) \longrightarrow G(r) \longrightarrow \mathrm{Gr}_2(\mathbb{F}^{r+1}).$$

Since the left S^1 action on the total space is free and preserves the fibres, we can divide by it in the total space and fibre, by Theorem 1.4 (i) obtaining the fibration

$$\Delta(1) \longrightarrow \Delta(r) \longrightarrow \mathrm{Gr}_2(\mathbb{F}^{r+1}). \tag{8}$$

Again we note that the base is 1-connected.

1.4 Homotopy orbits of spaces of geodesics

In this section we are going to study the so-called homotopy orbits of the spaces of geodesics we have studied so far. For this definition we need the following concepts: Let G be a group, and suppose we have a contractible space with a free G action. It turns out that all such spaces are homotopy equivalent, so we can define EG to be any such space. We can then define $BG = EG/G$ to be the classifying space of G. Note that this is a "working" definition; actually BG is defined for a category, but this is all I will need. For $G = S^1$ we find $ES^1 \simeq S^\infty$, since this is contractible. Thus we get $BS^1 \simeq S^\infty/S^1 = \mathbb{C}P^\infty$.

Definition 1.5. Let X be a topological space with a (left) action of S^1. We define the space of homotopy orbits of X by

$$X_{hS^1} = ES^1 \times_{S^1} X = ES^1 \times X / \left\{(e, tx) \sim (et, x), t \in S^1\right\}.$$

Projection on the first factor gives a map $X_{hS^1} \longrightarrow BS^1$, and for a cohomology theory h^* (we consider cohomology and K-theory), we get an induced map

$$h^*(BS^1) \longrightarrow h^*(X_{hS^1}).$$

As explained in the introduction, this gives $h^*(X_{hS^1})$ the structure of an $h^*(BS^1)$-module.

Recall that $G(r)$ is the space of simple parametrized geodesics with the free left action of S^1 given by rotation. The space of n-times iterated geodesics, $G_n(r)$, we have identified as an S^1-space with $G(r)^{(n)}$, which is $G(r)$ with the rotation action twisted by the nth power map $\mathcal{P}_n : S^1 \longrightarrow S^1$, see (3).

Proposition 1.6. *In the following commutative diagram, the vertical and* *horizontal maps are fibrations with 1-connected base spaces:*

$$\begin{tikzcd}
 & G(r) \arrow[d] & \\
BC_n \arrow[r] & ES^1 \times_{S^1} G(r)^{(n)} \arrow[r] \arrow[d] & \Delta(r) \arrow[d] \\
 & BS^1 \arrow[r, "B\mathcal{P}_n"] & BS^1
\end{tikzcd}$$

Here $C_n \subseteq S^1$ denotes the group of nth roots of unity.

Proof. To see that the vertical map is a fibration, use the product bundle $G(r)^{(n)} \longrightarrow ES^1 \times G(r)^{(n)} \xrightarrow{\mathrm{pr}_1} ES^1$, and divide out by the free action of S^1 on both total space and base, according to Theorem 1.4 (ii). Using the long exact homotopy sequence for the fibration $S^1 \longrightarrow ES^1 \longrightarrow BS^1$ shows that the base BS^1 is 1-connected.

The horizontal fibration is built up in steps: We start with the product fibre bundle,

$$ES^1 \longrightarrow ES^1 \times G(r)^{(n)} \xrightarrow{\mathrm{pr}_2} G(r)^{(n)}.$$

Clearly, $C_n \subseteq S^1$ acts freely on $ES^1 \times G(r)^{(n)}$, preserving the fibres. So by Theorem 1.4 (i), dividing out by C_n in the total space and fibre yields the fibration:

$$BC_n \longrightarrow ES^1 \times_{C_n} G(r)^{(n)} \longrightarrow G(r)^{(n)}.$$

We get $ES^1/C_n = BC_n$ because ES^1 is a contractible space upon which C_n acts freely, and so $ES^1 \simeq EC_n$. Now consider the quotient group S^1/C_n, which is isomorphic to S^1 by the n'th power map. Since C_n acts trivially on $G(r)^{(n)}$, we have an action of S^1/C_n on $G(r)^{(n)}$. By definition, this acts on $G(r)^{(n)}$ exactly as S^1 acts on $G(r)$, so $(S^1/C_n) \setminus G(r)^{(n)} \cong S^1 \setminus G(r)$. By Theorem 1.4 (ii), dividing out by this free action in the total and base spaces gives us the fibration

$$BC_n \longrightarrow \left(ES^1 \times_{C_n} G(r)^{(n)}\right)/(S^1/C_n) \longrightarrow S^1 \setminus G(r).$$

Now $\left(ES^1 \times_{C_n} G(r)^{(n)}\right)/(S^1/C_n) \cong ES^1 \times_{S^1} G(r)^{(n)}$, by the definition of the actions, so we get the desired fibration. As noted in Section 1.3, the base is 1-connected.

To get the commutative square, note that we have the homotopy equivalence $\mathrm{pr}_2 : ES^1 \times G(r) \longrightarrow G(r)$, since ES^1 is contractible. Since this is an S^1 map and S^1 acts freely on both spaces, we can use [tomDieck] Prop. 2.7 to conclude that $ES^1 \times_{S^1} G(r) \longrightarrow S^1 \setminus G(r) = \Delta(r)$ is also a homotopy equivalence. The upper vertical map in the square is defined as $\mathrm{pr}_2 : ES^1 \times_{S^1} G(r) \longrightarrow \Delta(r)$ using this homotopy equivalence. For the identification S^1/C_n with S^1 above, we used the nth power map $\mathcal{P}_n : S^1 \longrightarrow S^1$, so for the diagram to commutate, the lower horizontal map $BS^1 \longrightarrow BS^1$ must also be the one induced by $\mathcal{P}_n$. Note: This is well-defined on BS^1 because S^1 is commutative.

□

Remark 1.7. If we let $n = 1$, the vertical fibration becomes $G(r) \longrightarrow ES^1 \times_{S^1} G(r) \longrightarrow BS^1$. As noted in the proof, $ES^1 \times_{S^1} G(r) \longrightarrow S^1 \setminus G(r)$ is a homotopy equivalence. So, up to homotopy, we have in practice a fibration

$$G(r) \longrightarrow \Delta(r) \longrightarrow BS^1. \tag{9}$$

□

2 Cohomology of spaces of geodesics in $\mathbb{H}P^r$

2.1 The parametrized geodesics

In this section we find the cohomology of the space of parametrized geodesics on $\mathbb{H}P^r$, $G(r) = G(\mathbb{H}P^r)$, followed by some Lemmas necessary to determine the space of oriented, unparametrized geodesics, $\Delta(r) = \Delta(\mathbb{H}P^r) = S^1\backslash G(r)$.

Theorem 2.1. *As a graded ring,*

$$H^*(G(\mathbb{H}P^r);\mathbb{Z}) \cong \mathbb{Z}[y,\tau]/\left\langle (r+1)y^r, y^{r+1}, \tau^2 \right\rangle,$$

where $y \in H^4(G(\mathbb{H}P^r);\mathbb{Z})$ and $\tau \in H^{4r+3}(G(\mathbb{H}P^r);\mathbb{Z})$.

Let p be a prime number. Then

$$H^*(G(\mathbb{H}P^r);\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p[y,\sigma]/\left\langle y^{r+1}=0, \sigma^2=0\right\rangle, & p \mid r+1; \\ \mathbb{F}_p[y,\tau]/\left\langle y^{r}=0, \tau^2=0\right\rangle, & p \nmid r+1. \end{cases}$$

where $y \in H^4(G(\mathbb{H}P^r);\mathbb{F}_p)$, $\sigma \in H^{4r-1}(G(\mathbb{H}P^r);\mathbb{F}_p)$, $\tau \in H^{4r+3}(G(\mathbb{H}P^r);\mathbb{F}_p)$.

Proof. We use the diffeomorphism from (5), $G(r) \cong S(\tau)$, where $S(\tau)$ is the sphere bundle of the tangent bundle,

$$S^{4r-1} \longrightarrow S(\tau) \longrightarrow \mathbb{H}P^r.$$

Since $\mathbb{H}P^r$ is 1-connected, we can use Serre's spectral sequence,

$$H^p(\mathbb{H}P^r; H^q(S^{4r-1})) \Rightarrow H^{p+q}(S(\tau)) \tag{10}$$

(here the coefficients will be $\mathbb{Z}$ at first, and $\mathbb{F}_p$ to prove the last part) which has the following E_2 page:

$$\begin{array}{r|ccccc}
4r-1 & \sigma & y\sigma & y^2\sigma & & y^r\sigma \\
 & & & & & \\
0 & 1 & y & y^2 & & y^r \\
\hline
 & 0 & 4 & 8 & \dots & 4r
\end{array}$$

We can see for dimensional reasons that there can only be one non-trivial differential, namely $d_{4r}(\sigma)$. For the sphere bundle, it is a general theorem that this differential is multiplication by the Euler characteristic of the manifold, here $\mathbb{H}P^r$, so $d_{4r}(\sigma) = (r+1)y^r$. This is proved in [Milnor-Stasheff], Cor.

11.12 and Thm. 12.2. This is an injective map $\mathbb{Z} \longrightarrow \mathbb{Z}$, so when passing to the E_{4r+1} page, the result is

$$\begin{array}{c|cccccc}
4r-1 & 0 & y\sigma & y\cdot y\sigma & & y^{r-2}\cdot y\sigma & y^{r-1}\cdot y\sigma \\
 & & & & & & \\
0 & 1 & y & y^2 & & y^{r-1} & y^r \\
\hline
 & 0 & 4 & 8 & \dots & 4r-4 & 4r
\end{array}$$

As mentioned, there are no other non-trivial differentials, so this is E_∞. Also, there are no extension problems since there is at most one non-trivial group on each diagonal $p+q=n$, so $y\sigma$ defines a class in $H^{4r+3}(S(\tau);\mathbb{Z})$ which we call τ. We can then read off the classes $y \in \mathbb{H}^4(S(\tau);\mathbb{Z})$ and $\tau \in H^{4r+3}(S(\tau);\mathbb{Z})$ with the relations $y^{r+1}=0$, $(r+1)y^r=0$, and $\tau^2=0$.

To prove the result with $\mathbb{F}_p$ coefficients, we use the same spectral sequence (10), now with $\mathbb{F}_p$-coefficients. In case $p \mid r+1$, $d_2(\sigma)=0$, so there are no non-trivial differentials, and $E_\infty = E_2$. As above, there are no extension problems, and σ defines an element in $H^{4r-1}(S(\tau);\mathbb{F}_p)$. So we can read off the desired result. In case $p \nmid r+1$, $r+1$ is a unit in $\mathbb{F}_p$, so $d_2 : \mathbb{F}_p\sigma \longrightarrow \mathbb{F}_p y^r$ is an isomorphism. So when passing to the E_{4r+1} page, these two groups disappear. The result follows. □

Now we can deal with the smallest case, $\mathbb{H}P^1$, which we have shown in Example 1.1 is diffeomorphic to S^4. This is going to be useful, since we have the fibration $\Delta(\mathbb{H}P^1) \longrightarrow \Delta(\mathbb{H}P^r) \longrightarrow \mathrm{Gr}_2(\mathbb{H}^{r+1})$ from (8).

Lemma 2.2.
$$H^*(\Delta(\mathbb{H}P^1);\mathbb{Z}) \cong \mathbb{Z}[x,t]/\left\langle 2t - x^2, t^2 \right\rangle,$$
where $x \in H^2(\Delta(\mathbb{H}P^1);\mathbb{Z})$ *and* $t \in H^4(\Delta(\mathbb{H}P^1);\mathbb{Z})$.

Proof. We use the fibration $S^1 \longrightarrow G(\mathbb{H}P^1) \longrightarrow \Delta(\mathbb{H}P^1)$ from the S^1 action. Here we know the cohomology of the fibre and the total space, the latter from Theorem 2.1,

$$H^n(G(\mathbb{H}P^1)) = \begin{cases} \mathbb{Z}, & n = 0,7; \\ \mathbb{Z}/2\mathbb{Z}, & n=4; \\ 0, & \text{else}. \end{cases}$$

We can use the Serre's spectral sequence,

$$H^p(\Delta(\mathbb{H}P^1); H^q(S^1;\mathbb{Z})) \Rightarrow H^{p+q}(G(\mathbb{H}P^1);\mathbb{Z}),$$

to find the cohomology of the base. Let $\sigma \in H^1(S^1)$ denote a generator. The E_2 page has only two non-zero rows. We see that the only possible non-trivial differentials are d_2, so $E_3 = E_\infty$. We know the total space has nothing in degree 1, so there must be zero at $(1,0)$ since this cannot be killed by anything. So $H^1(\Delta(\mathbb{H}P^1)) = 0$, which means there is zero at $(1,1)$, too. Also, σ must be killed by an outgoing differential, so $d_2^{0,1}$ is injective. Actually it must be an isomorphism, otherwise something would survive in degree 2, and there is nothing. So we have a $H^2(\Delta(\mathbb{H}P^1);\mathbb{Z}) \cong \mathbb{Z}$ generated, say, by $x = d_2(\sigma)$. Let us take a look at the E_2 page as we know it now:

$$\begin{array}{c|cccccccccc}
1 & \sigma & 0 & \sigma x & ? & ? & ? & ? & ? & ? & \cdots \\
0 & 1 & 0 & x & ? & ? & ? & ? & ? & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots
\end{array}$$

Continuing in this fashion we see there is zero at $(3,0)$ since $H^3(G(\mathbb{H}P^1);\mathbb{Z}) = 0$, and so also at $(3,1)$. Likewise, there are zeroes at $(5,0)$ and $(5,1)$. Now consider $d_2^{2,1}$. This must be injective, since it starts in degree 3, where the total space has nothing. Also, $d_2^{2,1}$ ends at $(4,0)$, and must be such that we get $H^4(G(\mathbb{H}P^1);\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ when taking the cokernel of it. This means it must be multiplication by ± 2; we might as well say 2 for concreteness. So $H^4(\Delta(\mathbb{H}P^1);\mathbb{Z}) \cong \mathbb{Z}$ generated by some t, which we can choose such that $d_2(\sigma x) = 2t$. A quick summary:

$$\begin{array}{c|cccccccccc}
1 & \sigma & 0 & \sigma x & 0 & \sigma t & 0 & ? & ? & ? & \cdots \\
0 & 1 & 0 & x & 0 & t & 0 & ? & ? & ? & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots
\end{array}$$

Now we have gotten something at $(4,1)$, but the total space has zero in degree 5, so σt must be killed by the outgoing differential $d_2^{4,1}$. Again it must be an isomorphism. Note that by the derivation property of d_2,

$$d(\sigma t) = d(\sigma)t - \sigma d(t) = d(\sigma)t = xt$$

so xt is a generator of $H^6(\Delta(\mathbb{H}P^1);\mathbb{Z})$. This gives us a $\mathbb{Z}$ at $(6,1)$ generated by σxt. Now to see what further happens, we note that $\Delta(\mathbb{H}P^1)$ is at most 7-dimensional, since $G(\mathbb{H}P^1) = S(T(\mathbb{H}P^1))$ is a 7-manifold. So we know that $H^*(\Delta(\mathbb{H}P^1);\mathbb{Z})$ is zero above degree 7. This means that σxt cannot be killed, so it survives to E_∞, meaning there can be nothing else in degree 7. So from column 7 and onwards there are zeroes in the E_2 page. Now we know the full story:

$$\begin{array}{c|cccccccccc}
1 & \sigma & 0 & \sigma x & 0 & \sigma t & 0 & \sigma xt & 0 & 0 & \cdots \\
0 & 1 & 0 & x & 0 & t & 0 & xt & 0 & 0 & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots
\end{array}$$

To get to the bottom of the multiplicative structure we calculate:

$$2t = d(\sigma x) = d(\sigma)x - \sigma d(x) = d(\sigma)x = x^2.$$

For dimensional reasons $t^2 = 0$, and all other relations come from these two (e.g. $x^3 = x^2 \cdot x = 2xt$). This proves the result. □

We now turn to the general case of $\Delta(r)$. We have the fibration from (8),

$$\Delta(\mathbb{H}P^1) \longrightarrow \Delta(\mathbb{H}P^r) \longrightarrow \mathrm{Gr}_2(\mathbb{H}^{r+1}).$$

So in order to apply Serre's spectral sequence, we need to know the cohomology of $\mathrm{Gr}_2(\mathbb{H}^{r+1})$. This is taken care of by the following Lemma, which is the quaternion version of [Bökstedt-Ottosen] Thm. 3.1:

Lemma 2.3. *For $r \geq 1$,*

$$H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1}); \mathbb{Z}) \cong \mathbb{Z}[p_1, p_2] / \langle \varphi_r, \varphi_{r+1} \rangle ,$$

where p_1, p_2 are the Pontryagin classes of the standard bundle $\gamma_2 \searrow \mathrm{Gr}_2(\mathbb{H}^{r+1})$, and $\varphi_i = \varphi_i(p_1, p_2)$ is the polynomial given inductively by

$$\varphi_0 = 1, \quad \varphi_1 = p_1, \quad \varphi_i = -p_1\varphi_{i-1} - p_2\varphi_{i-2}, \ \textit{for } i \geq 2.$$

Proof. We use a result of Borel, [Borel] Prop. 31.1. Let $\gamma_2 \searrow \mathrm{Gr}_2(\mathbb{H}^{r+1})$ denote the standard 2-dimensional bundle, i.e. the fibre over $V \subseteq \mathbb{H}^{r+1}$ is V. Let p_i, $i \geq 0$ be the Pontryagin classes, $p_i \in H^{4i}(\mathrm{Gr}_2(\mathbb{H}^{r+1})$, which satisfy $p_i = 0$ for $i > 2$, since γ_2 is 2-dimensional. Let $\bar{\gamma}_{r-1}$ denote its $(r-1)$-dimensional orthogonal complement, i.e. the fibre over $V \subseteq \mathbb{H}^{r+1}$ is $V^\perp \subseteq \mathbb{H}^{r+1}$. Denote the Pontryagin classes of this bundle by $\bar{p}_j$, $j \geq 0$, $\bar{p}_j \in H^{4j}(\mathrm{Gr}_2(\mathbb{H}^{r+1}))$, and note that $\bar{p}_j = 0$ for $j > r-1$. Then $\gamma_2 \oplus \bar{\gamma}_{r-1} \cong \varepsilon^{r+1}$, the trivial bundle of dimension $r+1$. The sum formula for Pontryagin classes gives the relations

$$\sum_{i+j=k} p_i\bar{p}_j = \bar{p}_k + \bar{p}_{k-1}p_1 + \bar{p}_{k-2}p_2 = 0, \quad \text{for } k > 0 \tag{11}$$

Borel's theorem states that $H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1}); \mathbb{Z})$ is generated by the Pontryagin classes of γ_2 and $\bar{\gamma}_{r-1}$, subject to the relations mentioned above:

$$H^*\left(\mathrm{Gr}_2(\mathbb{H}^{r+1}); \mathbb{Z}\right) \cong \mathbb{Z}[p_i, \bar{p}_j \mid i, j > 0] / \langle \{p_i\}_{i>2}, \{\bar{p}_j\}_{j>r-1}, \big(\sum_{i+j=k} p_i\bar{p}_j\big)_{k>0} \rangle.$$

By (11) we see that we can inductively express $\bar{p}_k$ as a polynomial in p_1 and p_2. Call that polynomial φ_k, so $\bar{p}_k = \varphi_k(p_1, p_2)$, and we get from (11)

$$\varphi_0 = 1, \quad \varphi_1 = p_1, \quad \varphi_i = -p_1\varphi_{i-1} - p_2\varphi_{i-1}, i \geq 2.$$

Then we get

$$\begin{aligned} H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1});\mathbb{Z}) &\cong \mathbb{Z}[p_1,p_2,\bar{p}_j \mid j>0]/\left\langle \{\bar{p}_j\}_{j>r-1}, \Big(\sum_{i+j=k} p_i\bar{p}_j\Big)_{k>0}\right\rangle \\ &\cong \mathbb{Z}[p_1,p_2,\bar{p}_1,\bar{p}_2,\ldots]/\left\langle \{\bar{p}_j\}_{j>r-1}, \{\bar{p}_k-\varphi_k(p_1,p_2)\}_{k>0}\right\rangle \\ &\cong \mathbb{Z}[p_1,p_2]/\left\langle \varphi_k \mid k\geq r\right\rangle. \end{aligned}$$

From the inductive formula for φ_k it is seen that $\langle\varphi_k \mid k\geq r\rangle = \langle\varphi_r,\varphi_{r+1}\rangle$, and this proves the lemma. □

2.2 The unparametrized geodesics

Recall $H^*(BS^1)\cong H^*(\mathbb{C}P^\infty)\cong\mathbb{Z}[u]$ where u has degree 2; a fact that can be deduced from $H^*(\mathbb{C}P^n)\cong\mathbb{Z}[u]/\langle u^{n+1}\rangle$.

Theorem 2.4. *The space of unparametrized oriented geodesics, $\Delta(\mathbb{H}P^r)$, has the following cohomology:*

$$H^*(\Delta(\mathbb{H}P^r);\mathbb{Z})\cong\mathbb{Z}[x,t]/\langle Q_r,Q_{r+1}\rangle,$$

where $x\in H^2(\Delta(\mathbb{H}P^r);\mathbb{Z})$ is the image of the generator $u\in H^*(BS^1)\cong\mathbb{Z}[u]$ and $t\in H^4(\Delta(\mathbb{H}P^r);\mathbb{Z})$. Q_k for $k\in\mathbb{N}$ is a polynomial in x and t inductively given by

$$Q_0=1,\quad Q_1=2t-x^2,\quad Q_s=(2t-x^2)Q_{s-1}-t^2Q_{s-2},\ \textit{for } s\geq 2.$$

Note that Lemma 2.2 is a special case of this with $r=1$: $Q_1=2t-x^2$, and $Q_2=(2t-x^2)Q_1-t^2\equiv t^2 \pmod{Q_1}$. The proof of Theorem 2.4 for $\mathbb{H}P^r$ is not at all like the $\mathbb{C}P^r$ case, since $\Delta(\mathbb{H}P^r)$ is not isomorphic to $\mathbb{P}(\gamma_2)$, and the proof will take quite some time. First we show that the cohomology is a polynomial algebra generated by classes x and t as in the Theorem, modulo certain relations. It will follow from Lemma 2.3 that the polynomials Q_r, Q_{r+1} are among these relations. Then we use a purely algebraic counting argument to show that there can be no further relations.

Proposition 2.5 (Theorem 2.4, Part 1)**.** *There is a surjective map*

$$\mathbb{Z}[x,t]/\langle Q_r,Q_{r+1}\rangle\twoheadrightarrow H^*(\Delta(\mathbb{H}P^r);\mathbb{Z}).$$

Proof of Theorem 2.4, Part 1. We write down the E_2 page of the Serre's spectral sequence for the fibration $\Delta(\mathbb{H}P^1)\longrightarrow\Delta(\mathbb{H}P^r)\longrightarrow\mathrm{Gr}_2(\mathbb{H}^{r+1})$, using Lemma 2.2 and the above Lemma 2.3:

$$
\begin{array}{c|ccccc}
4 & t & & \vdots & & \\
2 & x & & xp_1 & & \cdots \\
0 & 1 & & p_1 & & p_2 \\
\hline
 & 0 & 2 & 4 & 6 & 8
\end{array}
$$

We see there can be no differentials for dimension reasons, so $E_2 = E_\infty$. Since x is the only element of degree 2 in E_∞, it defines a class $\overline{x} \in H^2(\Delta(\mathbb{H}P^r)); \mathbb{Z})$. We also have $\overline{p_i} \in H^{4i}(\Delta(\mathbb{H}P^r); \mathbb{Z})$ for $i = 1, 2$: the image of p_i under the map induced by $\Delta(\mathbb{H}P^r) \longrightarrow \mathrm{Gr}_2(\mathbb{H}^{r+1})$. But t is only defined up to higher filtration. That is, we can choose $\bar{t} \in H^4(\Delta(\mathbb{H}P^r))$ which hits $t \in H^4(\Delta(\mathbb{H}P^1))$, but for any $m \in \mathbb{Z}$, $\bar{t} + m\overline{p_1}$ also hits t. As an abelian group, $H^4(\Delta(\mathbb{H}P^r); \mathbb{Z}) \cong \mathbb{Z}\overline{p_1} \oplus \mathbb{Z}\bar{t}$, so there must be a relation

$$\bar{x}^2 = a\overline{p_1} + b\bar{t}. \tag{12}$$

We will show that we can choose $\bar{t}$ a representative for t in $H^4(\Delta(\mathbb{H}P^r); \mathbb{Z})$ in such a way that $\overline{p_1} = \bar{x}^2 - 2\bar{t}$.

To get more information about $H^*(\Delta(\mathbb{H}P^r))$, we use Serre's spectral sequence for the fibration from Remark 1.7, $G(\mathbb{H}P^r) \longrightarrow \Delta(\mathbb{H}P^r) \longrightarrow BS^1$. By Theorem 2.1, the E_2 page has only one non-trivial group in total degree 2, namely a $\mathbb{Z}$ generated by u from $H^*(BS^1) \cong \mathbb{Z}[u]$. As $\bar{x}$ also generates $H^2(\Delta(\mathbb{H}P^r))$, we must have $\bar{x} = \pm u$. We can simply choose $\bar{x}$ to be the image of u. Also, u^2 generates a $\mathbb{Z}$ in $H^4(\Delta(\mathbb{H}P^r); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, so in particular, $\bar{x}^2$ is not divisible by 2, which we will need shortly.

We can make the following diagram where the middle is $H^4(\Delta(\mathbb{H}P^r); \mathbb{Z})$:

$$
\begin{tikzcd}
 & & \mathbb{Z}x^2 \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \mathbb{Z}p_1 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z}t \arrow[r] & 0
\end{tikzcd}
$$

Since, in the fibre, we have the relation $x^2 = 2t$, the diagonal map sends x^2 to $2t$. This implies that $b = 2$ in (12). So we now have $\bar{x}^2 = a\overline{p_1} + 2\bar{t}$. Changing $\bar{t}$ by adding an integer multiple of $\overline{p_1}$ yields that we can obtain either of the two relations

$$\bar{x}^2 = \overline{p_1} + 2\bar{t}, \quad \text{or} \quad \bar{x}^2 = 2\bar{t},$$

depending on whether a is odd or even. As noted, $\bar{x}^2$ cannot be divisible by 2, so we can choose $\bar{t}$ as desired.

Now I will drop the bar, and simply refer to these classes as x, t, p_1 and p_2. We have found the relation $p_1 = x^2 - 2t$ in $H^4(\Delta(\mathbb{H}P^r))$, and since $H^4 \cong \mathbb{Z} \oplus \mathbb{Z}$, there can be no further relations in degree 4.

Lemma 2.6. *In the above setting, $p_2 = t^2$.*

Proof. Recall the notation from section 1.2,

$$V_2(\mathbb{H}^{r+1}) = \left\{(v,w) \in \mathbb{H}^{r+1} \times \mathbb{H}^{r+1} \mid \|v\| = \|w\| = 1, \langle v,w\rangle_{\mathbb{H}} = 0\right\}.$$

Also recall from (5) that

$$G(\mathbb{H}P^r) \cong PV_2(\mathbb{H}^{r+1}) = V_2(\mathbb{H}^{r+1})/\operatorname{diag} S^3,$$

identifying the unit sphere in $\mathbb{H}$ with S^3. We also have a right S^1 action on V_2, simply by restricting the S^3 action to S^1. Now we mod out by the left S^1 action of rotation first, defining $Y_2(\mathbb{H}^{r+1}) = S^1 \setminus V_2(\mathbb{H}^{r+1})$. As the two actions are on the right and left, respectively, they clearly commute. So $Y_2(\mathbb{H}^{r+1})/S^3 \cong \Delta(\mathbb{H}P^r)$. In order to investigate p_2, we rely on the results for $\mathbb{C}P^r$, so we also consider $V_2(\mathbb{C}^{r+1})$ and define $Y_2(\mathbb{C}^{r+1}) = S^1 \setminus V_2(\mathbb{C}^{r+1})$. We then consider the following diagram:

$$\begin{tikzcd}
\Delta(\mathbb{C}P^r) \cong Y_2(\mathbb{C}^{r+1})/S^1 \arrow[r, "i"] \arrow[d, "p_{\mathbb{C}}"'] & Y_2(\mathbb{H}^{r+1})/S^1 \arrow[r, "q"] & Y_2(\mathbb{H}^{r+1})/S^3 \cong \Delta(\mathbb{H}P^r) \arrow[d, "p_{\mathbb{H}}"] \\
\mathrm{Gr}_2(\mathbb{C}^{r+1}) \arrow[rr, "h"] & & \mathrm{Gr}_2(\mathbb{H}^{r+1})
\end{tikzcd} \tag{13}$$

All maps are the obvious ones: $p_{\mathbb{C}}$ and $p_{\mathbb{H}}$ are the standard maps taking the pair of vectors to their span, i is induced by the inclusion $\mathbb{C} \subseteq \mathbb{H}$, and q is the quotient map. The map h sends a 2-dimensional complex subspace V to $V \otimes_{\mathbb{C}} \mathbb{H}$. Clearly, the diagram is commutative.

We investigate this diagram on cohomology. First note that Serre's spectral sequence for the fibration $S^2 \longrightarrow Y_2(\mathbb{H}^{r+1})/S^1 \xrightarrow{q} Y_2(\mathbb{H}^{r+1})/S^3$ has all non-trivial groups in even total degree, so there are no differentials, and we see that the induced map q^* on cohomology is injective. The map i is defined on representatives, so we can look at the corresponding map $\tilde{i}$ on V_2. Now $V_2(\mathbb{C}^{r+1})$ fits into the fibration $S^{2r-1} \longrightarrow V_2(\mathbb{C}^{r+1}) \longrightarrow S^{2r+1}$ (similar for $V_2(\mathbb{H}^{r+1})$), by choosing a unit vector v and then a unit vector w in v's orthogonal complement. So these V_2-spaces are at least $(2r-2)$-connected. Thus $\tilde{i}$ on V_2 is highly connected. When dividing by the S^1 actions, right and then left, we note that they are free actions. So we can apply e.g. [tomDieck] II.2.7 to conclude that i in the diagram is as highly connected. Thus i^* is an isomorphism on cohomology in degrees less than $2r-2$.

The idea is to obtain a relation in $H^*(\Delta(\mathbb{C}P^r))$ by going around the diagram (13). To find $(p_{\mathbb{C}})^*$, we will use the computation from the complex case, and the results are found in [Bökstedt-Ottosen], Thm. 3.2 and Cor.

3.3. From here we get $H^*(\Delta(\mathbb{C}P^r)) \cong \mathbb{Z}[x_1, x_2]/\text{relations}$, where x_1, x_2 are in degree 2, and $(p_{\mathbb{C}})^*$ is given by $c_1 \mapsto x_1 + x_2$ and $c_2 \mapsto x_1 x_2$, c_i denoting the ith Chern class in $H^*(\mathrm{Gr}_2(\mathbb{C}^{r+1}))$.

To relate p_2 to the other classes in $H^*(\Delta(\mathbb{H}P^r))$, we must know their images in $H^*(\Delta(\mathbb{C}P^r))$ under $j^* = (q \circ i)^*$. The classes p_1, p_2 come from the Pontryagin classes in $H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1}))$, and we can use Cor. 15.5 from [Milnor-Stasheff] which relates the Pontryagin and Chern classes to find $h^*(p_1) = {c_1}^2 - 2c_2$ and $h^*(p_2) = {c_2}^2$ in $H^*(\mathrm{Gr}_2(\mathbb{C}^{r+1}))$. As noted, x is the class coming from the generator $u \in H^*(BS^1)$, and according to [Bökstedt-Ottosen] page 13, u maps to $x_1 - x_2$. As we have the relation $p_1 = x^2 - 2t$ in $H^*(\Delta(\mathbb{H}P^r))$, we get $j^*(2t) = j^*(x^2) - j^* p_1$ in $H^*(\Delta(\mathbb{C}P^r))$. So we can compute all our classes in terms of x_1 and x_2:

$$\begin{array}{lllll} j^*p_1 & = (p_{\mathbb{C}})^*({c_1}^2 - 2c_2) & = (x_1 + x_2)^2 - 2x_1x_2 & = {x_1}^2 + {x_2}^2, \\ j^*p_2 & = (p_{\mathbb{C}})^*({c_2}^2) & = (x_1x_2)^2, & \\ j^*x & = x_1 - x_2, & & \\ j^*(2t) & = j^*(x^2) - j^*(p_1) & = (x_1 - x_2)^2 - {x_1}^2 - {x_2}^2 & = -2x_1x_2. \end{array}$$

Since $H^*(\Delta(\mathbb{C}P^r))$ is torsion-free, we see $j^*t = -x_1x_2$, and thus $j^*(t^2) = j^*(p_2)$. This implies $t^2 = p_2$ in $H^*(\Delta(\mathbb{H}P^r))$, since q^* is injective and i^* is an isomorphism on cohomology in degree 8, when r is large ($r > 5$). By naturality, it is enough to consider large r, since the classes pull back under the inclusion $\mathbb{H}P^r \longrightarrow \mathbb{H}P^{r+1}$. □

To recapitulate, $H^*(\Delta(\mathbb{H}P^r)) \cong \mathbb{Z}[x, t]/\text{relations}$, and the classes p_1 and p_2 coming from $H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1}))$ are related to x and t by $p_1 = x^2 - 2t$ and $p_2 = t^2$. By Lemma 2.3, in $H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1}))$ we have the relations φ_r, φ_{r+1}, which are polynomials in p_1 and p_2. Substituting the expressions for p_1 and p_2, we obtain the following relations Q_r and Q_{r+1} in $H^*(\Delta(\mathbb{H}P^r))$, where Q_s is the polynomial in x and t given by:

$$Q_s(x,t) = \varphi_s(x^2 - 2t, t^2) = -(x^2 - 2t)\varphi_{s-1} - t^2\varphi_{s-2} = (2t - x^2)Q_{s-1} - t^2 Q_{s-2}.$$

This ends Part 1 of the proof. □

I now investigate the Q-polynomials in order to complete the proof of Theorem 2.4. Q_s is a polynomial in x and t, where x has degree 2 and t has degree 4. It is given inductively by:

$$Q_0 = 1, \quad Q_1 = 2t - x^2, \quad Q_r = (2t - x^2)Q_{r-1} - t^2 Q_{r-2} \text{ for } r \geq 2. \tag{14}$$

Note that Q_r is a homogenous polynomial when taking into account that x has degree 2 and t has degree 4. It then has degree $4r$. It will be useful to know an explicit formula, and this is provided by the following lemma:

Lemma 2.7. *For any $r \geq 0$,*

$$Q_r = \sum_{k=0}^{r} (-1)^k \binom{r+k+1}{r-k} t^{r-k} x^{2k}.$$

Proof. Not surprisingly, this is proved by induction in r. It is clearly true for $r = 0$ and $r = 1$. Let us denote the coefficient of $t^l x^m$ in Q_s by $a^s_{l,m}$. Then we can write the coefficient of $t^{r-k}x^{2k}$ in $Q_r = (2t - x^2)Q_{r-1} - t^2 Q_{r-2}$ as:

$$\begin{aligned} a^r_{r-k,2k} &= 2a^{r-1}_{r-k-1,2k} - a^{r-1}_{r-k,2k-2} - a^{r-2}_{r-k-2,2k} \\ &= 2a^{r-1}_{r-1-k,2k} - a^{r-1}_{r-1-(k-1),2(k-1)} - a^{r-2}_{r-2-k,2k}. \end{aligned}$$

By induction we can substitute $a^s_{s-k,2k}$ by $(-1)^k \binom{s+k+1}{s-k}$ if $s < r$. So:

$$\begin{aligned} a^r_{r-k,2k} &= 2(-1)^k \binom{r-1+k+1}{r-1-k} - (-1)^{k-1} \binom{r-1+k}{r-1-k+1} \\ &\quad -(-1)^k \binom{r-2+k+1}{r-2-k} \\ &= (-1)^k \left(2\binom{r+k}{r-k-1} + \binom{r+k-1}{r-k} - \binom{r+k-1}{r-k-2} \right). \end{aligned}$$

All we need to show is that

$$2\binom{r+k}{r-k-1} + \binom{r+k-1}{r-k} - \binom{r+k-1}{r-k-2} = \binom{r+k+1}{r-k},$$

and this is easily done by three times applying the Pascal's triangle formula, $\binom{m-1}{j-1} + \binom{m-1}{j} = \binom{m}{j}$. □

Part 2 of the proof of Theorem 2.4 consists in to showing that the two rings $\mathbb{Z}[x,t]/\langle Q_r, Q_{r+1}\rangle \twoheadrightarrow H^*(\Delta(\mathbb{H}P^r);\mathbb{Z})$ have the same size, and deducing that the map must be an isomorphism. This will be done in the following lemmas.

Lemma 2.8. *The map*

$$Q_r + Q_{r+1} : \mathbb{Z}[x,t]_{4r} \oplus \mathbb{Z}[x,t]_{4r-4} \longrightarrow \mathbb{Z}[x,t]_{8r},$$

given by $(f,g) \mapsto fQ_r + gQ_{r+1}$, is surjective.

Proof. Let $M_r \subseteq \mathbb{Z}[x,t]_{8r}$ denote the image of $Q_r + Q_{r+1}$. Recall that x has degree 2, t has degree 4, and the degree of Q_s is $4s$, so M_r is generated over $\mathbb{Z}$ by

$$Q_r t^{r-k} x^{2k}, \quad k = 0, \ldots, r; \qquad Q_{r+1} t^{r-1-k} x^{2k}, \quad k = 0, \ldots, r-1. \tag{15}$$

We use induction in r. The induction start, $r = 1$, is easy:

$$\begin{aligned} t^2 &= (2t - x)Q_1 - Q_2, \\ x^2 t &= 2t^2 - tQ_1, \\ x^4 &= Q_2 - 4x^2 t + 3t^2. \end{aligned}$$

Now assume $r \geq 2$. Now let us rewrite the generators of M_r in (15), trying to bring into play the inductive definition of the Q-polynomials:

$$Q_{r+1} = (2t - x^2)Q_r - t^2 Q_{r-1}.$$

We can add the generators as follows for $k = 0, \dots, r-1$:

$$\begin{aligned} & Q_r t^{r-(k+1)} x^{2(k+1)} + Q_{r+1} t^{r-1-k} x^{2k} - 2Q_r t^{r-k} x^{2k} \\ = \;& t^{r-1-k} x^{2k}(Q_{r+1} + x^2 Q_r - 2tQ_r) = -t^2 \cdot Q_{r-1} t^{r-1-k} x^{2k} \end{aligned}$$

Furthermore, we have the ones involving Q_r, slightly rewritten:

$$t^2 \cdot Q_r t^{r-k-2} x^{2k}, \quad k = 0, \dots, r-2.$$

Now, inductively we assume that $M_{r-1} = \mathbb{Z}[x,t]_{8(r-1)}$. This means that everything in $\mathbb{Z}[x,t]_{8(r-1)}$ can be expressed as $\mathbb{Z}$-linear combinations of

$$Q_{r-1} t^{r-1-k} x^{2k}, \quad k = 0, \dots, r-1; \qquad Q_r t^{r-2-k} x^{2k}, \quad k = 0, \dots, r-2.$$

We see that, if multiplied by t^2, these are exactly the elements we have found in $M_r \subseteq \mathbb{Z}[x,t]_{8r}$. This means by induction that every generator for $\mathbb{Z}[x,t]_{8r}$ which is divisible by t^2 is in M_r.

All we are missing are the generators x^{4r} and tx^{4r-2}. Using Lemma 2.7, we see that:

$$Q_r t x^{2r-2} = (-1)^r t x^{4r-2} + \underbrace{\sum_{k=0}^{r-1} (-1)^k \binom{r+k+1}{r-k} t^{r-k+1} x^{2k+2r-2}}_{\text{divisible by } t^2}.$$

So $tx^{4r-2} \in M_r$, since elements divisible by t^2 are in M_r. Similarly, writing out $Q_r x^{2r}$, we get $x^{4r} \in M_r$ as desired. This accounts for all the generators in $\mathbb{Z}[x,t]_{8r}$ and ends the proof of surjectivity. □

I now compute the size of the ring $\mathbb{Z}[x,t]/\langle Q_r, Q_{r+1}\rangle$. For the formulation of the lemma below, it will be convenient to use the notational tool of the Poincaré series. This is simply a short way of expressing the ranks of a graded R-module $A = \bigoplus_m A_m$. (In order for the rank to be well-defined, we can assume R is commutative; mostly we will have $R = \mathbb{Z}$.) The Poincaré series of A is then the formal expression $P_A(t) = \sum_m \operatorname{rank}(A_m) t^m$.

Lemma 2.9. *Write $A = \mathbb{Z}[x,t]/\langle Q_r, Q_{r+1}\rangle$. Then A is torsion free, and the Poincaré series of the graded ring A is given by*

$$P(t) = (1+t^2)\cdot \frac{1-t^{4r}}{1-t^4}\cdot \frac{1-t^{4(r+1)}}{1-t^4}.$$

Remark 2.10. This gives that the ranks of A in each degree are as follows:

0	2	4	6	8	$\cdots$	$4r-6$	$4r-4$	$4r-2$	$4r$	$4r+2$	$4r+4$	$\cdots$	$8r-4$	$8r-2$
1	1	2	2	3	$\cdots$	$r-1$	r	r	r	r	$r-1$	$\cdots$	1	1

where the degree is in the top row. Each rank is repeated twice, increasing by one from 1 to r up to the vertical line, and then decreasing by one from r to 1. For this, see the start of the proof below.

Proof. Let us try to write the Poincaré series differently. We calculate

$$\frac{1-t^{4r}}{1-t^4}\cdot \frac{1-t^{4(r+1)}}{1-t^4} = \left(\sum_{i=0}^{r-1} t^{4i}\right)\left(\sum_{j=0}^{r} t^{4j}\right) = \sum_{k=0}^{2r-1}\left(\sum_{i+j=k} t^{4k}\right) = \sum_{k=0}^{2r-1} a_k t^{4k},$$

where

$$a_k = \begin{cases} k+1, & k<r; \\ 2r-k, & k\geq r. \end{cases}$$

simply by counting the number of ways to write k as a sum of i and j. So we must show that the Poincaré series is

$$(1+t^2)\sum_{k=0}^{2r-1} a_k t^{4k}, \quad \text{where } a_k = \begin{cases} k+1, & k<r; \\ 2r-k, & k\geq r. \end{cases} \tag{16}$$

Let $A_s \subseteq \mathbb{Z}[x,t]_s$ denote the homogeneous polynomials in A of degree s. Since Q_r has degree $4r$, we must have $A_s = \mathbb{Z}[x,t]_s$ for $s < 4r$, since there are no relations. So A_s is torsion-free for $s<4r$. The generators of $\mathbb{Z}[x,t]_s$ are: For $s = 4k$, $\left\{t^{k-j}x^{2j} \mid j = 0,\dots,k\right\}$ and for $s = 4k+2$, $\left\{t^{k-j}x^{2j+1} \mid j = 0,\dots,k\right\}$, so the rank is $k+1$ in both cases. From this, the Poincaré series of $\mathbb{Z}[x,t]$ is $(1+t^2)\sum_{k=0}^{\infty}(k+1)t^{4k}$, so it is clear that $a_k = k+1$ for $k<r$ as claimed in (16).

Now we handle degrees $4r$ and $4r+2$. Here the only relations are Q_r and xQ_r, respectively. By Lemma 2.7, the coefficient of x^{2r} (resp. x^{2r+1}) in Q_r (resp. xQ_r) is ± 1, we get exactly one generator less than in $\mathbb{Z}[x,t]_{4r}$ (resp. $\mathbb{Z}[x,t]_{4r+2}$), which had rank $r+1$. This means A_{4r} and A_{4r+2} are torsion-free, and the rank is r in both cases, as (16) claims.

We now show that the A_{4r+2m} is torsion-free for $2 \leq m \leq 2r$. To do this, assume there was a torsion element $a \in \mathbb{Z}[x,t]_{4r+2m}$, i.e. $na = Q_r f + Q_{r+1} g$ for some $n \in \mathbb{Z}$. Multiplying by x^{2r-m} gives

$$nax^{2r-m} = Q_r f x^{2r-m} + Q_{r+1} g x^{2r-m} \in \mathbb{Z}[x,y]_{8r}. \tag{17}$$

Now, $ax^{2r-m} \in \mathbb{Z}[x,y]_{8r}$, so since $Q_r + Q_{r+1}$ is onto this by Lemma 2.8, we have

$$ax^{2r-m} = Q_r f' + Q_{r+1} g', \quad \text{for some } f', g'. \tag{18}$$

Multiplying this by n and comparing with (17) we get

$$(fx^{2r-m} - nf')Q_r = (-gx^{2r-m} + ng')Q_{r+1}. \tag{19}$$

Since $Q_r + Q_{r+1}$ is surjective onto $\mathbb{Z}[x,y]_{8r}$, Q_r and Q_{r+1} are relatively prime. We then conclude from (19) that x^{2r+m} divides f' and g'. So we can divide by x^{2r+m} in (18) and obtain the relation $a = Q_r f'' + Q_{r+1} g''$. So $a = 0$ in A_{4r+2m}, and there is no torsion.

For the last part, the surjectivity result of Lemma 2.8 implies $A_s = 0$ for $s \geq 8r$, as the Poincaré series states. We already calculated the rank of $\mathbb{Z}[x,t]_{4s}$ to be $s+1$, so we see that both $\mathbb{Z}[x,t]_{4r} \oplus \mathbb{Z}[x,t]_{4r-4}$ and $\mathbb{Z}[x,t]_{8r}$ have rank $2r+1$. Since we have shown A is torsion-free, this means that the map $Q_r + Q_{r+1} : \mathbb{Z}[x,t]_{4r} \oplus \mathbb{Z}[x,t]_{4r-4} \twoheadrightarrow \mathbb{Z}[x,t]_{8r}$ must also be injective. This implies that for any m such that $2 \leq m \leq 2r$, the map

$$Q_r + Q_{r+1} : \mathbb{Z}[x,t]_{2m} \oplus \mathbb{Z}[x,t]_{2m-4} \longrightarrow \mathbb{Z}[x,t]_{4r+2m}$$

is also injective, since we can multiply a relation $Q_r f + Q_{r+1} g = 0$ in $\mathbb{Z}[x,t]_{4r+2m}$ by x^{2r-m}, and get a similar relation in $\mathbb{Z}[x,t]_{8r}$, where $Q_r + Q_{r+1}$ is injective. Therefore,

$$\begin{aligned} \operatorname{rank}(A_{4r+2m}) &= \operatorname{rank} \operatorname{Cok}(Q_r + Q_{r+1}) \\ &= \operatorname{rank} \mathbb{Z}[x,t]_{4r+2m} - \operatorname{rank}\left(\mathbb{Z}[x,t]_{2m} \oplus \mathbb{Z}[x,t]_{2m-4}\right). \end{aligned}$$

These ranks we already know. If $m = 2l$ or $m = 2l+1$ we get in either case:

$$\operatorname{rank}(A_{4r+2m}) = r + l + 1 - (l+1) - l = r - l, \quad \text{for } 2 \leq m \leq 2r$$

which, substituting $k = r + l$, is $2r - k$, as claimed in (16). $\square$

Now we can finish the proof of Theorem 2.4:

Proof of Theorem 2.4, Part 2. Picking up where we left in Part 1, we have a surjective map

$$\mathbb{Z}[x,t]/\langle Q_r, Q_{r+1}\rangle \twoheadrightarrow H^*(\Delta(\mathbb{H}P^r);\mathbb{Z}). \tag{20}$$

By Lemma 2.9 and 2.8 we have computed the ranks of the free, graduated $\mathbb{Z}$-module $\mathbb{Z}[x,t]/\langle Q_r, Q_{r+1}\rangle$. It has the Poincaré series

$$P_{\mathbb{Z}[x,t]/\langle Q_r,Q_{r+1}\rangle}(t) = (1+t^2)\cdot\frac{1-t^{4r}}{1-t^4}\cdot\frac{1-t^{4(r+1)}}{1-t^4}.$$

If $H^*(\Delta(\mathbb{H}P^r);\mathbb{Z})$ has the same Poincaré series, the surjective map (20) must be an isomorphism. We compute the ranks via the spectral sequence of the fibration (8), $\Delta(\mathbb{H}P^1) \longrightarrow \Delta(\mathbb{H}P^r) \longrightarrow \mathrm{Gr}_2(\mathbb{H}^{r+1})$. We see that the non-trivial part of the E_2 page sits in even total degree, so $E_\infty = E_2$, and we can compute the Poincaré series of the total space,

$$P_{H^*(\Delta(\mathbb{H}P^r))}(t) = P_{H^*(\Delta(\mathbb{H}P^1))}(t)\cdot P_{H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1}))}(t).$$

Here we know by Lemma 2.2

$$H^n(\Delta(\mathbb{H}P^1);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n = 0,2,4,6; \\ 0, & \text{otherwise.}\end{cases}$$

so its Poincaré series is $P_{H^*(\Delta(\mathbb{H}P^1))}(t) = 1+t^2+t^4+t^6 = (1-t^8)/(1-t^2)$. Also by Lemma 2.3

$$H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1});\mathbb{Z}) \cong \mathbb{Z}[p_1,p_2]/\langle \varphi_r, \varphi_{r+1}\rangle .$$

To compute the Poincaré series, one proceeds as in Lemmas 2.8 and 2.9. Lemma 2.9 does not cover the Grassmannian case, for when I tried stating and proving a more general lemma that could handle both cases, everything got extremely complicated. So I simply state the result for the Grassmannian, the proof of which is just like Lemma 2.9:

$$P_{H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1}))}(t) = \frac{1-t^{4r+4}}{1-t^4}\cdot\frac{1-t^{4r}}{1-t^8}.$$

Then

$$\begin{aligned} P_{H^*(\Delta(\mathbb{H}P^r))}(t) &= P_{H^*(\Delta(\mathbb{H}P^1))}(t)\cdot P_{H^*(\mathrm{Gr}_2(\mathbb{H}^{r+1}))}(t) \\ &= \frac{1-t^8}{1-t^2}\cdot\frac{1-t^{4r+4}}{1-t^4}\cdot\frac{1-t^{4r}}{1-t^8} \\ &= (1+t^2)\cdot\frac{1-t^{4r+4}}{1-t^4}\cdot\frac{1-t^{4r}}{1-t^4} = P_{\mathbb{Z}[x,t]/\langle Q_r,Q_{r+1}\rangle}(t). \end{aligned}$$

This finishes the proof. $\square$

2.3 Equivariant cohomology of spaces of geodesics

Using our previous computations (Theorems 2.1 and 2.4) and Serre's spectral sequence, we will be able to compute the equivariant cohomology of the space of geodesics, $G(\mathbb{H}P^r)^{(n)}$.

We first consider the case $p \nmid n$, since this is the easiest. We show:

Proposition 2.11. *For $p \nmid n$:*

$$H^m(BC_n; \mathbb{F}_p) = 0, \quad \textit{for } m > 0.$$

Proof. We are going to use that $EC_n \longrightarrow BC_n$ is a covering, since C_n is discrete. In general, given a k-sheet covering $\pi : E \longrightarrow B$ (assume B connected), one can construct a so-called transfer map. By barycentric subdivision one knows that it is enough to consider very small simplices in B. Therefore, given a simplex in B we can assume it is contained in a neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open sets mapped homeomorphically to U by π. Then we can pull the simplex in U back by π, yielding k copies of the simplex in E, which we formally add, giving a chain map $\tau : C_m(B) \longrightarrow C_m(E)$. This induces the transfer map $\tau^* : H^m(E) \longrightarrow H^m(B)$ on cohomology. From the definition, $\pi_\sharp \circ \tau$ is multiplication by k, and so $\tau^*\pi^*$ is also multiplication by k. In our case, $EC_n \longrightarrow BC_n$ is an n-sheet covering, and so the composition

$$H^m(BC_n; \mathbb{F}_p) \xrightarrow{\tau^*} H^m(EC_n; \mathbb{F}_p) \xrightarrow{\pi^*} H^m(BC_n; \mathbb{F}_p)$$

is multiplication by n. Since we are using $\mathbb{F}_p$-coefficients and $p \nmid n$, this is an isomorphism. On the other hand, for $m > 0$, the middle term is zero, since EC_n is contractible. Thus $H^m(BC_n; \mathbb{F}_p) = 0$ for $m > 0$. □

With this we can prove:

Theorem 2.12. *For $p \nmid n$, the equivariant cohomology with $\mathbb{F}_p$ coefficients of the n-twisted space of geodesics on $\mathbb{H}P^r$ is*

$$H^*((G(\mathbb{H}P^r)^{(n)})_{hS^1}; \mathbb{F}_p) \cong \mathbb{F}_p[x,t]/\langle Q_r, Q_{r+1}\rangle,$$

where x has degree 2, and t has degree 4, and x is the image of the generator $u \in H^2(BS^1)$ under the map $\Delta(\mathbb{H}P^r) \longrightarrow BS^1$ in (9).

Proof. We use the Serre's spectral sequence of the fibration from Prop. 1.6:

$$BC_n \longrightarrow ES^1 \times_{S^1} G(r)^{(n)} \longrightarrow \Delta(r).$$

Proposition 2.11 above now immediately implies that

$$H^*((G(r)^{(n)})_{hS^1}; \mathbb{F}_p) = H^*(ES^1 \times_{S^1} G(r)^{(n)}; \mathbb{F}_p) \cong H^*(\Delta(r); \mathbb{F}_p)$$

The theorem is now proved by our computation in Theorem 2.4. □

The case $p \mid n$ requires more work, and one needs to take into account whether or not $p \mid r+1$. But first we need a computation of $H^*(BC_n;\mathbb{F}_p)$:

Proposition 2.13. *For $p \mid n$,*

$$H^*(BC_n;\mathbb{F}_p) \cong \mathbb{F}_p[u,e]/\left\langle e^2\right\rangle.$$

Proof. Use Theorem 1.4 (i) on the fibration $S^1 \longrightarrow ES^1 \longrightarrow BS^1$ to divide out the action of $C_n \subseteq S^1$, and obtain a fibration

$$S^1 \longrightarrow BC_n \longrightarrow BS^1. \tag{21}$$

Here we have identified the quotient group S^1/C_n with S^1 itself via the nth power map $z \mapsto z^n$. We will apply Serre's spectral sequence.

First, though, we will find $H^1(BC_n;\mathbb{F}_p)$. Since C_n is discrete, $EC_n \longrightarrow BC_n$ is the universal covering. From covering space theory, $\pi_1(BC_n) \cong C_n$, and since this is abelian, it follows that $H_1(BC_n;\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Using the Universal Coefficient theorem, we can compute $H^1(BC_n;\mathbb{F}_p)$. Note that $H_0(BC_n) = \mathbb{Z}$, so $\mathrm{Ext}(H_0(BC_n),\mathbb{F}_p) = 0$, and therefore, since $p \mid n$:

$$H^1(BC_n;\mathbb{F}_p) \cong \mathrm{Hom}(H_1(BC_n),\mathbb{F}_p) \cong \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{F}_p,$$

Now we turn to Serre's spectral sequence for the fibration (21), with $E_2^{p,q} = H^p(BS^1;H^q(S^1;\mathbb{F}_p)) = H^p(BS^1,\mathbb{F}_p) \otimes H^q(S^1;\mathbb{F}_p)$. Note that the only possible non-trivial differential is d^2, since the E^2 page has only two non-zero rows. Knowing that $H^1(BC_n;\mathbb{F}_p) \cong \mathbb{F}_p$, we conclude that the first differential $d_2^{0,1}$ must be a map $\mathbb{F}_p \longrightarrow \mathbb{F}_p$ with kernel isomorphic to $\mathbb{F}_p$. This forces $d_2(e) = 0$, where e generates $H(S^1;\mathbb{F}_p)$. Using the derivation property:

$$d(eu^j) = d(e)u^j \pm ed(u^j) = 0.$$

So all differentials are zero, the spectral sequence collapses, and $E_\infty = E_2$. There are no extension problems, since each diagonal $p+q = *$ contains at most one non-zero group, so $H^*(BC_n;\mathbb{F}_p) = E_\infty$, as desired. □

Theorem 2.14. *Let p be a prime number and $n \in \mathbb{N}$ such that $p \mid n$. As $\mathbb{F}_p[u]$-modules, the following holds:*

(i) *Suppose $p \nmid r+1$. Then*

$$H^*\left(\left(G(\mathbb{H}P^r)^{(n)}\right)_{hS^1};\mathbb{F}_p\right) \cong \mathbb{F}_p[u]\left\{1,y,y^2,\ldots,y^{r-1},\tau,\tau y,\ldots,\tau y^{r-1}\right\}.$$

(ii) *Suppose $p \mid r+1$. Then*

$$H^*\left(\left(G(\mathbb{H}P^r)^{(n)}\right)_{hS^1};\mathbb{F}_p\right) \cong \mathbb{F}_p[u]\left\{1,y,y^2,\ldots,y^{r},\sigma,\sigma y,\ldots,\sigma y^{r}\right\}.$$

where y has degree 4, τ has degree $4r+3$ and σ has degree $4r-1$.

Proof. In the beginning, the proofs of the two cases are the same. Consider the spectral sequence for the fibration from Prop. 1.6

$$G(r) \longrightarrow ES^1 \times_{S^1} G(r)^{(n)} \longrightarrow BS^1. \tag{22}$$

According to our computation of the cohomology of the fibre in Theorem 2.1, neither the fibre nor the base has anything in cohomology of degree 1. This means that $H^1((G(r)^{(n)})_{hS^1}) = 0$. We can use this when considering the spectral sequence for the other fibration from Prop. 1.6:

$$BC_n \longrightarrow ES^1 \times_{S^1} G(r)^{(n)} \longrightarrow \Delta(r).$$

According to Prop. 2.13, $E_2^{q,s} = H^q(\Delta(r); H^s(BC_n; \mathbb{F}_p)$ looks as follows:

$$\begin{array}{c|ccccc}
3 & ue & & uex & & uex^2, uet & \dots \\
2 & u & & ux & & ux^2, ut & \\
\hline
1 & e & & ex & & ex^2, et & \dots \\
0 & 1 & & x & & x^2, t & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \dots
\end{array} \tag{23}$$

Let us denote the two lower rows of the E_2 page by F. Then the next two rows (rows 2 and 3) consists of uF, the next two are u^2F, etc. Consider the differential d_2 as a map $d_2 : eH^*(\Delta(r)) \longrightarrow H^*(\Delta(r))$ from row 1 to row 0. Then, using the derivation property of the differentials we see that d_2 is multiplication with $d_2(e)$. When passing from the E_2 to the E_3 page, F will be replaced by two rows, $\operatorname{Cok} d_2$ and $\operatorname{Ker} d_2$, uF will be replaced by $u\operatorname{Cok} d_2$ and $u\operatorname{Ker} d_2$, etc.

So to determine the E_3 page, we need to find $d_2(e)$. As noted, the total space has $H^1 = 0$, so $d_2^{0,1} : E_2^{0,1} \longrightarrow E_2^{2,0}$ must be an injective map, hence an isomorphism. This forces $d_2(e) = \text{unit} \cdot x$; we might as well say $d_2(e) = x$. So d_2 is multiplication by x, and we must determine $\operatorname{Cok}(x)$ and $\operatorname{Ker}(x)$. Using Theorem 2.4, we see that

$$\operatorname{Cok}(x) \cong \mathbb{F}_p[x,t]/\langle x, Q_r, Q_{r+1}\rangle \cong \mathbb{F}_p[t]/\langle Q_r(0,t), Q_{r+1}(0,t)\rangle. \tag{24}$$

Now by Lemma 2.7, $Q_r(0,t) = (r+1)t^r$ and $Q_{r+1}(0,t) = (r+2)t^{r+1}$. This is where we must distinguish between the two cases.

But let us first investigate the kernel. I have tried to diagram the dimensions of $\mathbb{F}_p[x,t]/\langle Q_r, Q_{r+1}\rangle$ using Remark 2.10, with boldface indicating the degrees where, for dimension reasons, the kernel must be non-trivial. The degrees are in the top row:

0	2	4	6	8	$\cdots$	$4r$	$4r{+}2$	$4r{+}4$	$4r{+}6$	$4r{+}8$	$4r{+}10$	$4r{+}12$	$\cdots$
1	1	2	2	3	$\cdots$	r	$\boldsymbol{r}$	$r-1$	$\boldsymbol{r-1}$	$r-2$	$\boldsymbol{r-2}$	$r-3$	$\cdots$

The pattern is (hopefully) clear: There must be a part of the kernel in degrees $4(r+i)-2$ for $i=1,...,r$. In particular, the dimension is at least r. Now, for the rest of the proof, we need to handle the two cases separately.

Case (i)**:** $p \nmid r+1$**.** In this case, $r+1$ is a unit in $\mathbb{F}_p$, so (24) becomes $\mathrm{Cok}(x) \cong \mathbb{F}_p[t]/\langle t^r \rangle$. In particular, the dimension of $\mathrm{Cok}(x)$ is r, generated by $1, t, \ldots, t^{r-1}$.

Since $\dim \mathrm{Ker}(x) = \dim \mathrm{Cok}(x) = r$, we have determined above that the kernel is in degrees $4(r+i)-2$ for $i=1,...,r$. In each degree, the kernel is one-dimensional, say generated by φ_i in degree $4(r+i)-2$. So we can write down the E_3 page:

	0	2	4	6	8	...	4r−4	4r−2	4r	4r+2	4r+4	4r+6	...	8r−2
3										$u\varphi_1$		$u\varphi_2$	$\cdots$	$u\varphi_r$
2	u		ut		ut^2	$\cdots$	ut^{r-1}							
1										φ_1		φ_2	$\cdots$	φ_r
0	1		t		t^2	$\cdots$	t^{r-1}							

Because there are no further differentials on t and u, and the differentials satisfy the derivation property, we see that the spectral sequence collapses from the E_3 page. Now let us compare this to the spectral sequence for the fibration $G(r) \longrightarrow ES^1 \times_{S^1} G(r)^{(n)} \longrightarrow BS^1$ from (22) considered in the beginning, which also converges to $H^*((G(r)^{(n)})_{hS^1}; \mathbb{F}_p)$. Since

$$H^*(G(r); \mathbb{F}_p) \cong \mathbb{F}_p[y, \tau]/\left\langle y^r = 0, \tau^2 = 0 \right\rangle,$$

where y has degree 4 and τ has degree $4r+3$, we get the E_2 page,

$$E_2^{*,*} \cong \mathbb{F}_p[y, \tau]/\left\langle y^r = 0, \tau^2 = 0 \right\rangle \otimes \mathbb{F}_p[u]$$

Comparing this to the E_3 page above, we see that we have in each case $2r$ generators which are multiplied by 1, u, u^2, etc. This means, since the first spectral sequence collapses, that this second one must also collapse. Consequently we can read off that $H^*(G(r)^{(n)}_{hS^1}; \mathbb{F}_p)$ as an $\mathbb{F}_p[u]$-module is generated by

$$\left\{1, y, y^2, \ldots, y^{r-1}, \tau, \tau y, \ldots, \tau y^{r-1}\right\}$$

Case (ii)**:** $p \mid r+1$**.** In this case, $r+1$ is zero in $\mathbb{F}_p$, but $r+2$ is a unit, so (24) becomes:

$$\mathrm{Cok}(x) \cong \mathbb{F}_p[t]/t^{r+1}.$$

In particular, the dimension of $\mathrm{Cok}(x)$ is $r+1$, generated by $1, t, \ldots, t^r$.

Consequently, $\dim \operatorname{Ker}(x) = r+1$, so we need to find an additional element in the kernel. By Lemma 2.7, Q_r is the polynomial

$$Q_r = (r+1)t^r - \binom{r+2}{r-1} t^{r-1}x^2 + \cdots \pm x^{2r},$$

so x divides Q_r in $\mathbb{F}_p[x,t]$. This means we have an element $\varphi_0 = Q_r/x$ in degree $4r-2$ which is in the kernel of x. So together with the elements $\varphi_1, \ldots, \varphi_r$ from before, we have found generators of the kernel.

As in Case (i), we see that the spectral sequence collapses from the E_3 page. Comparing with the E_2 page of the fibration (22), and using that since $p \mid r+1$,

$$H^*(G(r);\mathbb{F}_p) \cong \mathbb{F}_p[y,\sigma]/\left\{y^{r+1}=0, \sigma^2=0\right\},$$

we conclude as above that $H^*({G(r)^{(n)}}_{hS^1};\mathbb{F}_p)$ as an $\mathbb{F}_p[u]$ module is generated by

$$\left\{1, y, y^2, \ldots, y^r, \sigma, \sigma y, \ldots, \sigma y^r\right\}. \qquad \square$$

Corollary 2.15. *For the Serre spectral sequence of the fibration*

$$G(\mathbb{H}P^r) \longrightarrow {G(\mathbb{H}P^r)^{(n)}}_{hS^1} \longrightarrow BS^1$$

the following holds: If $p \mid n$, it collapses from the E_2 page. If $p \nmid n$ the inclusion of the fibre induces a surjective map on even degree cohomology

$$H^{2*}({G(\mathbb{H}P^r)^{(n)}}_{hS^1};\mathbb{F}_p) \longrightarrow H^{2*}(G(\mathbb{H}P^r);\mathbb{F}_p)$$

Proof. The case $p \mid n$ follows directly from the proof of Theorem 2.14 above. For the case $p \nmid n$, we must check that the classes y^j from Theorem 2.1 are in the image of the inclusion of the fibre. To do this, we consider the E_2 page of the spectral sequence, and must show that the classes y^j survive to E_∞. Since the differentials are derivations, $d_s(y^j) = jy^{j-1}d_s(y)$, and so it suffices to show y survives. Clearly it does, since any differential starting at y ends in total degree 5, and there are no non-trivial classes in total degree 5. $\square$

3 K-theory of spaces of geodesics in $\mathbb{C}P^r$

Let $G(r) = G(\mathbb{C}P^r)$ be the space of simple, closed, parametrized geodesics in $\mathbb{C}P^r$, and let $\Delta(r) = S^1 \setminus G(r)$ be the quotient space under the rotation action. In this chapter we obtain K-theoretic analogues of the results for cohomology from the previous chapter.

By K-theory we mean complex K-theory, i.e. $K^0(X)$ for a CW-complex X is the group completion of the semi-group of complex vector bundles with base space X. Define $K^*(X)$ for a general space X as follows: Chose any CW complex Y weakly equivalent to X, put $K(X) = K(Y)$. This is well defined, since two choices of Y will be homotopy equivalent, and K-theory is homotopy invariant. We most often employ the $\mathbb{Z}/2\mathbb{Z}$-grading from Bott-periodicity, writing $K^*(X) = K^0(X) \oplus K^1(X)$.

3.1 The unparametrized geodesics

Recall the model for $\Delta(r)$ from the end of section 1.2. We had γ_2, the standard 2-dimensional bundle over the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{r+1})$ and $p : \mathbb{P}(\gamma_2) \longrightarrow \mathrm{Gr}_2(\mathbb{C}^{r+1})$ the associated projective bundle. Then we had a composite map (6), which is an S^1-equivariant diffeomorphism

$$\varphi : \Delta(r) \longrightarrow \mathbb{P}(\gamma_2)$$

Take the standard line bundle γ_1 over $\mathbb{P}(\gamma_2)$. The pullback $\varphi^*(\gamma_1)$ of γ_1 under φ is a line bundle we will denote X. We consider also the conjugate line bundle $\gamma_1^\perp$ to γ_1 over $\mathbb{P}(\gamma_2)$, i.e. $\gamma_1 \oplus \gamma_1^\perp = p^*\gamma_2$. The pullback $\varphi^*(\gamma_1^\perp)$ of this bundle to $\Delta(r)$ we will denote Y. In $K^0(\Delta(r))$ we define the classes $x = [X] - 1$ and $y = [Y] - 1$.

Theorem 3.1. *Let $x, y \in K^0(\Delta(r))$ be the classes defined above. Then*

$$\begin{aligned} K^0(\Delta(r)) &\cong \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle, \\ K^1(\Delta(r)) &= 0, \end{aligned}$$

where Q_s for $s \in \mathbb{N}$ is the homogeneous polynomial in x, y given by

$$Q_s(x,y) = \sum_{j=0}^{s} x^j y^{s-j}.$$

Note that these polynomials are not the same is in the cohomology case, but I use the same notation, since they play precisely the same role.

Proof. We apply the Atiyah-Hirzebruch spectral sequence, Theorem 1.3

$$H^*(\Delta(r); K^*(*)) \Rightarrow K^*(\Delta(r)). \tag{25}$$

Since we know the cohomology of $\Delta(r)$ from [Bökstedt-Ottosen],

$$H^*(\Delta(r)) \cong \mathbb{Z}[x_1, x_2]/ \langle Q_r, Q_{r+1} \rangle ,$$

and x_1, x_2 have degree 2, we see that all differentials in (25) are trivial, so that

$$E_\infty = E_2 \cong \mathbb{Z}[x_1, x_2]/ \langle Q_r, Q_{r+1}\rangle \otimes \mathbb{Z}[\beta, \beta^{-1}],$$

where β denotes the Bott element. This shows that $K^1(\Delta(r)) = 0$, and $K^0(\Delta(r))$ is free abelian of the same rank as $H^*(\Delta(r))$.

We use the Chern character,

$$\mathrm{ch} : K^0(X) \longrightarrow H^*(X; \mathbb{Q}),$$

which is a ring homomorphism. By construction, $x_1 = c_1(X)$ and $x_2 = c_1(Y)$ are the first Chern classes of X and Y, cf. [Bökstedt-Ottosen] Thm. 3.2, so since X, Y are line bundles, we get

$$\begin{aligned} \mathrm{ch}(x) &= \mathrm{ch}(X) - 1 = \exp(c_1(X)) - 1 = \exp(x_1) - 1, \\ \mathrm{ch}(y) &= \mathrm{ch}(Y) - 1 = \exp(c_1(Y)) - 1 = \exp(x_2) - 1. \end{aligned}$$

There is a relation between the Chern character ch and the Atiyah-Hirzebruch spectral sequence, by [Atiyah-Hirzebruch] Cor. 2.5. We see that $\mathrm{ch}(x^i y^j) = \mathrm{ch}(x)^i \mathrm{ch}(y)^j = x_1^i x_2^j +$ higher terms, where "higher terms" means terms in higher filtration, which in this case is equivalent to higher total degree in x_1, x_2. By *(iii)* in the corollary, this shows that the ring homomorphism $\mathbb{Z}[x, y] \longrightarrow K^*(\Delta(r))$ is surjective.

This means we can use x, y as polynomial generators for $K^*(X)$, and it remains to determine the relations. Again we use the Chern character, this time after tensoring with $\mathbb{Q}$:

$$\mathrm{ch} : K^0(X) \otimes \mathbb{Q} \longrightarrow H^*(X, \mathbb{Q})$$

which is then a ring isomorphism. We now want to prove that $\mathrm{ch}(x)$ and $\mathrm{ch}(y)$ satisfy the relations Q_r, Q_{r+1}. If we can prove this, we are done: Since the Chern character is an isomorphism after tensoring with $\mathbb{Q}$, and the groups are torsion-free, there can be no further relations in $K^0(S(\tau)/S^1)$, since this has the same rank as $H^*(S(\tau)/S^1) \cong \mathbb{Z}[x_1, x_2]/ \langle Q_r, Q_{r+1}\rangle$.

So we need to prove that $Q_s(\exp(x_1)-1, \exp(x_2)-1) = 0$ if $Q_s(x_1, x_2) = 0$ for $s = r, r+1$. Recalling that the ideals $\langle Q_r, Q_{r+1}\rangle$ and $\left\langle Q_r, x_1^{r+1}, x_2^{r+1}\right\rangle$

coincide, we first get that $(\exp(x_i) - 1)^{r+1} = x_i^{r+1}(1 + \text{higher terms}) = 0$. Consider the quotient map

$$R = \mathbb{Q}[x_1, x_2]/\left\langle x_1^{r+1}, x_2^{r+1}\right\rangle \longrightarrow \mathbb{Q}[x_1, x_2]/\left\langle Q_r, Q_{r+1}\right\rangle = S,$$

which has kernel $I = \langle Q_r \rangle$. Given a power series without constant term, $g(z) = a_1 z + a_2 z^2 + \cdots$, we can define $g_* : R \longrightarrow R$ by $x_i \mapsto g(x_i)$ for $i = 1, 2$. In our case, $g(z) = \exp(z) - 1$. If we can prove that $g_* I \subseteq I$, the map g_* will be well-defined as a map $S \longrightarrow S$, as shown below:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "g_*"] & R \arrow[r] \arrow[d, "g_*"] & S \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & I \arrow[r] & R \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

We will show $I = \mathrm{Ker}(x_1 - x_2)$. Consider a homogeneous polynomial $f \in R$ of degree m. It suffices to take $m \geq r$, for if f had lower degree, it could not be in $I = \langle Q_r \rangle$, since Q_r has degree r. Then, using $x_1^{r+1} = x_2^{r+1} = 0$, we can write

$$f = \sum_{i=m-r}^{r} c_i x_1^i x_2^{m-i} \quad \Rightarrow \quad (x_1 - x_2) f = \sum_{i=m-r+1}^{r} (c_{i-1} - c_i) x_1^i x_2^{m-i}.$$

By [Bökstedt-Ottosen] Lemma 3.4, $f \in I$ if and only if $c_{m-r} = \ldots = c_r$, and we conclude $I = \mathrm{Ker}(x_1 - x_2)$. This implies $g_* I \subseteq \mathrm{Ker}(g_* x_1 - g_* x_2)$. So we calculate

$$g_* x_1 - g_* x_2 = \sum_{i \geq 1} a_i (x_1^i - x_2^i) = (x_1 - x_2) \sum_i a_i \left(\sum_{k=0}^{i-1} x_1^k x_2^{i-k-1} \right).$$

This shows $g_* I \subseteq \mathrm{Ker}(g_* x_1 - g_* x_2) \subseteq \mathrm{Ker}(x_1 - x_2) = I$, as desired. □

Remark 3.2. Let $M = K^*(\Delta(r)) = \mathbb{Z}[x, y]/\left\langle Q_r, Q_{r+1}\right\rangle$. We often use filtration arguments, so let us fix the notation now. Let $M_j \subseteq M$ be the group generated by monomials in x, y of total degree at least j, i.e. $M_j = \mathbb{Z}[x, y]_{\geq j}/\left\langle Q_r, Q_{r+1}\right\rangle$. This makes sense since Q_r, Q_{r+1} are homogeneous. Then $0 = M_{2r} \subseteq M_{2r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$ is a filtration of M.

3.2 Equivariant K-theory of spaces of geodesics

Recall the commutative diagram of fibrations from Prop. 1.6,

$$\begin{tikzcd} & S(\tau) \arrow[d] & \\ BC_n \arrow[r] & ES^1 \times_{S^1} G(r)^{(n)} \arrow[r] \arrow[d] & \Delta(r) \arrow[d] \\ & BS^1 \arrow[r, "B\mathcal{P}_n"] & BS^1 \end{tikzcd}$$

Here the map $B\mathcal{P}_n : BS^1 \longrightarrow BS^1$ is induced by the nth power map $\mathcal{P}_n : S^1 \longrightarrow S^1$, $z \mapsto z^n$, and $C_n \subseteq S^1$ denotes the group of nth roots of unity. Taking the K-theory gives the commutative square

$$\begin{tikzcd} K^*(ES^1 \times_{S^1} G(r)^{(n)}) & K^*(\Delta(r)) \arrow[l] \\ K^*(BS^1) \arrow[u] & K^*(BS^1) \arrow[l, "B\mathcal{P}_n"'] \arrow[u] \end{tikzcd} \tag{26}$$

We see we will need to know the K-theory of classifying spaces in order to proceed, and luckily there is a general theorem due to Atiyah about this, which I will now explain and use. So let G be a compact Lie group. The *representation ring* $R(G)$ is defined as the Groethendieck group completion of the semigroup of representations of G under direct sum. This becomes a ring via the tensor product. We can define a map

$$\begin{array}{rcl} R(G) & \longrightarrow & K^0(BG), \\ V & \mapsto & \{EG \times_G V \searrow BG\}\,; \end{array} \tag{27}$$

and extend by the Groethendieck construction. Define the *augmentation ideal*, $I = I(G) \subseteq R(G)$ by

$$I = \operatorname{Ker}\left\{ R(G) \xrightarrow{\dim} \mathbb{Z} \right\}.$$

We define the completion to be the inverse limit,

$$\widehat{R(G)}_I = \varprojlim_k R(G)/I^k,$$

and can now state the theorem, se [Atiyah2] Thm. 7.2 for G a finite group, and [Atiyah-Hirzebruch] Thm. 4.6 for G a connected compact Lie group:

Theorem 3.3 (Atiyah)**.** *Let G be a compact Lie group. Then*

(i) $K^0(BG) \cong \widehat{R(G)}_I$,

(ii) $K^1(BG) = 0$.

I will now use this theorem to determine $K^*(BS^1)$ and $K^*(BC_n)$.

Lemma 3.4. *Let $T : S^1 \hookrightarrow \mathbb{C}^*$ be the natural 1-dimensional representation of S^1, and let $t = [T] - 1 \in K^0(BS^1)$. Then*

$$R(S^1) = \mathbb{Z}[T, T^{-1}], \quad I = \langle T - 1 \rangle, \quad K^0(BS^1) \cong \widehat{R(S^1)}_I = \mathbb{Z}[[t]].$$

Proof. First note that a representation $\rho : S^1 \longrightarrow GL_n(\mathbb{C})$ can be conjugated to $\rho : S^1 \longrightarrow U(n)$, by choosing an inner product on $\mathbb{C}^n$ (all of which are conjugate) which is S^1-invariant. So it suffices to look at representations $\rho : S^1 \longrightarrow U(n)$. Now $\rho(t) \in U(n)$ (for $t \in [0, 2\pi]$) is diagonizable, $\rho(t) \sim \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$. This also diagonalizes $\rho(kt)$, $k \geq 1$, so if we choose t rationally independent of π, this diagonalization works for a dense subset of S^1. So by continuity we can diagonalize $\rho(t)$ for all t simultaneously, and so ρ is given by $\mathrm{diag}(\rho_1(t), \ldots, \rho_n(t))$, where $\rho_k : S^1 \longrightarrow S^1$ is a homomorphism. This means $\rho_k(z) = z^{m_k}, m_k \in \mathbb{Z}$. Using the natural representation $T : z \mapsto z$, and its inverse $T^{-1} : z \mapsto z^{-1}$, we can reformulate this by saying that every representation of S^1 has the form $\sum_{i=-N}^{N} n_i T^i$, $n_i \geq 0$. The Groethendieck construction yields

$$R(S^1) = \left\{ \sum_{i=-N}^{N} n_i T^i \mid n_i \in \mathbb{Z} \right\} = \mathbb{Z}[T, T^{-1}].$$

Now to the augmentation ideal. By definition

$$I = \left\{ \sum_{i=-N}^{N} n_i T^i \mid \sum_{i=-N}^{N} n_i = 0 \right\}.$$

Clearly, $T - 1 \in I$, and also, $\sum_{i=-N}^{N} n_i T^i \in I$ is divisible by $T - 1$, because the sum of the coefficients is zero. So $I = \langle T - 1 \rangle$. Now $I^k = \left\langle (T-1)^k \right\rangle$, and $R(S^1)/I^k$ has generators $1, T-1, (T-1)^2, \ldots, (T-1)^{k-1}$. Consequently, putting $t = [T] - 1$, we get

$$K^0(BS^1) \cong \widehat{R(S^1)}_I = \mathbb{Z}[[t]].$$

□

Lemma 3.5. *Let $n \in \mathbb{N}$ be a number with prime factorisation $n = \prod_{p|n} p^{i(p)}$. Then*

$$K^0(BC_n) \cong \mathbb{Z} \oplus \bigoplus_{p|n} (\hat{\mathbb{Z}}_p)^{p^{i(p)}-1},$$

where $\hat{\mathbb{Z}}_p$ denotes the p-adic integers.

Proof. Let W be the natural 1-dimensional representation of $C_n \subseteq \mathbb{C}^*$. As in the proof of Lemma 3.4 above, we only need look at representations $\rho : C_n \longrightarrow U(m)$ and diagonalize, so that $\rho = \operatorname{diag}(\rho_1, \dots, \rho_m)$. Here each $\rho_j : C_n \longrightarrow S^1$ is a group homomorphism, and so is a power of W, with the relation $W^n = 1$. Consequently $R(C_n) = \mathbb{Z}[W]/\langle W^n - 1\rangle$. The augmentation ideal is $I = \langle W - 1\rangle$ for the same reason as before, and we must compute the inverse limit $\varprojlim_k R(C_n)/I^k$. This we propose to do in two steps:

First assume $n = p^i$. Then C_{p^i} is a p-group, and according to [Atiyah2] the I-adic and p-adic topologies on $I = I(C_{p^i})$ are equivalent, so that

$$K^0(BC_{p^i}) \cong \widehat{R(C_{p^i})}_I = \mathbb{Z} \oplus \widehat{I(C_{p^i})}_I \cong \mathbb{Z} \oplus \widehat{I(C_{p^i})}_p.$$

To calculate this, let $w = W - 1$, and note that $I(C_{p^i}) = \langle w\rangle$ in the ring $\mathbb{Z}[w]/\left\langle (w+1)^{p^i} = 1\right\rangle$, and so $I(C_{p^i}) \cong \mathbb{Z}^{p^i - 1}$. Thus $\widehat{I(C_{p^i})}_p \cong (\hat{\mathbb{Z}}_p)^{p^i-1}$.

Now take any $n \in \mathbb{N}$. Observe that $C_{p^{i(p)}}$, where $n = p^{i(p)}m$ with $\gcd(p, m) = 1$, are exactly the Sylow p subgroups of C_n. Then by [Atiyah2] Prop. 4.10, there is an injective map

$$K^0(BC_n) \longrightarrow \bigoplus_{p|n} K^0(BC_{p^{i(p)}}),$$

and in particular

$$\widehat{I(C_n)}_{I(C_n)} \longrightarrow \bigoplus_{p|n} \widehat{I(C_{p^{i(p)}})}_{I(C_{p^{i(p)}})}$$

is injective. By using that $C_n \cong \prod_{p|n} C_{p^{i(p)}}$ by the Chinese Remainder Theorem, it is easily seen that this map is an isomorphism, so that

$$K^0(BC_n) \cong \mathbb{Z} \oplus \bigoplus_{p|n} \widehat{I(C_{p^{i(p)}})} \cong \mathbb{Z} \oplus \bigoplus_{p|n} (\hat{\mathbb{Z}}_p)^{p^{i(p)}-1},$$

by the result for p^i above. $\square$

With these results, let us first take a look at the $K^*(BS^1)$-module structure on $K^*(X_{hS^1})$, where X is an S^1-space, as described in Section 1.4. Following the notation in Lemma 3.4, we have the canonical representation T of S^1, which by (27) gives a bundle over BS^1, which we also call T. On K-theory, T defines a class in $K^*(BS^1)$, and $K^*(BS^1) = \mathbb{Z}[[t]]$, where $t = T-1$. Using the projection $\mathrm{pr}_1 : X_{hS^1} \longrightarrow BS^1$, we get classes $\mathrm{pr}_1^*(T)$ and $\mathrm{pr}_1^*(t)$ in $K^*(X_{hS^1})$. We will suppress the map pr_1 from the notation, and simply call these classes T and t again.

We can now determine the $K^*(BS^1)$ module structure on $\Delta(r) \simeq G(r)_{hS^1}$:

Lemma 3.6. *The $K^*(BS^1) = \mathbb{Z}[[t]]$ module structure on $K(\Delta(r))$ is given by $t \mapsto (x-y)/(y+1)$. In particular, t^{2r} acts as* 0.

Proof. We use the results from cohomology, where the $H^*(BS^1) = \mathbb{Z}[u]$ module structure on $H^*(G(r)/S^1) = \mathbb{Z}[x_1, x_2]/\langle Q_r, Q_{r+1}\rangle$ is given by $u \mapsto x_1 - x_2$, cf. [Bökstedt-Ottosen] Cor. 3.7. Recall that $x = [X]-1$, $y = [Y]-1$, where $x_1 = c_1(X)$ and $x_2 = c_1(Y)$ are the first Chern classes. Also $u = c_1(T)$. The first Chern class gives a group isomorphism from complex line bundles over $\Delta(r)$ to $H^2(\Delta(r))$, so since

$$c_1(T \otimes Y) = c_1(T) + c_1(Y) = u + x_2 = x_1 = c_1(X).$$

we get $T \otimes Y = X$. Then we calculate

$$(T-1) \otimes (Y-1) = T \otimes Y - Y - T + 1 = (X-1) - (Y-1) - (T-1)$$

Isolating $T-1$ gives

$$(T-1) = ((X-1) - (Y-1)) \otimes Y^{-1}.$$

In $K^*(\Delta(r))$ this equality gives $t = (x-y)(y+1)^{-1}$, as desired. Since in $K(\Delta(r)) \cong \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle$ all non-zero elements have a total degree in x, y which is less than $2r$, we see that $t^{2r} = (x-y)^{2r}(y+1)^{-2r} = 0$. □

Now we prove the main Theorem of this section, but first we introduce a bit of notation: We write $K^*_{hS^1}(X)$ for $K^*(ES^1 \times_{S^1} X)$, when X is an S^1-space. Recall the diagram (26)

$$\begin{array}{ccc} K^*_{hS^1}(G(r)^{(n)}) & \longleftarrow & K^*(\Delta(r)) \\ \uparrow & & \uparrow \\ K^*(BS^1) & \xleftarrow{B\mathcal{P}_n} & K^*(BS^1) \end{array}$$

This gives a map

$$K^*(BS^1)^{(n)} \otimes_{K^*(BS^1)} K^*(\Delta(r)) \longrightarrow K^*_{hS^1}(G(r)^{(n)})$$

where the $K^*(BS^1)^{(n)}$ denotes that the map $B\mathcal{P}_n$ should be applied in the tensor product, as the diagram indicates.

Theorem 3.7. *Let $n \in \mathbb{N}$. Then the map*

$$K^*(BS^1)^{(n)} \otimes_{R(S^1)} K^*(\Delta(r)) \longrightarrow K^*_{hS^1}(G(r)^{(n)})$$

is an isomorphism of rings. In particular, $K^1_{hS^1}(G(r)^{(n)}) = 0$.

To fix the notation and avoid long, cumbersome expressions, put

$$\begin{aligned} R = R(S^1) &= \mathbb{Z}[U, U^{-1}], & \hat{R} &= K^0(BS^1) = \mathbb{Z}[[u]], \quad u = U - 1. \\ S = R(S^1) &= \mathbb{Z}[T, T^{-1}], & \hat{S} &= K^0(BS^1) = \mathbb{Z}[[t]], \quad t = T - 1. \\ & & M &= K^*(\Delta(r)) = \mathbb{Z}[x, y]/\langle Q_r, Q_{r+1}\rangle . \end{aligned}$$

Here S is an R-module by the map $U \mapsto T^n$, and likewise $\hat{S}$ is an $\hat{R}$-module by $u \mapsto (t+1)^n - 1$. By Lemma 3.6, M is an $\hat{R}$-module by $u \mapsto (x-y)/(1+y)$, and thus an R-module by $U \mapsto (x-y)/(1+y) + 1$.

The Theorem says that $\hat{S}^{(n)} \otimes_R M \cong K_{hS^1}(G(r)^{(n)})$. The reason for restricting to R instead of $\hat{R}$ is given by the following lemma, which also shows that for the isomorphism, this restriction does not matter.

Lemma 3.8. *$\hat{S}$ is a flat R-module, and*

$$\hat{S} \otimes_{\hat{R}} N \cong \hat{S} \otimes_R N.$$

for any finitely generated $\hat{R}$-module N where u^m acts as 0 on N for some m. In particular this holds for the filtration modules M_j from Remark 3.2, for $M = M_{2r+1}$, and for the quotients M_j/M_{j+1}.

Proof. Clearly, S is a free R-module (with basis $\{1, U, \ldots, U^{n-1}\}$), so S is flat over R. Since S is Noetherian, $\hat{S}$ is flat over S, see [Atiyah-MacDonald], Prop. 10.14. By the natural isomorphism, for any R-module M,

$$\hat{S} \otimes_R M \cong \hat{S} \otimes_S S \otimes_R M,$$

we see that $\hat{S}$ is flat over R.

Take N as in the lemma. Then the completion by the ideal $I = \langle u \rangle \subseteq \hat{R}$ gives

$$\hat{N} = \lim_{\overleftarrow{k}} N/u^k N = N.$$

Also by [Atiyah-MacDonald], Prop. 10.13, since R is Noetherian and N is finitely generated, $\hat{N} \cong \hat{R} \otimes_R N$. Combining these two facts yields the isomorphism

$$\hat{S} \otimes_{\hat{R}} N \cong \hat{S} \otimes_{\hat{R}} \hat{N} \cong \hat{S} \otimes_{\hat{R}} (\hat{R} \otimes_R N) \cong \hat{S} \otimes_R N.$$

Now consider the $\hat{R}$-module M_j. Since u acts as $(x-y)/(1+y)$, and M_j consists of polynomials degree at least j, u^{2r+1} acts as zero. For the quotient M_j/M_{j+1}, u itself acts as zero. So the requirements of N holds for these modules. □

We will use the filtration M_j of M to prove the Theorem, so we need a Lemma which proves the Theorem in the case $M = \mathbb{Z}$:

Lemma 3.9. *The following map is an isomorphism:*

$$K^*(BS^1) \otimes_{R(S^1)} \mathbb{Z} \longrightarrow K^*(BC_n).$$

Proof. Let $A = S \otimes_R \mathbb{Z}$ and $B = \hat{S} \otimes_R \mathbb{Z}$, and let $A \longrightarrow B$ be the map induced by the completion $S \longrightarrow \hat{S}$. We now define another map

$$A \longrightarrow R(C_n) = \mathbb{Z}[W]/\langle W^n - 1\rangle, \quad T \mapsto W.$$

This is clearly an isomorphism, and preserves the augmentation ideal. Consider the diagram:

$$\begin{tikzcd} B \arrow[d] & A \arrow[l] \arrow[r, "\cong"] \arrow[d] & R(C_n) \arrow[d] \\ \hat{B} & \hat{A} \arrow[l] \arrow[r, "\cong"] & K^0(BC_n) \end{tikzcd}$$

Here the vertical arrows denote completion with respect to the augmentation ideals; respectively tB, $(T-1)A$, and $\langle W-1 \rangle)$. To prove the Lemma, we must show $B \cong \hat{A}$. First note that $\hat{A} \longrightarrow \hat{B}$ is an isomorphism, since for any k, the map given by $T \mapsto t+1$, is an isomorphism:

$$A/(T-1)^k = \mathbb{Z}[T]/\left\langle T^n - 1, (T-1)^k \right\rangle \longrightarrow \mathbb{Z}[t]/\left\langle (t+1)^n - 1, t^k \right\rangle = B/t^k B.$$

Next we show that $B \longrightarrow \hat{B}$ is an isomorphism. To show this, consider the exact sequence given by multiplication by $u - 1 \in R$,

$$0 \longrightarrow R \xrightarrow{u-1} R \longrightarrow \mathbb{Z} \longrightarrow 0\,.$$

Since $\hat{S}$ is flat over R, we obtain a new exact sequence,

$$0 \longrightarrow \hat{S} \otimes_R R \xrightarrow{1 \otimes (u-1)} \hat{S} \otimes_R R \longrightarrow \hat{S} \otimes_R \mathbb{Z} \longrightarrow 0\,,$$

which, after applying the natural isomorphism, becomes

$$0 \longrightarrow \hat{S} \xrightarrow{(t+1)^n-1} \hat{S} \longrightarrow \hat{S} \otimes_R \mathbb{Z} \longrightarrow 0\,. \tag{28}$$

Completing this with respect to the ideal $\langle t \rangle$, which is an exact functor, we obtain yet another exact sequence

$$0 \longrightarrow \lim_{\leftarrow} \hat{S}/\left\langle t^k \right\rangle \xrightarrow{(t+1)^n-1} \lim_{\leftarrow} \hat{S}/\left\langle t^k \right\rangle \longrightarrow \lim_{\leftarrow}(\hat{S} \otimes_R \mathbb{Z})/\left\langle t^k \right\rangle \longrightarrow 0\,.$$

Recall $\hat{S} = \mathbb{Z}[[t]]$. After applying the isomorphism $\lim_{\leftarrow} \hat{S}/\left\langle t^k \right\rangle \cong \hat{S}$, we get the exact sequence,

$$0 \longrightarrow \hat{S} \xrightarrow{(t+1)^n-1} \hat{S} \longrightarrow \lim_{\leftarrow}(\hat{S} \otimes_R \mathbb{Z})/\left\langle t^k \right\rangle \longrightarrow 0\,. \tag{29}$$

Comparing (28) and (29), we see that $B \cong \hat{B}$. As already noted, this means that $\hat{A} \cong B$, and this proves the result. $\square$

Now we can prove the main Theorem 3.7:

Proof of Theorem 3.7. First we claim that the map

$$K^*(BS^1) \otimes_{\mathbb{Z}} K^*_{hS^1}(G(r)) \longrightarrow K^*_{hS^1}(G(r)^{(n)}) \tag{30}$$

is surjective. To see this, we first note that the map $K^*(BC_n) \longrightarrow K^*(BS^1)$ is surjective. This follows from the fact that the map of representation rings, $R(C_n) \longrightarrow R(S^1)$ is surjective, since any representation of C_n can be extended to a representation of S^1. Now to prove surjectivity of (30), we use a filtration argument in the spectral sequence

$$H^*(\Delta(r); K^*(BC_n)) \Rightarrow K^*_{hS^1}(G(r)^{(n)}).$$

This collapses, since everything sits in even degrees. As in the proof of Theorem 3.1, we now use Cor. 2.5 of [Atiyah-Hirzebruch], so let A denote the image of $K^*(BS^1) \otimes_{\mathbb{Z}} K^*_{hS^1}(G(r))$ in $K^*_{hS^1}(G(r)^{(n)})$. In filtration degree 0 we have $K^*(BC_n)$. As already shown $K^*(BS^1)$ is surjective onto this, so the lowest filtration can be hit. Anything else in $H^*(\Delta(r); K^*(BC_n))$ is generated by monomials $x_1^i x_2^j$, and we have $x^i y^j \in A$ with $\mathrm{ch}(x^i y^j) = x_1^i x_2^j +$ higher terms. This shows that $A = K^*_{hS^1}(G(r)^{(n)})$, so (30) is surjective.

Now we show that the map is injective. We will use a filtration argument, where we filter $M = K^0(S(\tau)/S^1)$ as in Remark 3.2. We look at the exact sequence,

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_i/M_{i+1} \longrightarrow 0.$$

As $\hat{S}$ is flat over R by Lemma 3.8, we get the exact sequence

$$0 \longrightarrow \hat{S} \otimes_R M_{i+1} \longrightarrow \hat{S} \otimes_R M_i \longrightarrow \hat{S} \otimes_R M_i/M_{i+1} \longrightarrow 0. \tag{31}$$

We first apply this to K-theory with $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ coefficients. For the field $\mathbb{F}_p$, we have by the Universal Coefficient Theorem, $K^*(X; \mathbb{F}_p) \cong K^*(X) \otimes \mathbb{F}_p$. Clearly the filtration $M'_i = M_i \otimes \mathbb{F}_p$ works for $\mathbb{F}_p$ coefficients, so we can use the result above. But since $\mathbb{F}_p$ is a field, the exact sequence (31) splits, so we can do a counting argument quite easily. Observe that $M'_i/M'_{i+1} = \mathbb{F}_p[x, y]_i/\langle Q_r, Q_{r-1}\rangle = (\mathbb{F}_p)^{n_i}$, where $n_i \in \mathbb{N}$. By Lemma 3.9, we know

$$\hat{S} \otimes_R M'_i/M'_{i+1} \cong (K^0(BC_n; \mathbb{F}_p))^{n_i}. \tag{32}$$

and in addition, $K^0(BC_n; \mathbb{F}_p)$ is a finite number of copies of $\mathbb{F}_p$, so it makes sense to count them. Also $M'_{2r-1} = \mathbb{F}_p$, so $\hat{S} \otimes_R M'_{2r-1} \cong K^0(BC_n; \mathbb{F}_p)$. So inductively, since $M \otimes \mathbb{F}_p$ is a graded ring with a total of $r(r+1)$ copies of $\mathbb{F}_p$, then

$$\hat{S} \otimes_R M \otimes \mathbb{F}_p \cong (K^0(BC_n; \mathbb{F}_p))^{r(r+1)}.$$

We compare this with $K^*(G(r)^{(n)}; \mathbb{F}_p)$ via the spectral sequence for the vertical fibration in Prop. 1.6:

$$E_2 = H^*(\Delta(r); K^*(BC_n; \mathbb{F}_p)) \Rightarrow K^*(G(r)^{(n)}; \mathbb{F}_p).$$

We see everything sits in even degrees in E_2, so there are no differentials, and, working over a field $\mathbb{F}_p$, we can simply count the dimension of $K^0(G(r)^{(n)}; \mathbb{F}_p)$ as the sum of the dimensions of $E_2^{m,n}$ on the diagonal $m + n = 0$. Since $H^*(\Delta(r); \mathbb{F}_p) \cong \mathbb{F}_p[x, y]/\langle Q_r, Q_{r+1}\rangle$ also has a total of $r(r+1)$ copies of $\mathbb{F}_p$, again by Lemma 3.5, we get,

$$K^0(G(r)^n; \mathbb{F}_p) \cong (K^0(BC_n; \mathbb{F}_p))^{r(r+1)}.$$

So the map of $\mathbb{F}_p$-vector spaces

$$\hat{S} \otimes_R M \otimes \mathbb{Z}_p = K^0(BS^1) \otimes_{R(S^1)} K^0(S(\tau)/S^1; \mathbb{Z}_p) \longrightarrow K^0(S(\tau)^{(p)}_{hS^1}; \mathbb{Z}_p)$$

is a surjection between spaces of the same dimension, and is thus an isomorphism, and this holds for every prime number p.

Now we compare $\mathbb{Z}$- and $\mathbb{F}_p$-coefficients (for a prime p) by the diagram

$$\begin{tikzcd} K^0(\Delta(r)) \arrow[r] & K^0(G(r)^{(n)}; \mathbb{F}_p) \\ \hat{S} \otimes_R M \arrow[r] \arrow[u, "\varphi"] & \hat{S} \otimes_R M \otimes \mathbb{F}_p \arrow[u, "\cong"'] \end{tikzcd} \tag{33}$$

Assume $a \in \hat{S} \otimes_R F$ is in the kernel of φ. Then, by the diagram, a reduced mod p is zero, so $a = p \cdot a_1$ for some a_1. But then, since $K^0(\Delta(r))$ is torsion free, $a_1 \in \ker(\varphi)$, so $a_1 = p \cdot a_2$, etc. Consequently, if $a \in \ker(\varphi)$, then a is divisible by p infinitely often. Recall that this holds for any prime p, and thus also for n, so a is infinitely often divisible by n.

Now take a look at the filtration again

$$0 \longrightarrow \hat{S} \otimes_R M_{i-1} \longrightarrow \hat{S} \otimes_R M_i \longrightarrow \hat{S} \otimes_R M_i/M_{i-1} \longrightarrow 0. \tag{34}$$

If $a \in \hat{S} \otimes_R M_i$ is divisible by n infinitely often, then the image in

$$\hat{S} \otimes_R M_i/M_{i-1} \cong \mathbb{Z}^N \oplus \bigoplus_{p|n} (\hat{\mathbb{Z}}_p)^{N_p}$$

is zero (the isomorphism is Lemma 3.5 and Lemma 3.9). So a comes from a' in $\hat{S} \otimes_R M_{i-1}$ and a' is also infinitely often divisible by n. So inductively a comes from $a_0 \in \hat{S} \otimes_R F_0 \cong \mathbb{Z} \oplus \bigoplus_{p|n} (\hat{\mathbb{Z}}_p)^{p^i-1}$, and a_0 is divisible by n infinitely often, and so $a_0 = 0$, which implies $a = 0$.

This shows that the kernel of φ is zero, and thus the map

$$\varphi : K^0(BS^1) \otimes_{R(S^1)} K^0(S(\tau)/S^1) \longrightarrow K^0(S(\tau)^{(p)}_{hS^1}) \tag{35}$$

is an isomorphism. $\square$

4 The free loop space and Morse theory

Now we turn to study the free loop space $L(\mathbb{F}P^r)$, where as usual $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$. First a definition:

Definition 4.1. Let X be a topological space. The space

$$LX = \{f : [0,1] \longrightarrow X \mid f(0) = f(1), f \text{ is continuous}\},$$

with the compact-open topology, is called the free loop space of X.

We are going to use Morse theory to study LM for a smooth manifold M, where we will take $M = \mathbb{F}P^r$. It is a fact that it does not change the homotopy type of LM if we require all $f \in LM$ to be differentiable, or even smooth, so we do that.

Now let us consider how one could do Morse theory on the free loop space LM as well the space of homotopy orbits LM_{hS^1}, where M denotes a compact n-dimensional manifold. For details, I refer to [Klilngenberg1], and [Bökstedt-Ottosen], especially chapters 7 and 8. LM is not a finite-dimensional manifold, but one can make a model of LM which is a so-called Hilbert manifold, cf. [Klilngenberg1] §1.2, meaning there are charts on LM making it locally homeomorphic to a Hilbert space. The tangent space of a loop $f \in LM$ is the space $\Gamma(f)$ of vector fields along f. Let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric on M. Now the tangent space $T_f LM$ carries the structure of a Hilbert space via

$$\langle \xi, \eta \rangle_c = \int_{S^1} \big(\langle \xi(t), \eta(t) \rangle + c \langle \nabla \xi(t), \nabla \eta(t) \rangle\big) dt, \tag{36}$$

where $\xi, \eta \in T_f LM$ are vector fields along f in LM, and ∇ denotes the covariant derivative along f. The constant $c \in \mathbb{R}$ makes the inner product vary. This is necessary to ensure that the n-fold iteration map, $\mathcal{P}_n$, becomes an isometry

$$\mathcal{P}_n^* = D_f(\mathcal{P}_n) : T_f LM \longrightarrow T_{\mathcal{P}_n f} LM, \quad \mathcal{P}_n^*(\xi(z)) = \xi(z^n)$$

since $\langle \mathcal{P}_n^* \xi, \mathcal{P}_n^* \eta \rangle_1 = \langle \xi, \eta \rangle_{n^2}$, see [Bökstedt-Ottosen] §7.

We are going to do Morse theory via the energy function

$$E : LM \longrightarrow \mathbb{R}, \qquad f \mapsto \int_{S^1} |f'(t)|^2 \, dt.$$

For each $a \in \mathbb{R}$, we set $\mathcal{F}(a) = E^{-1}(]-\infty, a]) \subseteq LM$. The critical points of E are the closed geodesics on M. We shall assume that the critical points are collected on compact submanifolds, each of which satisfy the Bott non-degeneracy condition. This strong condition is needed for the Morse theory machinery, and it is satisfied for $M = \mathbb{F}P^r$, and more generally for symmetric spaces, [Ziller]. Call the critical values $0 = \lambda_0 < \lambda_1 < \ldots$, and consider the filtration

$$\mathcal{F}(\lambda_0) \subseteq \mathcal{F}(\lambda_1) \subseteq \cdots \subseteq LM. \tag{37}$$

This filtration is equivariant with respect to the S^1 action. This means it induces a filtration of LM_{hS^1},

$$\mathcal{F}(\lambda_0)_{hS^1} \subseteq \mathcal{F}(\lambda_1)_{hS^1} \subseteq \cdots \subseteq LM_{hS^1}. \tag{38}$$

The non-degeneracy condition ensure that each critical submanifold $N(\lambda) = E^{-1}(\lambda)$ is finite-dimensional, and the tangent bundle $T(LM)|_{N(\lambda)} \subseteq T(LM)$ splits S^1-equivariantly:

$$T(LM)|_{N(\lambda)} \cong \mu^-(\lambda) \oplus \mu^0(\lambda) \oplus \mu^+(\lambda),$$

into the bundles of negative, zero-, and positive directions, respectively, and the negative bundle $\mu^-(\lambda)$ is finite-dimensional. To ease the notation, write $\mathcal{F}_n = \mathcal{F}(\lambda_n)$ and $\mu_n^- = \mu^-(\lambda_n)$. The main result of Morse theory in this setting is proved by Klingenberg in [Klilngenberg1], §2.4: There is an S^1-equivariant homotopy equivalence

$$\mathcal{F}_n)/\mathcal{F}_{n-1} \simeq Th(\mu_n^-). \tag{39}$$

We want a similar result for $(LM)_{hS^1}$, so we consider the quotients of the filtration (38):

$$ES^1 \times_{S^1} \mathcal{F}_n / ES^1 \times_{S^1} \mathcal{F}_{n-1} \cong ES^1_+ \wedge_{S^1} \mathcal{F}_n/\mathcal{F}_{n-1},$$

where $ES^1_+ \wedge_{S^1} X = (ES^1_+ \wedge X)/S^1$ is the smash product modded out by the diagonal S^1 action. The obvious map defined on representatives is a homeomorphism. Thus by the Morse theorem in (39),

$$ES^1 \times_{S^1} \mathcal{F}_n / ES^1 \times_{S^1} \mathcal{F}_{n-1} \simeq ES^1_+ \wedge_{S^1} Th(\mu_n^-).$$

We can use [Bökstedt-Ottosen] Lemma 5.1 to find that

$$(\mathcal{F}_n)_{hS^1}/(\mathcal{F}_{n-1})_{hS^1} \simeq ES^1_+ \wedge_{S^1} Th(\mu_n^-) \cong Th((\mu_n^-)_{hS^1}) \tag{40}$$

where for an S^1-vector bundle ξ given by a projection map $p : E \longrightarrow B$ we denote by ξ_{hS^1} the bundle with projection $\mathrm{id} \times p : ES^1 \times_{S^1} E \longrightarrow ES^1 \times_{S^1} B$. This means we also have a Morse theorem for the S^1-equivariant filtration.

4.1 The negative bundle

In [Bökstedt-Ottosen] Lemma 5.1, it is shown that the negative bundle $(\mu_n^-)_{hS^1}$ is an oriented vector bundle if μ_n^- is. But to use the Thom isomorphism in K-theory we need to know that the negative bundle is complex, or more precisely

Proposition 4.2. *The negative bundle μ_n^- for the energy filtration of $L\mathbb{C}P^r$ can be written as $\varepsilon \oplus \nu$, where ε is a trivial real S^1-line bundle, and ν is a complex S^1 vector bundle. Consequently, the negative bundle $(\mu_n^-)_{hS^1}$ for the energy filtration of $L\mathbb{C}P^r{}_{hS^1}$ can also be written as $\varepsilon \oplus \nu_{hS^1}$.*

Proof. There is a Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $T\mathbb{C}P^r$, and the Riemannian metric is $\langle \cdot, \cdot \rangle = \mathrm{Re}(\langle \cdot, \cdot \rangle_{\mathbb{C}})$. The tangent space $T_f L\mathbb{C}P^r$ is a complex vector space, and it carries the structure of a Hilbert space via

$$\langle \xi, \eta \rangle = \int_{S^1} \left(\langle \xi(t), \eta(t) \rangle + \langle \nabla \xi(t), \nabla \eta(t) \rangle \right) dt,$$

where $\xi, \eta \in T_f L\mathbb{C}P^r$ are vector fields along f in $L\mathbb{C}P^r$, and ∇ denotes the covariant derivative along f. Since $\langle \cdot, \cdot \rangle = \mathrm{Re}(\langle \cdot, \cdot \rangle_{\mathbb{C}})$, we get

$$\langle z\xi, z\eta \rangle = \langle \xi, \eta \rangle \quad \text{for } z \in S^1. \tag{41}$$

If f is a critical point of the energy functional E (a geodesic), then the tangent space of $L\mathbb{C}P^r$ splits as

$$T_f L\mathbb{C}P^r = \Gamma(\mathbb{R}f') \oplus \Gamma(\mathbb{R}if') \oplus \Gamma((f')^\perp) \tag{42}$$

where e.g. $\Gamma(\mathbb{R}f') \subseteq \Gamma(f)$ denotes the vector fields ξ along f with $\xi(t) \in \mathbb{R}f'(t) \subseteq T_{f(t)}\mathbb{C}P^r$. We can use the inner product to represent the Hessian $H = D^2E$ of E by a linear operator $A = A_f$ on $T_f L\mathbb{C}P^r$, by requiring $\langle A\xi_1, \xi_2 \rangle = H(\xi_1, \xi_2)$. Then we get by (41) that $\bar{z}Az = A$ for $z \in S^1$, which implies that A is complex linear.

According to Klingenberg, [Klilngenberg1] Thm. 2.4.2,

$$A_f = \mathrm{id} - (1 - \nabla^2)^{-1} \circ (\tilde{K}_f + 1),$$

where

$$\begin{aligned} \tilde{K}_f(\xi)(t) &= R(\xi(t), f'(t))f'(t) \\ &= \pi^2 \left(f'(t)\langle \xi(t), f'(t) \rangle - 2f'(t)\langle f'(t), \xi(t) \rangle + \xi(t)\langle f'(t), f'(t) \rangle \right) \end{aligned}$$

Note the factor π^2; it appears because our metric on $\mathbb{C}P^r$ is scaled so that the "circumference" is 1, not π. This gives us the following eigenvalue equation $A\xi = \lambda\xi$:

$$(\lambda - 1)\nabla^2 \xi = (\tilde{K}_f + \lambda)\xi \tag{43}$$

The negative bundle consists of solutions to this equation with $\lambda < 0$. Notice that by the formula for A, it preserves the decomposition (42), since covariant derivative commutes with the complex structure on $T\mathbb{C}P^r$. Thus we can solve (43) in the three spaces separately.

(i) $\xi \in \Gamma(\mathbb{R}f')$: Then $\xi(t) = g(t)f'(t)$ where $g : [0,1] \longrightarrow \mathbb{R}$ is a smooth function with $g(0) = g(1)$. Then $\tilde{K}_f(t) = 0$, and equation (43) becomes

$$(\lambda - 1)g'' = \lambda g \quad \Leftrightarrow \quad g'' = \tfrac{\lambda}{\lambda-1} g \quad \Rightarrow \quad g = 0$$

since $\lambda < 0$ and g must be periodic. So we have no non-trivial solutions.

(ii) $\xi \in \Gamma((f')^\perp)$: Since $(f')^\perp$ is a complex vector space, and A is complex linear as noted, we see that $A\xi = \lambda\xi$ implies $A(i\xi) = \lambda(i\xi)$. So this space of solutions has a complex structure.

(iii) $\xi \in \Gamma(\mathbb{R}if')$: Then $\xi(t) = g(t)if'(t)$, where $g : [0,1] \longrightarrow \mathbb{R}$ is a smooth function with $g(0) = g(1)$. Then $\tilde{K}_f(t) = 4\pi^2 \left\| f'(t) \right\|^2 \xi(t) = 4\pi^2 n^2 \xi(t)$, since f is a geodesic of length n. The equation (43) then becomes

$$(\lambda - 1)g'' = (4\pi^2 n^2 + \lambda)g \quad \Leftrightarrow \quad g'' = \frac{4\pi^2 n^2 + \lambda}{\lambda - 1} g$$

To get a periodic solution g, we must have $\frac{4\pi^2 n^2 + \lambda}{\lambda - 1} \leq 0$, i.e. $\lambda \geq -4\pi^2 n^2$. For $\lambda = -4\pi^2 n^2$ we must have g constant, and this gives the trivial real line bundle ε. If $-4\pi^2 n^2 < \lambda < 0$ we have the solution set spanned over $\mathbb{R}$ by

$$g_1^K(t) = \cos(K \cdot 2\pi t), \text{ and } g_2^K(t) = \sin(K \cdot 2\pi t), \quad t \in [0,1]$$

where

$$K = \sqrt{-\frac{4\pi^2 n^2 + \lambda}{2\pi(\lambda - 1)}}, \quad \text{and } K \in \mathbb{N},$$

since the functions must be periodic with period 1. This happens if and only if

$$\lambda = \frac{4\pi^2(K^2 - n^2)}{4\pi^2 K^2 + 1},$$

so for a fixed n we get solutions with $\lambda < 0$ for $K = 1, \dots, n-1$. This space of solutions can be given a complex structure J by rotating $t \mapsto t - \frac{1}{4K}$, where $t \in [0,1]$, i.e.

$$J(g_1^K) = g_2^K, \quad J(g_2^K) = -g_1^K.$$

and extending linearly. Clearly J satisfies $J^2 = -\mathrm{id}$.

This gives the bundle ν, which is clearly an S^1 bundle, with the S^1 action given by rotation.

Now let us see that the result for μ_n^- implies that for $(\mu_n^-)_{hS^1}$. The bundle $(\mu_n^-)_{hS^1}$ is defined so that the pullback of $(\mu_n^-)_{hS^1}$ agrees with $\mathrm{pr}^*(\mu_n^-)$ in the following diagram,

$$\begin{tikzcd} \mu_n^- \arrow[d] & \mathrm{pr}^*(\mu_n^-) \arrow[l] \arrow[r] \arrow[d] & (\mu_n^-)_{hS^1} \arrow[d] \\ G_n(r) & ES^1 \times G_n(r) \arrow[l, "\mathrm{pr}"'] \arrow[r] & ES^1 \times_{S^1} G_n(r) \end{tikzcd}$$

where $G_n(r)$ denotes the space of n-times iterated geodesics. Since $\mu_n^- = \varepsilon \oplus \nu$ is a decomposition in S^1-bundles, we automatically get the decomposition for $(\mu_n^-)_{hS^1}$. $\square$

4.2 The power map

We consider the nth power map $\mathcal{P}_n : L\mathbb{F}P^r \longrightarrow L\mathbb{F}P^r$, which iterates a loop n times: For $f : S^1 \longrightarrow L\mathbb{F}P^r$, $\mathcal{P}_n(f)(z) = f(z^n)$ for $z \in S^1 \subseteq \mathbb{C}$. When restricting to the energy filtration, we get $\mathcal{P}_n : \mathcal{F}_i \longrightarrow \mathcal{F}_{ni}$, which gives diagrams

$$\begin{tikzcd} \mathcal{F}_i \arrow[r] \arrow[d, "\mathcal{P}_n"] & \mathcal{F}_{i+1} \arrow[r] \arrow[d, "\mathcal{P}_n"] & \mathcal{F}_{i+1}/\mathcal{F}_i \arrow[d, "\mathcal{P}_n"] \\ \mathcal{F}_{ni} \arrow[r] & \mathcal{F}_{n(i+1)} \arrow[r] & \mathcal{F}_{n(i+1)}/\mathcal{F}_{ni} \end{tikzcd}$$

We now compare this to the n-twisted action of S^1 on $\mathcal{F}_i$. We see that we get an S^1-equivariant map $\mathcal{P}_n : \mathcal{F}_i^{(n)} \longrightarrow \mathcal{F}_{ni}$, and consequently a diagram of S^1-maps

$$\begin{tikzcd} \mathcal{F}_i^{(n)} \arrow[r] \arrow[d, "\mathcal{P}_n"] & \mathcal{F}_{i+1}^{(n)} \arrow[r] \arrow[d, "\mathcal{P}_n"] & \mathcal{F}_{i+1}^{(n)}/\mathcal{F}_i^{(n)} \arrow[d, "\mathcal{P}_n"] \\ \mathcal{F}_{ni} \arrow[r] & \mathcal{F}_{n(i+1)} \arrow[r] & \mathcal{F}_{n(i+1)}/\mathcal{F}_{ni} \end{tikzcd} \tag{44}$$

In particular when $i = 0$, since the action on $\mathcal{F}_0$ is trivial, we get a map

$$\mathcal{P}_n : \mathcal{F}_1^{(n)}/\mathcal{F}_0 \longrightarrow \mathcal{F}_n/\mathcal{F}_0. \tag{45}$$

We can compose with the inclusion map $\mathcal{F}_n \longrightarrow \mathcal{F}_\infty$ to get

$$\mathcal{P}_n : \mathcal{F}_1^{(n)}/\mathcal{F}_0 \longrightarrow \mathcal{F}_\infty/\mathcal{F}_0. \tag{46}$$

This will be very useful in section 6.

4.3 The Morse theory spectral sequence

To avoid excessive use of parentheses, write $L\mathbb{F}P^r{}_{hS^1}$ for $(L(\mathbb{F}P^r))_{hS^1}$. To prove convergence of the Morse spectral sequences, we will need the following:

Lemma 4.3. *Given k, there is m such that the inclusions $\mathcal{F}_m \longrightarrow L\mathbb{F}P^r$ and $(\mathcal{F}_m)_{hS^1} \longrightarrow L\mathbb{F}P^r{}_{hS^1}$ induce isomorphism on π_j and H_j, for all $j \leq k$.*

Proof. First we show that the homology groups of LM and LM_{hS^1} are finitely generated in each degree when $M = \mathbb{F}P^r$ (we say LM and LM_{hS^1} are of finite type): By Serre's spectral sequence for the fibration $\Omega M \longrightarrow PM \longrightarrow M$ we see that ΩM is of finite type, and then the spectral sequence for the fibration $\Omega M \longrightarrow LM \longrightarrow M$ shows that LM is of finite type. The fibration $LM \longrightarrow LM_{hS^1} \longrightarrow BS^1$ then shows LM_{hS^1} is of finite type. For the filtration spaces $\mathcal{F}_m$, $(\mathcal{F}_m)_{hS^1}$, we can use the same fibrations if we restrict the spaces LM, ΩM, PM to curves of maximal energy m^2. The same argument works for homotopy groups, using the long exact sequence for a fibration instead of Serre's spectral sequence.

We first show the lemma for homology groups. Write $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ to cover both situations, $\mathcal{F}_i \subseteq L\mathbb{F}P^r$ and $(\mathcal{F}_i)_{hS^1} \subseteq L\mathbb{F}P^r{}_{hS^1}$. Let k be given, and consider numbers m, M with $k \leq m \leq M$, and with the following properties:

(*i*) $H_k(X_m) \longrightarrow H_k(X)$ is surjective.

(*ii*) $\mathrm{Ker}(H_k(X_m) \longrightarrow H_k(X)) = \mathrm{Ker}(H_k(X_m) \longrightarrow H_k(X_M))$.

A simplex $\Delta^k \longrightarrow X$ is compact, so it has finite energy. Take m such that m^2 is bigger than the maximum energy over the finitely many generators of $H_k(X)$, then the inclusion $X_m \longrightarrow X$ induces a surjective map on H_k. We see we can chose m as in (*i*). Given this m, we consider $\mathrm{Ker}(H_k(X_m) \longrightarrow H_k(X))$, which is finitely generated, since $H_k(X_m)$ is. Such a generator is a formal sum of simplices $\Delta^k \longrightarrow X_m$, which, when included in X, is the boundary of some formal sum of $(k+1)$-simplices. Again by compactness, these have finite energy, and we can choose $M \geq m$ as desired.

Consider a pair (X_{i+1}, X_i) in the chain $X_m \longrightarrow X_{m+1} \longrightarrow \cdots \longrightarrow X_M$. By Morse theory we know the quotient X_{i+1}/X_i is homotopy equivalent to the Thom space of a bundle of dimension at least $2ri$, and such a Thom space can be given the cell structure with one 0-cell, and all other cells of dimension at least $2ri$. So by cellular homology, the relative homology groups satisfy:

$$H_j(X_{i+1}, X_i) = 0, \text{ for } j < 2ri. \tag{47}$$

Then by the long exact sequence for homology groups, the maps $H_k(X_i) \longrightarrow H_k(X_{i+1})$ are isomorphisms, since $k \leq m \leq 2ri - 2$ for $i \geq m$. This means $H_k(X_m) \xrightarrow{\cong} H_k(X_M)$, so by (ii), the map $H_k(X_m) \longrightarrow H_k(X)$ is injective, and thus by (i) an isomorphism.

To show the Lemma for homotopy groups, do the same for π_j in place of H_j. Use Hurewicz on (47) to get $\pi_j(X_{i+1}, X_i) = 0$ for $j < 2ri$, then conclude as above. □

We now state the result about Morse spectral sequences. In cohomology, we need both the S^1-equivariant and the non-equivariant case, but in K-theory we need only the S^1-equivariant case:

Theorem 4.4. *There are convergent spectral sequences in cohomology,*

$$\begin{aligned} E_s^{n,q}(\mathcal{M})(L\mathbb{H}P^r) &\Rightarrow H^{n+q}(L\mathbb{H}P^r) \\ E_s^{n,q}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1}) &\Rightarrow H^{n+q}(L\mathbb{H}P^r{}_{hS^1}) \end{aligned}$$

with E_1 pages given by, for $n \geq 1$, respectively,

$$\begin{aligned} E_1^{n,q} \cong \tilde{H}^{n+q}(Th(\mu_n^-)) &\cong H^{n+q-(4r+2)n+4r-1}(G_n(\mathbb{H}P^r)), \\ E_1^{n,q} \cong \tilde{H}^{n+q}(Th(\mu_n^-)_{hS^1}) &\cong H^{n+q-(4r+2)n+4r-1}(G_n(\mathbb{H}P^r)), \end{aligned}$$

and for $n = 0$, $E^{0,q} = H^q(\mathbb{H}P^r)$ and $E^{0,q} = H^q(BS^1 \times \mathbb{H}P^r)$, respectively.

There is a strongly convergent spectral sequence in K-theory,

$$E_s^{n,q}(\mathcal{M})(L\mathbb{C}P^r{}_{hS^1}) \Rightarrow K^{n+q}(L\mathbb{C}P^r{}_{hS^1})$$

with E_1 page given by $E_1^{0,q} = K^q(BS^1) \otimes K^q(\mathbb{C}P^r)$, and

$$E_1^{n,q} \cong \tilde{K}^{n+q}(Th(\mu_n^-)_{hS^1}) \cong K^{n+q-2r(n-1)-1}(G_n(\mathbb{C}P^r)_{hS^1}), \quad \textit{for } n \geq 1,$$

where $G_n(\mathbb{F}P^r)$ denotes the space of geodesics of length n for $n \geq 1$.

Proof. A closed, simple geodesic has energy 1, and when iterated n times has energy n^2. So the critical values are $0 < 1^2 < 2^2 < 3^2 < \ldots$, and we denote $\mathcal{F}(n^2)$ by $\mathcal{F}_n$. Using the energy filtrations (37) and (38), respectively, we make an exact couple via the long exact sequences for the pair $(\mathcal{F}_n, \mathcal{F}_{n-1})$, and $((\mathcal{F}_n)_{hS^1}, (\mathcal{F}_{n-1})_{hS^1})$, respectively. For details about this process, the reader can see e.g. [Hatcher2], §1.1. This gives rise to a spectral sequence $\{E_r^{p,q}(\mathcal{M})\}_r$, which we call a Morse spectral sequence. The process which constructs a spectral sequence from the exact pairs works for any cohomology theory, so we get spectral sequences in both cohomology and K-theory. By construction together with the homotopy equivalences from Morse theory, (39) and (40), the E_1 page is given by, for $n \geq 1$,

$$\begin{aligned} E_1^{n,q}(\mathcal{M})(LM) &= \tilde{H}^{n+q}(\mathcal{F}_n/\mathcal{F}_{n-1}) \cong \tilde{H}^{n+q}(Th(\mu_n^-)); \\ E_1^{n,*}(\mathcal{M})(LM_{hS^1}) &= \tilde{H}^*((\mathcal{F}_n)_{hS^1}/(\mathcal{F}_{n-1})_{hS^1}) \cong \tilde{H}^*(Th(\mu_n^-)_{hS^1}); \end{aligned}$$

and similar for K-theory. The negative bundle μ_n^- is a bundle over the critical submanifold $N(n^2)$, which is the space $G_n(r)$ of geodesics of length n. It follows that $(\mu_n^-)_{hS^1}$ is a bundle over $G_n(\mathbb{F}P^r)_{hS^1}$.

For $n = 0$, $\mathcal{F}_0$ is space of loops of energy zero, i.e. the constant loops, so $\mathcal{F}_0 = \mathbb{F}P^r$ itself, and the S^1 action is trivial, so $ES^1 \times_{S^1} \mathcal{F}_0 = BS^1 \times \mathbb{F}P^r$. The result follows for $n = 0$.

Now let $n \geq 1$, and consider first $\mathbb{H}P^r$. The negative bundle μ_n^- is found in [Bökstedt-Ottosen2], Thm. 6.2, and here one can see it is oriented and has dimension $(4r+2)n - 4r + 1$. By [Bökstedt-Ottosen] Lemma 5.2, $(\mu_n^-)_{hS^1}$ is also oriented. So we can use the Thom isomorphism, which gives:

$$\begin{aligned} E_1^{n,q}(\mathcal{M})(L\mathbb{H}P^r) &\cong \tilde{H}^{n+q}(Th(\mu_n^-)) \cong H^{n+q-(4r+2)n+4r-1}(G_n(\mathbb{H}P^r)); \\ E_1^{n,q}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1}) &\cong \tilde{H}^{n+q}(Th(\mu_n^-)_{hS^1}) \cong H^{n+q-(4r+2)n+4r-1}(G_n(\mathbb{H}P^r)_{hS^1}); \end{aligned}$$

Similarly for K-theory, but here we use Prop. 4.2 to get that the bundles μ_n^- and $(\mu_n^-)_{hS^1}$ are both the sum of a trivial real line bundle with a complex bundle. This means we can use the Thom isomorphism for K-theory. From [Bökstedt-Ottosen2] Thm. 6.1, we see that the negative bundles μ_n^- and $(\mu_n^-)_{hS^1}$ have dimension $2r(n-1)+1$ for $n \geq 1$.

For the convergence, note that the cohomology Morse spectral sequence is a first quadrant spectral sequence. By [Hatcher2] Prop. 1.2 the criterion for convergence is that the inclusions $\mathcal{F}_n \hookrightarrow L\mathbb{H}P^r$, resp. $(\mathcal{F}_n)_{hS^1} \hookrightarrow L\mathbb{H}P^r{}_{hS^1}$, induce isomorphism on $H^q(-;\mathbb{F}_p)$ if n is large enough compared to q. By the universal coefficient theorem it suffices to show this on $H_q(-;\mathbb{F}_p)$, and this is proved in Lemma 4.3.

The K-theory Morse spectral sequence is not first quadrant, so the convergence question is more subtle. Note that, if we take a finite filtration $(\mathcal{F}_0)_{hS^1} \subseteq \cdots \subseteq (\mathcal{F}_n)_{hS^1}$, the corresponding Morse spectral sequence converges to $K^*((\mathcal{F}_n)_{hS^1})$. The Morse spectral sequence then determines the inverse limit of the $K^*((\mathcal{F}_n)_{hS^1})$. There is a surjective map

$$K^*(L\mathbb{C}P^r{}_{hS^1}) \longrightarrow \varprojlim_n K^*((\mathcal{F}_n)_{hS^1}),$$

and we say say the spectral sequence converges strongly, if this map is an isomorphism. This requires some work, and will be shown in the lemmas below. $\square$

To show convergence of the Morse spectral sequence in K-theory, let $X_0 \subset X_1 \subset \ldots$, and $X = \cup X_i$. We want to find conditions that ensure

$$i : K^*(X) \xrightarrow{\cong} \varprojlim_i K^*(X_i) \tag{*}$$

when $X = L\mathbb{C}P^r{}_{hS^1}$. As mentioned in the proof above, the map is i surjective, so the question is injectivity.

Lemma 4.5. *Let $X = ES^1 \times_{S^1} L\mathbb{C}P^r$. Let X_n denote the n-skeleton of X. Then $(^*)$ holds.*

Proof. First note that the lemma is equivalent to saying that the Atiyah-Hirzebruch spectral sequence for X converges strongly. We have $K^0(X) = [X, \mathbb{Z} \times BU]$ and $K^1(X) = [X, U]$, so a class in K-theory can be considered a (homotopy class of a) map from X to either $Y = \mathbb{Z} \times BU$ or $Y = U$. A class in the kernel of i is then a map $X \longrightarrow Y$ whose restriction to each X_n is null-homotopic. Such a map is called a *phantom map*, and we denote by $\mathrm{Ph}(X, Y)$ the set of homotopy classes of phantom maps $X \longrightarrow Y$. Their existence is studied in [McGibbon-Roitberg], who give the following criterion (Thm. 1): The following are equivalent:

(i) $\mathrm{Ph}(X, Y) = 0$ for every Y with finitely generated homotopy groups.

(ii) There exists a map from ΣX to a wedge of spheres that induces an isomorphism in rational homology.

A map as in (ii) we call a rational equivalence. Note that $\mathbb{Z} \times BU$ and U have finitely generated homotopy groups. Let us apply this to $X = ES^1 \times_{S^1} Z$, where we will specialize to $Z = L\mathbb{C}P^r$.

First we consider the bundle $\xi = p^*T$ over X, the pullback of the standard line bundle $T \longrightarrow BS^1$ under the map $p : ES^1 \times_{S^1} Z \longrightarrow BS^1$. We use the cofiber sequence

$$S(\xi) \longrightarrow D(\xi) \longrightarrow Th(\xi) \longrightarrow \Sigma S(\xi). \tag{48}$$

We claim it suffices to show the result for $Th(\xi)$ instead of X: $K^*(X) \cong K^*(Th(\xi))$ by Thom isomorphism, and the cell structure on X gives rise to a natural cell structure on $Th(\xi) \searrow X$, where n-cells in X correspond to $(n+2)$-cells in $Th(\xi)$. So we also get an isomorphism of the inverse systems $\{K^*(X_n)\}$ and $\{K^*(Th(\xi)_n)\}$ such that the obvious diagram commutes:

$$\begin{tikzcd}
K^*(X) \arrow[r, "i"] \arrow[d, "\cong"] & \varprojlim K^*(X_n) \arrow[d, "\cong"] \\
K^*(Th(\xi)) \arrow[r, "i"] & \varprojlim K^*(Th(\xi)_n)
\end{tikzcd}$$

So we investigate (48). We have of course $D(\xi) \simeq X = ES^1 \times_{S^1} Z$, and we will show that $S(\xi) \cong ES^1 \times Z$: First note

$$S(\xi) = \left\{ ([e,z],t) \in ES^1 \times_{S^1} Z \times T \mid \|e\| = 1, \|t\| = 1, t \in \mathrm{span}_{\mathbb{C}} e \right\},$$

where we consider $e \in ES^1 = S^\infty \subseteq \mathbb{C}^\infty$ and $t \in T \subseteq \mathbb{C}^\infty$, by viewing $BS^1 = \mathbb{C}P^\infty$ as complex lines in $\mathbb{C}^\infty$. For $([e,z],t) \in S(\xi)$, we see that there is $s \in S^1$ with $es = t$. We can then construct a homeomorphism

$$F : S(\xi) \longrightarrow ES^1 \times Z, \quad F([e,z],t) = (t, s^{-1}z). \tag{49}$$

This is well-defined, with inverse $G(t,z) = ([t,z],t)$.

Now let $Z = L\mathbb{C}P^r$. By [Bökstedt-Ottosen2] Theorem 6.1, there is a homotopy equivalence $\Sigma L\mathbb{C}P^r \longrightarrow \Sigma(\mathbb{C}P^r) \vee \bigvee_i \Sigma Th(\mu_i^-)$, which is the splitting result for the non-equivariant case. So clearly, the Atiyah-Hirzebruch spectral sequence converges in this case, i.e. there are no phantom maps from $L\mathbb{C}P^r$, so by the criterion, there is rational equivalence from $\Sigma L\mathbb{C}P^r$ to a wedge of spheres. Since $S(\xi) \cong ES^1 \times L\mathbb{C}P^r \simeq L\mathbb{C}P^r$, we see that we have a rational equivalence f_2 from $\Sigma S(\xi)$ to a wedge of spheres. By (48) this gives a map from $Th(\xi)$ to a wedge of spheres,

$$Th(\xi) \xrightarrow{f_1} \Sigma(\xi) \xrightarrow{f_2} \bigvee_i S^{n_i} . \tag{50}$$

Let us consider the inclusion $L\mathbb{C}P^r \longrightarrow ES^1 \times_{S^1} L\mathbb{C}P^r$. One can investigate this map on rational cohomology using Serre's spectral sequence for the fibration $L\mathbb{C}P^r \longrightarrow ES^1 \times_{S^1} L\mathbb{C}P^r \longrightarrow BS^1$. This is done in [Bökstedt-Ottosen] Prop. 15.2, and it emerges that $E_\infty = E_3$ with all non-trivial groups in either $E_3^{0,*} \subseteq E_2^{0,*} = H^*(L\mathbb{C}P^r;\mathbb{Q})$ or $E_3^{*,0} = H^*(BS^1;\mathbb{Q})$. This implies that the combined map

$$\tilde{H}^*(L\mathbb{C}P^r;\mathbb{Q}) \oplus \tilde{H}^*(BS^1;\mathbb{Q}) \longrightarrow \tilde{H}^*(ES^1 \times_{S^1} L\mathbb{C}P^r;\mathbb{Q}) \tag{51}$$

is surjective.

In (48), use the homotopy equivalences $S(\xi) \cong ES^1 \times L\mathbb{C}P^r$ and $D(\xi) \simeq ES^1 \times_{S^1} L\mathbb{C}P^r$, and project on the first factor to get

$$\begin{tikzcd}
S(\xi) \arrow[r] \arrow[d] & D(\xi) \arrow[r] \arrow[d] & Th(\xi) \arrow[d] \\
ES^1 \arrow[r] & BS^1 \arrow[r] & BS^1/ES^1 \simeq BS^1
\end{tikzcd}$$

which gives a map $g_1 : Th(\xi) \longrightarrow BS^1$. Note that the Atiyah-Hirzebruch spectral sequence for BS^1 converges, so by the criterion, there is a rational equivalence $g_2 : \Sigma BS^1 \longrightarrow \bigvee_j S^{n_j}$.

Combining with (50), we can make a composite map

$$\varphi:\ \Sigma Th(\xi) \xrightarrow{\Delta} \Sigma Th(\xi) \vee \Sigma Th(\xi) \xrightarrow{f_1 \vee g_1} \Sigma^2 S(\xi) \vee \Sigma BS^1 \xrightarrow{f_2 \vee g_2} \bigvee_k S^{n_k}$$

Here $f_2 \vee g_2$ is a rational equivalence, and by (51), $\Delta^* \circ (f_1 \vee f_2)^*$ is surjective on reduced cohomology with rational coefficients. So the composite map φ^* is surjective on rational cohomology, and by collapsing some of the spheres, we can ensure it becomes injective. We have constructed the desired rational equivalence. $\square$

Lemma 4.6. *If X_i is a sequence of subcomplexes of the CW complex $X = L\mathbb{C}P^r{}_{hS^1}$, and if for every k there is an m such that the k-skeleton $\mathrm{Sk}^k(X) \subseteq X_m$, then condition $(^*)$ applies.*

Proof. We must show that the map

$$K^*(X) \longrightarrow \varprojlim_i K^*(X_i)$$

is injective. Let a be in the kernel of this map. Because of our condition on the filtration, a will restrict trivially to each skeleton. Then Lemma 4.5 shows that a vanishes. $\square$

Now consider the general case. By Lemma 4.3, the condition on π_j is satisfied for $X = L\mathbb{C}P^r{}_{hS^1}$.

Lemma 4.7. *If X_i is a sequence of subspaces of X as above, and if for every k there is an m such that $\pi_j(X_m) \to \pi_j(X)$ is an isomorphism for $j \leq k$, then condition $(^*)$ applies.*

Proof. Using relative CW approximation (see [Hatcher1] Prop. 4.13), we can inductively construct a sequence of CW complexes Y_i such that the following ladder commutes,

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & \cdots \\ X_0 \arrow[r] & X_1 \arrow[r] & \cdots \end{tikzcd}$$

and such that the vertical maps are weak homotopy equivalences. Furthermore, for a given k we have by assumption that there is m such that $\pi_j(X_m) \to \pi_j(X)$ is an isomorphism for $j \leq k$, and this means we can ensure that all Y_n for $n \geq m$ are constructed from Y_{n-1} by adding cells of dimension greater than k. So letting $Y = \cup_i Y_i$, we have that for each k there is an m such that $\mathrm{Sk}^k(Y) \subseteq Y^m$.

The map $Y \to X$ is a weak homotopy equivalence. Noting that a weak homotopy equivalence preserves K-theory, the lemma follows from the previous one. $\square$

5 S^1-equivariant cohomology of $L\mathbb{H}P^r$

5.1 The Morse spectral sequences

For $L\mathbb{H}P^r{}_{hS^1}$, the Morse spectral sequence is as follows:

Theorem 5.1. *The Morse spectral sequence $E_r^{*,*}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1})$ is a spectral sequence of $H^*(BS^1;\mathbb{F}_p) = \mathbb{F}_p[u]$-modules, and it has the following E_1 page: Assume $p \mid r+1$. Then*

$$\begin{array}{lcll} E_1^{0,*} & = & \mathbb{F}_p[u,y]/\langle y^{r+1}\rangle ; & \\ E_1^{pm+k,*} & = & \alpha_{pm+k}\mathbb{F}_p[u,t]/\langle Q_r, Q_{r+1}\rangle , & \text{for } m \geq 0,\ 1 < k < p-1; \\ E_1^{pm,*} & = & \alpha_{pm}\mathbb{F}_p[u]\{1, y, \dots, y^r, \sigma, \dots, \sigma y^r\} & \text{for } m \geq 1. \end{array}$$

Assume $p \nmid r+1$. Then

$$\begin{array}{lcll} E_1^{0,*} & = & \mathbb{F}_p[u,y]/\langle y^{r+1}\rangle ; & \\ E_1^{pm+k,*} & = & \alpha_{pm+k}\mathbb{F}_p[u,t]/\langle Q_r, Q_{r+1}\rangle , & \text{for } m \geq 0,\ 1 < k < p-1; \\ E_1^{pm,*} & = & \alpha_{pm}\mathbb{F}_p[u]\{1, y, \dots, y^{r+1}, \tau, \dots, \tau y^{r+1}\} & \text{for } m \geq 1. \end{array}$$

In filtration $n = pm+k$, the element $\alpha_{pm+k}u^it^j$ has total degree $(4r+2)n - 4r + 2i + 4j + 1$. In filtration $n = pm$, the generators are free $\mathbb{F}_p[u]$-module generators, which have the following degrees:

Class	*Case*	*Total degree*
$\alpha_{pm}y^i$	$p \mid r+1, 0 \leq i \leq r$	$(4r+2)pm - 4r + 4i + 1$
$\alpha_{pm}y^i$	$p \nmid r+1, 0 \leq i \leq r-1$	$(4r+2)pm - 4r + 4i + 1$
$\alpha_{pm}y^i\sigma$	$p \mid r+1, 0 \leq i \leq r$	$(4r+2)pm + 4i$
$\alpha_{pm}y^i\tau$	$p \nmid r+1, 0 \leq i \leq r-1$	$(4r+2)pm + 4i + 4$

Note that the columns $E_1^{pm,*}$, $m \geq 0$, are infinite, while the class $\alpha_{pm+k}u^it^j$ in $E_1^{pm+k,*}$ is zero when $i \geq 4r$ or $j \geq 2r$.

Remark 5.2. The symbol α_n refers to the Thom isomorphism. The notation $\alpha_n x$ etc. denotes the cup product with the Thom class of μ_n^- in the critical submanifold $N(n^2)$. The product is not defined in the spectral sequence, and so it is a bit of abuse of notation. But it is a very practical way of keeping track of the dimension shift and should be read as such. □

Proof. The Morse spectral sequence is described in Theorem 4.4. We use cohomology with $\mathbb{F}_p$ coefficients. First take filtration $n = 0$. Then $G_0(\mathbb{H}P^r)_{hS^1} = \mathbb{H}P^r{}_{hS^1}$ itself, and the S^1 action is trivial. Thus

$$\begin{aligned} E_1^{0,*}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1}) &\cong H^*(\mathbb{H}P^r{}_{hS^1};\mathbb{F}_p) = H^*(BS^1 \times \mathbb{H}P^r;\mathbb{F}_p) \\ &\cong H^*(BS^1;\mathbb{F}_p) \otimes H^*(\mathbb{H}P^r;\mathbb{F}_p) \cong \mathbb{F}_p[u] \otimes \mathbb{F}_p[x]/\langle x^r\rangle . \end{aligned}$$

Now take $n \geq 1$. From Theorem 4.4,

$$E_1^{n,*}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1}) \cong H^{n+*-((4r+2)n-4r+1)}(G(\mathbb{H}P^r)^{(n)}_{hS^1};\mathbb{F}_p).$$

Now we can use the previous results about the spaces of geodesics, Theorems 2.12 and 2.14. For the case $n = pm + k$ we know from Theorem 2.12 that u maps to x, and so the $\mathbb{F}_p[u]$-module structure is that multiplication by u equals multiplication by x. This is incorporated in the notation by writing u for the class previously named x. The last part of the theorem is Lemma 2.9. □

The next Lemma is based upon [Bökstedt-Ottosen], Lemma 9.8:

Lemma 5.3. *In the Morse spectral sequence for $L\mathbb{H}P^r{}_{hS^1}$, all differentials starting in odd total degree are trivial.*

Proof. This is mostly seen for dimensional reasons. Using the table in Theorem 5.1, we see that elements of odd total degree in the spectral sequence have the form $\alpha_n y^i u^j$ or $\alpha_n u^i t^j$. Because of the derivation property of the differentials, it is enough to consider the $\mathbb{F}_p[u]$ generators, i.e. $\alpha_{pm} y^i$ and $\alpha_{pm+k} t^j$ for $m \geq 0$.

So let us prove that $d_s(\alpha_{pm} y^i)$ is trivial ($s \geq 1$). This has total degree $(4r+2)pm-4r+4i+2$ and filtration degree $pm+s$. By the table in Theorem 5.1, observe that a non-trivial class of filtration n and even total degree exists if and only if $p \mid n$. Furthermore, in case $p \mid n$ we can determine the class of filtration n with lowest total degree. If $p \mid (r+1)$, this class is $\alpha_n\sigma$ of total degree $(4r+2)n$, and if $p \nmid n$ this class is $\alpha_n\tau$ of total degree $(4r+2)n+4$. So if $d_s(\alpha_{pm} y^i)$ is non-trivial, its total degree must be at least the total degree mentioned above. That is,

$$(4r+2)pm - 4r + 4i + 2 \geq \begin{cases} (4r+2)(pm+s), & p \mid (r+1); \\ (4r+2)(pm+s)+4, & p \nmid (r+1). \end{cases}$$

Suppose $p \mid r+1$. Then we can reduce the inequality to

$$-4r + 4i + 2 \geq (4r+2)s \quad \Leftrightarrow \quad (-4r+2)(s+1) + 4i \geq 0.$$

This is easier to satisfy if s is small and i is large, so we try $s = 1$ (minimum) and $i = r$ (maximum), obtaining the equality $2(-4r+2)+4r = -4r+4 \geq 0$, which only holds for $r = 1$. In this case we have equality. If $s > 1$ or $i < r$, there are no solutions. So the question is whether $d_1(\alpha_{pm} y)$ can be a non-trivial class of even total degree in filtration $n = pm+1$, and it cannot, since then, as noted earlier, p should divide $pm+1$. If $p \nmid r+1$ there are no solutions at all.

Now take the case $\alpha_{pm+k}t^j$. Then $d_s(\alpha_{pm+k}t^j)$ has filtration degree $pm + k + s$ and total degree $(4r+2)(pm+k) - 4r + 4j + 2$, which is even. By the same observation as before, if $d_s(\alpha_{pm+k}t^j)$ were to be non-trivial, its total degree must satisfy

$$(4r+2)(pm+k) - 4r + 4j + 2 \geq \begin{cases} (4r+2)(pm+s+k), & p \mid (r+1); \\ (4r+2)(pm+s+k)+4, & p \nmid (r+1). \end{cases}$$

Like before, we reduce for $p \mid r+1$:

$$-4r + 4j + 2 \geq (4r+2)s \quad \Leftrightarrow \quad (4r+2)(s+1) - 4 \leq 4j$$

Recall $s \geq 1$, so to satisfy this, $j \geq 2r$. But then the class $\alpha_{pm+k}t^j$ is zero, according to the last part of Theorem 5.1. Likewise for $p \mid r+1$. This proves the Lemma. □

We are going to need an overview of the size of the E_1 page of the Morse spectral sequence.

Lemma 5.4. *The Poincaré series $P(t)$ of $E_1(L(\mathbb{H}P^r)_{hS^1})$ is given by for $p \nmid r+1$:*

$$\frac{1 - t^{4r+4} + \frac{t^3}{1-t^{4r+2}}(1-t^{4r})(1-t^{4r+4}) + \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}}(1-t^{4r})(t^{4r+3}+t^{4r+4})}{(1-t^2)(1-t^4)}.$$

and for $p \mid r+1$,

$$\frac{1 - t^{4r+4} + \frac{t^3}{1-t^{4r+2}}(1-t^{4r})(1-t^{4r+4}) + \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}}(1-t^{4r+4})(t^{4r-1}+t^{4r})}{(1-t^2)(1-t^4)}.$$

Proof. I only prove this for $p \nmid r+1$. The other case is exactly the same. We first find the Poincaré series for $E_1^{n,*}$.

- $n = 0$: By Theorem 5.1, since $E_1^{0,*}$ is a free $\mathbb{F}_p[u]$-module,

$$P(E_1^{0,*})(t) = P(\mathbb{F}_p[u]) \cdot P(\mathbb{F}_p[x]/\langle x^r \rangle) = \frac{1}{1-t^2} \cdot \frac{1-t^{4(r+1)}}{1-t^4}.$$

- $p \nmid n$: By Theorem 5.1

$$\begin{aligned} P(E_1^{n,*})(t) &= t^{4r(n-1)+2n+1} \cdot P(\mathbb{F}_p[t,u]/\langle Q_r, Q_{r+1}\rangle) \\ &= t^{4r(n-1)+2n+1}(1+t^2) \cdot \frac{1-t^{4r}}{1-t^4} \cdot \frac{1-t^{4(r+1)}}{1-t^4} \\ &= t^{4r(n-1)+2n+1} \frac{(1-t^{4r})(1-t^{4r+4})}{(1-t^2)(1-t^4)}, \end{aligned}$$

  using Lemma 2.9 to find $P(\mathbb{F}_p[t,u]/\langle Q_r, Q_{r+1}\rangle)$.

- $p \mid n$: According to Theorem 5.1, we obtain

$$\begin{aligned} P(E_1^{n,*})(t) &= t^{4r(n-1)+2n+1} \cdot P(\mathbb{F}_p[u]\left\{1, y, \ldots, y^{r+1}, \tau, \ldots, \tau y^{r+1}\right\}) \\ &= t^{4r(n-1)+2n+1} \frac{1}{1-t^2} \cdot \frac{(1-t^{4r})(1+t^{4r+3})}{1-t^4}. \end{aligned}$$

since y has degree 4 and τ has degree $4r+3$.

We must sum over $n \geq 1$ to calculate $P(E_1)(t)$. Only the factor $t^{4r(n-1)+2n+1}$ depends on n, so we sum that first, in the two cases $p \mid n$ and $p \nmid n$:

$$\sum_{n\geq 1, p\mid n} t^{4r(n-1)+2n+1} = \sum_{m\geq 1} t^{4r(mp-1)+2mp+1} = \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}}.$$

Using this, we can compute

$$\sum_{n\geq 1,\, p\nmid n} t^{4r(n-1)+2n+1} = \sum_{n\geq 1} t^{4r(n-1)+2n+1} - \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}} = \frac{t^3}{1-t^{4r+2}} - \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}}.$$

Combining the results above and summing over $n \geq 1$ then yields:

$$\begin{aligned} P(E_1)(t) &= P(E_1^{0,*})(t) + \sum_{n\geq 1, p\mid n} P(E_1^{n,*})(t) + \sum_{n\geq 1,\, p\nmid n} P(E_1^{n,*})(t) \\ &= \frac{1}{(1-t^2)(1-t^4)} \cdot \Bigg(1 - t^{4(r+1)} + \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}}(1-t^{4r})(1+t^{4r+3}) \\ &+ \left(\frac{t^3}{1-t^{4r+2}} - \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}}\right)(1-t^{4r})(1-t^{4r+4})\Bigg) = \end{aligned}$$

$$\frac{1 - t^{4r+4} + \frac{t^3}{1-t^{4r+2}}(1-t^{4r})(1-t^{4r+4}) + \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}}(1-t^{4r})(t^{4r+3}+t^{4r+4})}{(1-t^2)(1-t^4)}.$$

□

Remark 5.5. Later we are going to need the odd and even parts of E_1, i.e. $E_1^{\text{odd}} = \bigoplus_{p+q \text{ odd}} E_1^{p,q}$, and likewise for E_1^{even}. Notice that

$$K(t) := \frac{t^{p(4r+2)-4r+1}}{1-t^{p(4r+2)}}$$

has odd degree. Then we get from the above Lemma that for $p \nmid r+1$,

$$\begin{aligned} P(E_1^{\text{even}})(t) &= \frac{1 - t^{4r+4} + K(t)(1-t^{4r})t^{4r+3}}{(1-t^2)(1-t^4)}; \\ P(E_1^{\text{odd}})(t) &= \frac{1-t^{4r}}{(1-t^2)(1-t^4)} \left(\frac{(1-t^{4r+4})t^3}{1-t^{4r+2}} + K(t)t^{4r+4}\right). \end{aligned}$$

Similarly for $p \mid r+1$,

$$
\begin{aligned}
P(E_1^{\text{even}})(t) &= \frac{1-t^{4r+4}}{(1-t^2)(1-t^4)}\left(1+K(t)t^{4r-1}\right);\\
P(E_1^{\text{odd}})(t) &= \frac{1-t^{4r+4}}{(1-t^2)(1-t^4)}\left(\frac{(1-t^{4r})t^3}{1-t^{4r+2}}+K(t)t^{4r}\right).
\end{aligned}
$$

□

For comparison purposes we are also going to need the non-equivariant case, $H^*(L\mathbb{H}P^r)$.

Theorem 5.6. *Let $E_s^{*,*} = E_s^{*,*}(\mathcal{M})(L\mathbb{H}P^r)$. Assume $p \mid r+1$. Then*

$$
\begin{aligned}
E_1^{0,*} &= \mathbb{F}_p[y]/\langle y^{r+1}\rangle;\\
E_1^{n,*} &= \alpha_n\mathbb{F}_p[y,\sigma]/\langle y^{r+1},\sigma^2\rangle \quad \text{for } n\geq 1.
\end{aligned}
$$

Assume $p \nmid r+1$. Then

$$
\begin{aligned}
E_1^{0,*} &= \mathbb{F}_p[y]/\langle y^{r+1}\rangle;\\
E_1^{n,*} &= \alpha_n\mathbb{F}_p[y,\tau]/\langle y^{r},\tau^2\rangle \quad \text{for } n\geq 1.
\end{aligned}
$$

where $|x|=4$, $|\sigma|=4r-1$,$|\tau|=4r+3$,$|\alpha_n|=(4r+2)n-4r+1$.

This spectral sequence collapses from the E_1 page. This determines $H^*(L\mathbb{H}P^r;\mathbb{F}_p)$ as an abelian group, and it has the following Poincaré series: For $p \nmid r+1$,

$$P_{H^*(L\mathbb{H}P^r)}(t) = \frac{1-t^{4r+4}}{1-t^4}+\frac{(1-t^{4r})(1+t^{4r+3})t^3}{(1-t^4)(1-t^{4r+2})};$$

and for $p \mid r+1$,

$$P_{H^*(L\mathbb{H}P^r)}(t) = \frac{1-t^{4r+4}}{1-t^4}+\frac{(1-t^{4r+4})(1+t^{4r-1})t^3}{(1-t^4)(1-t^{4r+2})}.$$

The map induced by inclusion

$$i^*: E_1^{n,\text{odd}-n}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1}) \longrightarrow E_1^{n,\text{odd}-n}(\mathcal{M})(L\mathbb{H}P^r)$$

is surjective.

Proof. The computation of E_1 via Morse theory is just like the proof of the equivariant case, Theorem 5.1. That the spectral sequence collapses follows from a splitting result for $L\mathbb{H}P^r$. Such a result can be found in [Ziller].

For the computation of the Poincaré series, since the spectral sequence collapses, we can compute $P_{H^*(L\mathbb{H}P^r)} = P_{E_\infty} = P_{E_1}$. We reuse the computations from the proof of Lemma 5.4. Consider the case $p \nmid r+1$. (The case $p \mid r+1$ is similar.) In filtration $n > 0$ we have,

$$P(E_1^{n,*})(t) = t^{4r(n-1)+2n+1} \cdot \frac{1-t^{4r}}{1-t^4}(1+t^{4r+3}).$$

And so

$$\begin{aligned} P(E_1)(t) &= \frac{1-t^{4r+4}}{1-t^4} + \sum_{n>0}\left(t^{4r(n-1)+2n+1} \cdot \frac{1-t^{4r}}{1-t^4}(1+t^{4r+3})\right) \\ &= \frac{1-t^{4r+4}}{1-t^4} + \frac{(1-t^{4r})(1+t^{4r+3})t^3}{(1-t^4)(1-t^{4r+2})}. \end{aligned}$$

For the surjectivity, we prove for every $n \in \mathbb{N}$ that the map

$$E_1^{n,\mathrm{odd}-n}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1}) \longrightarrow E_1^{n,\mathrm{odd}-n}(\mathcal{M})(L\mathbb{H}P^r)$$

is surjective. For $n=0$ the target space is zero, so the result is trivial. For $n>0$, the degree of the Thom class α_n is odd, so by the formula for the E_1 page, the question is whether $i^* : H^{\mathrm{even}}(G(\mathbb{H}P^r)^{(n)}_{hS^1}) \longrightarrow H^{\mathrm{even}}(G(\mathbb{H}P^r)^{(n)})$ is surjective. This follows from Corollary 2.15. □

Remark 5.7. We also need the odd and even parts, so I will do that computation now. For $p \nmid r+1$,

$$P^{\mathrm{odd}}_{H^*(L\mathbb{H}P^r)}(t) = \frac{(1-t^{4r})t^3}{(1-t^4)(1-t^{4r+2})};$$

and

$$\begin{aligned} P^{\mathrm{even}}_{H^*(L\mathbb{H}P^r)}(t) &= \frac{1-t^{4r+4}}{1-t^4} + \frac{(1-t^{4r})t^{4r+6}}{(1-t^4)(1-t^{4r+2})} \\ &= 1 + \frac{(1-t^{4r})t^4}{(1-t^4)(1-t^{4r+2})}. \end{aligned} \tag{52}$$

Note that

$$t \cdot P(H^{\mathrm{odd}}(L\mathbb{H}P^r))(t) = P(H^{\mathrm{even}}(L\mathbb{H}P^r))(t) - 1, \tag{53}$$

and that

$$P^{\mathrm{odd}}_{H^*(L\mathbb{H}P^r)}(t) = t^3(1+t^4+\cdots+t^{4r-4})\sum_{n=0}^{\infty} t^{n(4r+2)} \tag{54}$$

has all coefficients equal to 0 or 1, and the difference in degree between the 1-coefficients is at least four. We have the same properties when $p \mid r+1$, and for future reference, when $p \mid r+1$,

$$P^{\text{odd}}_{H^*(L\mathbb{H}P^r)}(t) = \frac{(1-t^{4r+4})t^3}{(1-t^4)(1-t^{4r+2})} = t^3(1+t^4+\cdots+t^{4r})\sum_{n=0}^{\infty} t^{n(4r+2)} \quad (55)$$

□

Corollary 5.8. *For the energy filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq L\mathbb{H}P^r$, the dimension of $H^{\text{odd}}(\mathcal{F}_m)$ as an $\mathbb{F}_p$ vector space is as follows:*

$$\dim H^{\text{odd}}(\mathcal{F}_m) = \begin{cases} m(r+1), & p \mid r+1; \\ mr, & p \nmid r+1. \end{cases}$$

Proof. The Morse spectral sequence $\{E_s^{*,*}\} = \{E_s^{*,*}(\mathcal{M})(L\mathbb{H}P^r)\}$ induced by the energy filtration of $L\mathbb{H}P^r$ collapses from the E_1 page by Theorem 5.6 above. This means that $E_\infty = E_1$. Comparing with the spectral sequence $\{E_s(\mathcal{F}_m)\}$ of the finite filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_m$ we see that its E_1 page is the same as $E_1(\mathcal{M})(L\mathbb{H}P^r)$ up to filtration m. So by naturality, both spectral sequences collapse from the E_1 page, and $E_\infty(\mathcal{F}_m)$ equals $E_\infty(L\mathbb{H}P^r)$ up to filtration m. So we can calculate the dimension of $H^{\text{odd}}(\mathcal{F}_m)$ as an $\mathbb{F}_p$ vector space:

$$\begin{aligned} \dim H^{\text{odd}}(\mathcal{F}_m) &= \dim E_\infty^{m,\text{odd}-m}(\mathcal{F}_m) + \cdots + \dim E_\infty^{1,\text{odd}-1}(\mathcal{F}_m) \\ &= \begin{cases} m(r+1), & p \mid r+1; \\ mr, & p \nmid r+1. \end{cases} \end{aligned}$$

Here the last equality is from (54) and (55). □

To squeeze the last information out of the Morse spectral sequences, we are going to use localization. The general setup is as follows: Given an R module M and a multiplicative set $U \subseteq R$ (i.e. if $u, v \in U$ then $uv \in U$), we define M localized away from U as

$$M[U^{-1}] = \left\{ \frac{m}{u} \mid m \in M, u \in U \right\} / \sim$$

where $\frac{m}{u} \sim \frac{m'}{u'}$ if there is $v \in U$ such that $vu'm = vum'$. It is an elementary algebraic fact that localization away from $U \subseteq R$ is an exact functor on R-modules.

We are going to use $U = \{u^n \mid n \in \mathbb{N}\} \subseteq \mathbb{F}_p[u]$, where u as usually denotes our generator $u \in H^2(BS^1; \mathbb{F}_p)$, such that $H^*(BS^1; \mathbb{F}_p) \cong \mathbb{F}_p[u]$. The main localization result here is [Bökstedt-Ottosen] Theorem 8.3, which I state without proof:

Theorem 5.9. *There is an isomorphism of spectral sequences*

$$E_*(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1})\left[\frac{1}{u}\right] \cong E_*(\mathcal{M})(L\mathbb{H}P^r) \otimes \mathbb{F}_p[u, u^{-1}].$$

when re-indexing the columns: filtration pm goes to filtration m for $m \in \mathbb{N}$.

Note: This implies that the localized spectral sequence $E_*(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1})\left[\frac{1}{u}\right]$ collapses from the E_p page, since $E_*(\mathcal{M})(L\mathbb{H}P^r)$ collapses from the E_1 page.

5.2 The Main Theorem

To prove the Main Theorem, we follow the method used in [Bökstedt-Ottosen] §13, adopting the strategy and proofs to the quaternion case. We need all the information that we have hitherto deduced from the Morse spectral sequences. For convenience, we collect the necessary structural facts below:

SF(1) Classes of even total degree only occur in $E_*^{pm,*}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1})$, $m \geq 0$.

SF(2) $E_*^{pm,*}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1})$ is a free $\mathbb{F}_p[u]$-module. If $p \nmid n$, $E_*^{n,*}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1})$ is a finite dimensional $\mathbb{F}_p$ vector space.

SF(3) Non-trivial differentials in $E_*(\mathcal{M})(L\mathbb{H}P^r_{hS^1})$ start in even total degree.

SF(4) The inclusion $j : (\mathcal{F}_n)_{hS^1} \longrightarrow L\mathbb{H}P^r_{hS^1}$ induces a surjective map on cohomology, $j^* : H^{\text{odd}}(L\mathbb{H}P^r{}_{hS^1}) \longrightarrow H^{\text{odd}}((\mathcal{F}_n)_{hS^1})$.

SF(5) $E_1^{n,2i+1-n}(\mathcal{M})(L\mathbb{H}P^r) = 0$ if one of the following hold: $p \mid r+1$ and $i > (2r+1)n$, or $p \nmid r+1$ and $i > (2r+1)n - 2$.

SF(6) The map $i^* : H^{\text{odd}}(L\mathbb{H}P^r{}_{hS^1}) \longrightarrow H^{\text{odd}}(L\mathbb{H}P^r)$ is surjective.

Proof. SF(1) and SF(2) is Theorem 5.1. SF(3) is Lemma 5.3. For SF(4), we consider the map between the two Morse spectral sequences converging to $H^*(L\mathbb{H}P^r{}_{hS^1}; \mathbb{F}_p)$ resp. $H^*((\mathcal{F}_n)_{hS^1}; \mathbb{F}_p)$ induced by the two energy filtrations. By SF(3) every differential starting in odd total degree is trivial, so the map is seen to be surjective on H^{odd}.

To prove SF(5), we use Theorem 5.6 to find the maximal degree of a non-trivial element of odd total degree in filtration n. We get:

$$\begin{array}{rrcl} p \mid r+1: & |\alpha_n x^r| & = & (4r+2)n - 4r + 1 + 4r = (4r+2)n + 1 \\ p \nmid r+1: & |\alpha_n x^{r-1}| & = & (4r+2)n - 4r + 1 + 4(r-1) = (4r+2)n - 3 \end{array}$$

It follows that $E_1^{n,2i+1-n}(\mathcal{M})(L\mathbb{H}P^r) = 0$ if

$$\begin{array}{lrclcl} p \mid r+1: & 2i+1 & > & (4r+2)n+1 & \Longleftrightarrow & i > (2r+1)n, \\ p \nmid r+1: & 2i+1 & > & (4r+2)n-3 & \Longleftrightarrow & i > (2r+1)n-2. \end{array}$$

To prove SF(6), we first recall that by Theorem 5.6, the induced map $i^* : E_1^{\mathrm{odd}}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1}) \longrightarrow E_1^{\mathrm{odd}}(\mathcal{M})(L\mathbb{H}P^r)$ is surjective. Since every differential in $E_s(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1})$ starting in odd total degree is trivial, the map $i^* : E_\infty^{\mathrm{odd}}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1}) \longrightarrow E_\infty^{\mathrm{odd}}(\mathcal{M})(L\mathbb{H}P^r)$ is also surjective. It is a general fact for spectral sequences that the induced map on their limits is then also surjective, and this is easily seen by a filtration argument. This means that $i^* : H^{\mathrm{odd}}(L\mathbb{H}P^r{}_{hS^1}) \longrightarrow H^{\mathrm{odd}}(L\mathbb{H}P^r)$ is surjective. $\square$

We first prove the Main Theorem for the odd part of the cohomology. There are two kinds of $\mathbb{F}_p[u]$ generators, torsion and free, and we need to use the S^1 transfer map τ to find the first kind. Let $i : L\mathbb{H}P^r \longrightarrow ES^1 \times_{S^1} L\mathbb{H}P^r = L\mathbb{H}P^r{}_{hS^1}$ be the inclusion. Then it follows from [Bökstedt-Ottosen] Thm. 14.1 that the S^1 action differential d is composed as follows

$$\begin{tikzcd} H^{*+1}(L\mathbb{H}P^r) \arrow[rr, "d"] \arrow[dr, "\tau"'] & & H^*(L\mathbb{H}P^r) \\ & H^*(L\mathbb{H}P^r{}_{hS^1}) \arrow[ur, "i^*"'] & \end{tikzcd} \tag{56}$$

In general, for a space X with an action $\mu : S^1 \times X \longrightarrow X$, the map d is given by

$$\begin{array}{ccccc} H^{n+1}(X) & \longrightarrow & H^{n+1}(S^1 \times X) & \longrightarrow & H^{n+1}(X) \oplus H^n(X) \\ a & \mapsto & \mu^*(a) & \mapsto & (a, d(a)) \end{array}$$

where the last map is the Künneth formula. For ease of reference, in the Lemma below I have collected all the facts I need about the action differential. First some notation:

$$\begin{aligned} \mathcal{IF} = \mathcal{IF}(r,p) &= \{(4r+2)i + 4j \mid \delta \le j \le r, 0 \le i, p \mid (r+1)i + j\} \setminus \{0\}, \\ \mathcal{IT} = \mathcal{IT}(r,p) &= \{(4r+2)i + 4j \mid \delta \le j \le r, 0 \le i, p \nmid (r+1)i + j\}; \end{aligned}$$

where

$$\delta = \begin{cases} 1, & p \nmid r+1; \\ 0, & p \mid r+1. \end{cases}$$

Set $\mathcal{IA} = \mathcal{IF} \cup \mathcal{IT}$. Then define power series by

$$P_{\mathcal{I}}(t) = \sum_{n=0}^{\infty} a_n t^n, \quad \text{where } a_n = \begin{cases} 1, & n \in \mathcal{I}(r,p); \\ 0, & n \notin \mathcal{I}(r,p). \end{cases} \tag{57}$$

for $\mathcal{I} = \mathcal{IF}, \mathcal{IT}, \mathcal{IA}$. By [Bökstedt-Ottosen] Lemma 11.4, $\mathcal{IF} \cap \mathcal{IT} = \emptyset$, so we get $P_{\mathcal{IA}} = P_{\mathcal{IF}} + P_{\mathcal{IT}}$. Also note that by (54),

$$P_{H^{\mathrm{odd}}(L\mathbb{H}P^r)}(t) = \frac{1}{t} P_{\mathcal{IA}}(t). \tag{58}$$

The following Lemma on the action differential is proved in [Bökstedt-Ottosen] lemma 11.6.

Lemma 5.10 (The Action Differential)**.** *Put* $H^* = H^*(L\mathbb{H}P^r)$ *and let* $k \in \mathbb{N}$.

(i) $\mathrm{Ker}(d : H^{2k} \longrightarrow H^{2k-1})$ *is either a trivial or a 1-dimensional vector space. It is non-trivial if and only if* $2k \in \mathcal{IF}(r,p)$.

(ii) $\mathrm{Im}(d : H^{2k} \longrightarrow H^{2k-1})$ *is either a trivial or a 1-dimensional vector space. It is non-trivial if and only if* $2k \in \mathcal{IT}(r,p)$.

(iii) *The cokernel of the map*

$$d : \bigoplus_{0 \le k \le (2r+1)mp-\delta} H^{2k+2} \longrightarrow \bigoplus_{0 \le k \le (2r+1)mp-\delta} H^{2k+1}$$

has dimension rm *if* $p \nmid r+1$, *and dimension* $(r+1)m$ *if* $p \mid r+1$.

The next two Lemmas specify the $\mathbb{F}_p[u]$ generators for $H^*(L\mathbb{H}P^r{}_{hS^1}; \mathbb{F}_p)$:

Lemma 5.11. *There is a graded subgroup* $\mathcal{T}^* \subseteq H^{\mathrm{odd}}(L\mathbb{H}P^r{}_{hS^1})$ *such that*

(i) $u\mathcal{T}^* = 0$.

(ii) *The restricted inclusion map* $i^*|_{\mathcal{T}^*} : H^*(L\mathbb{H}P^r{}_{hS^1})|_{\mathcal{T}^*} \longrightarrow H^*(L\mathbb{H}P^r)$ *is injective.*

(iii) *The image* $i^*(\mathcal{T}^*) \subseteq H^*(L\mathbb{H}P^r)$ *equals the image* $d(H^{*+1}(L\mathbb{H}P^r)) \subseteq H^*(L\mathbb{H}P^r)$.

Proof. We use property (iii) to construct $\mathcal{T}^*$. We choose a graded subgroup $\overline{\mathcal{T}}^* \subseteq H^{*+1}(L\mathbb{H}P^r)$, such that d maps $\overline{\mathcal{T}}^*$ isomorphically onto $\mathrm{Im}\, d$. This we can do simply by lifting each generator of $\mathrm{Im}\, d \subseteq H^*(L\mathbb{H}P^r)$ to $H^{*+1}(L\mathbb{H}P^r)$. Now we put $\mathcal{T}^* = \tau(\overline{\mathcal{T}}^*)$. Then (iii) follows by construction, since $i^*(\mathcal{T}^*) = i^* \circ \tau^*(\overline{\mathcal{T}}^*) = d(\overline{\mathcal{T}}^*)$ by the diagram (56). Also (ii) holds, since i^* restricted to $\mathcal{T}^*$ corresponds to $i^* \circ \tau = d$ restricted to $\overline{\mathcal{T}}^*$, and we chose $\overline{\mathcal{T}}^*$ such that d was an isomorphism of $\overline{\mathcal{T}}^*$ onto its image. As for property (i), this holds because $u\tau = 0$ according to [Bökstedt-Ottosen] Thm. 14.1. This is because the transfer map τ appears right after multiplication by u in the Gysin exact sequence. □

Remark 5.12. By definition of $\mathcal{T}^*$ it follows from Lemma 5.10 (ii) that the non-trivial part of $\mathcal{T}^*$ sits in degree $2k-1$ if and only if $2k \in \mathcal{IT}(r,p)$. Using the notation in (57), we can write down the Poincaré series of $\mathcal{T}^*$:

$$P_{\mathcal{T}^*}(t) = \frac{1}{t} P_{\mathcal{IT}}(t).$$

□

Lemma 5.13. *There is a graded subgroup* $\mathcal{U}^* \subseteq H^{\mathrm{odd}}(L\mathbb{H}P^r{}_{hS^1})$ *such that*

(i) The composition

$$\mathcal{T}^* \oplus \mathcal{U}^* \hookrightarrow H^{\mathrm{odd}}(L\mathbb{H}P^r_{hS^1}) \xrightarrow{i^*} H^{\mathrm{odd}}(L\mathbb{H}P^r)$$

is an isomorphism.

(ii) The restriction

$$\mathcal{U}^{2i+1} \longrightarrow H^{2i+1}(L\mathbb{H}P^r{}_{hS^1}) \xrightarrow{j^*} H^{2i+1}((\mathcal{F}_{pm})_{hS^1})$$

is trivial if either $p \mid r+1$ *and* $i > (2r+1)pm$*, or* $p \nmid r+1$ *and* $i > (2r+1)pm - 2$.

Proof. Again we first specify a subgroup $\overline{\mathcal{U}}^* \subseteq H^{\mathrm{odd}}(L\mathbb{H}P^r)$, by demanding that it must be a complementary subgroup of $i^*(\mathcal{T}^*)$, so that we have the $\mathbb{F}_p$ vector space isomorphism $H^{\mathrm{odd}}(L\mathbb{H}P^r) \cong i^*(\mathcal{T}^*) \oplus \overline{\mathcal{U}}^*$. The idea is to find $\mathcal{U}^* \subseteq H^*(L\mathbb{H}P^r{}_{hS^1})$ such that i^* maps it isomorphically to $\overline{\mathcal{U}}^*$. This can be done since i^* is surjective by SF(6).

We now use the Gysin sequence, see [Bökstedt-Ottosen] Thm. 14.1, to make the following diagram with exact rows:

$$\begin{tikzcd}
H^{2i-1}(L\mathbb{H}P^r{}_{hS^1}) \arrow[r, "\cdot u"] \arrow[d, "j^*"] & H^{2i+1}(L\mathbb{H}P^r_{hS^1}) \arrow[r, two heads, "i^*"] \arrow[d, "j^*"] & H^{2i+1}(L\mathbb{H}P^r) \arrow[d] \\
H^{2i-1}((\mathcal{F}_{pm})_{hS^1}) \arrow[r, "\cdot u"] & H^{2i+1}((\mathcal{F}_{pm})_{hS^1}) \arrow[r] & H^{2i+1}(\mathcal{F}_{pm})
\end{tikzcd} \qquad (59)$$

The vertical maps j^* are surjective according to SF(4). By SF(6), the upper horizontal map i^* is surjective.

Under the assumption in *(ii)*, we get from SF(5) that $H^{2i+1}(\mathcal{F}_n, \mathcal{F}_{n-1}) = E_1^{n,2i+1-n} = 0$ for $0 \le n \le pm$. Using the long exact sequence for the pair $(\mathcal{F}_n, \mathcal{F}_{n-1})$ for $n = 0, 1, \dots, pm$ gives a series of injective maps,

$$H^{2i+1}(\mathcal{F}_{pm}) \hookrightarrow H^{2i+1}(\mathcal{F}_{pm-1}) \hookrightarrow \cdots H^{2i+1}(\mathcal{F}_0) \hookrightarrow H^{2i+1}(\mathcal{F}_{-1}) = 0.$$

This means $H^{2i+1}(\mathcal{F}_{pm}) = 0$. So $\overline{\mathcal{U}}^{2i+1}$ is in the kernel of the right vertical map. To ensure that $\mathcal{U}^{2i+1}$ is also in the kernel of the middle vertical map j^*, we use diagram chase. The image $j^*(\mathcal{U}^{2i+1})$ maps to zero, so it comes from $H^{2i-1}(\mathcal{F}_{pm})$. The left j^* map is onto this, so we can lift it, map it into $H^{2i+1}(L\mathbb{H}P^r)$, and subtract it from the original $\mathcal{U}^{2i+1}$. This gives a choice of $\mathcal{U}^{2i+1}$ that satisfies both *(i)* and *(ii)*. ∎

Remark 5.14. By property (i) of $\mathcal{U}^*$, we can calculate its Poincaré series

$$P_{\mathcal{U}^*}(t) = P_{H^{\mathrm{odd}}(L\mathbb{H}P^r)}(t) - P_{\mathcal{T}^*}(t) = \frac{1}{t}(P_{\mathcal{IA}}(t) - P_{\mathcal{IT}}(t)) = \frac{1}{t}P_{\mathcal{IF}}(t),$$

where we have used Remark 5.12 and (58). □

Remark 5.15. We will need the dimension of parts of $\mathcal{U}^*$. As $\mathcal{T}^* \oplus \mathcal{U}^* \overset{i^*}{\cong} H^{\mathrm{odd}}(L\mathbb{H}P^r)$, and $i^*(\mathcal{T}^*) = \operatorname{Im} d \subseteq H^{\mathrm{odd}}(L\mathbb{H}P^r)$, we can compute the dimension of $\mathcal{U}^*$ as the dimension of the cokernel of the action differential d. For this we can use Lemma 5.10 (iii) and (iv), and get

$$\begin{aligned} p \nmid r+1 \quad &: \quad \dim\Big(\bigoplus_{k \le (2r+1)mp-1} \mathcal{U}^{2k-1}\Big) = rm, \\ p \mid r+1 \quad &: \quad \dim\Big(\bigoplus_{k \le (2r+1)pm} \mathcal{U}^{2k-1}\Big) = (r+1)m. \end{aligned}$$

Now we can prove the Main Theorem for the odd degree cohomology:

Theorem 5.16. *The map of $\mathbb{F}_p[u]$-modules,*

$$h_1 \oplus h_2 : (\mathbb{F}_p[u] \otimes \mathcal{U}^*) \oplus \mathcal{T}^* \longrightarrow H^{\mathrm{odd}}(L\mathbb{H}P^r{}_{hS^1})$$

induced by the inclusions of $\mathcal{U}^*$ and $\mathcal{T}^*$, is an isomorphism of $\mathbb{F}_p[u]$-modules.

Expressed in terms of generators, $H^{\mathrm{odd}}(L\mathbb{H}P^r{}_{hS^1})$ is isomorphic as a graded $\mathbb{F}_p[u]$-module to

$$\bigoplus_{2k \in \mathcal{IF}} \mathbb{F}_p[u] f_{2k-1} \oplus \bigoplus_{2k \in \mathcal{IT}} (\mathbb{F}_p[u]/\langle u \rangle)\, t_{2k-1},$$

where the lower index denotes the degree of the generators.

Proof. From Lemma 5.11 (i) we see that $\mathcal{T}^*$ is actually an $\mathbb{F}_p[u]$-submodule of $H^{\mathrm{odd}}(L\mathbb{H}P^r_{hS^1})$, and so the inclusion $h_2 : \mathcal{T}^* \longrightarrow H^{\mathrm{odd}}(L\mathbb{H}P^r_{hS^1})$ is an $\mathbb{F}_p[u]$-linear map. On the contrary we just consider $\mathcal{U}^*$ as a subgroup, and make the $\mathbb{F}_p[u]$-module $\mathbb{F}_p[u] \otimes \mathcal{U}^*$. There is then a unique way to extend the inclusion of $\mathcal{U}^*$ to an $\mathbb{F}_p[u]$-linear map $h_1 : \mathbb{F}_p[u] \otimes \mathcal{U}^* \longrightarrow H^{\mathrm{odd}}(L\mathbb{H}P^r_{hS^1})$.

First we remark that $h_1 \oplus h_2$ is surjective. To see this we use part of the Gysin exact sequence, see (59), where the rightmost zero is SF(6):

$$H^{2i-1}(L\mathbb{H}P^r{}_{hS^1}) \xrightarrow{u} H^{2i+1}(L\mathbb{H}P^r_{hS^1}) \xrightarrow{i^*} H^{2i+1}(L\mathbb{H}P^r) \longrightarrow 0.$$

This is a sequence of $\mathbb{F}_p$ vector spaces, so it suffices to show that we can hit the image $u(H^{2i-1}(L\mathbb{H}P^r{}_{hS^1}))$ and the cokernel $H^{2i+1}(L\mathbb{H}P^r{}_{hS^1})/\ker(i^*) \cong$

$H^{2i+1}(L\mathbb{H}P^r)$. The cokernel can be hit according to (i) in Lemma 5.13. We now use induction in the degree $2i+1$. The induction start is trivial. We get inductively that the image $u(H^{2i-1}(L\mathbb{H}P^r{}_{hS^1}))$ can be hit by $u\big((\mathbb{F}_p[u]\otimes \mathcal{U}^*)\oplus\mathcal{T}^*\big)\subseteq(\mathbb{F}_p[u]\otimes\mathcal{U}^*)\oplus\mathcal{T}^*$, where the last inclusion follows from Lemma 5.11. So it remains to show that $h_1\oplus h_2$ is injective.

The idea of the proof is now to show that map $h_1\oplus h_2$ localized away from u, which we denote $(h_1\oplus h_2)[\frac{1}{u}]$, is injective. Again by Lemma 5.11 (i) we see that when localizing away from u, $\mathcal{T}^*$ vanishes. So we look at h_1, and by Lemma 5.13 there is a commutative diagram,

$$\begin{array}{ccc} \mathbb{F}_p[u]\otimes\bigoplus_i\mathcal{U}^{2i+1} & \xrightarrow{\quad h_1\quad} & H^{\mathrm{odd}}(L\mathbb{H}P^r_{hS^1}) \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{proj}} & & \Big\downarrow{\scriptstyle j^*} \\ \mathbb{F}_p[u]\otimes\bigoplus_{i\leq(2r+1)pm-\delta}\mathcal{U}^{2i+1} & \xrightarrow{\quad \overline{h}_1\quad} & H^{\mathrm{odd}}((\mathcal{F}_{pm})_{hS^1}) \end{array} \tag{60}$$

where

$$\delta=\begin{cases}1, & p\nmid r+1;\\ 0, & p\mid r+1.\end{cases}$$

The map j^* is surjective according to SF(4).

Localizing away from u can be done by tensoring with $\mathbb{F}_p[u,u^{-1}]$ over $\mathbb{F}_p[u]$. Since $h_1\oplus h_2$ is surjective, and localization is exact, $(h_1\oplus h_2)[\frac{1}{u}]$ is also surjective. As noted, h_2 vanishes when localizing away from u, so we conclude that

$$h_1[\frac{1}{u}]:\mathbb{F}_p[u,u^{-1}]\otimes\mathcal{U}^*\longrightarrow H^{\mathrm{odd}}(L\mathbb{H}P^r_{hS^1})[\frac{1}{u}]$$

is surjective. When localizing, we conclude from the diagram (60) that

$$\overline{h}_1[\frac{1}{u}]:\mathbb{F}_p[u,u^{-1}]\otimes\bigoplus_{0\leq i\leq(2r+1)pm-\delta}\mathcal{U}^{2i+1}\longrightarrow H^{\mathrm{odd}}((\mathcal{F}_{pm})_{hS^1})[\frac{1}{u}]$$

is also surjective.

To show $\overline{h}_1[\frac{1}{u}]$ as injective, we will prove that the domain and target spaces are isomorphic as abstract modules. So we first study the domain of $\overline{h}_1[\frac{1}{u}]$. The dimension of the $\mathcal{U}^*$ part is calculated in Remark 5.15, and tensoring with $\mathbb{F}_p[u,u^{-1}]$ we obtain the rank:

$$\mathrm{rank}\Big(\mathbb{F}_p[u,u^{-1}]\otimes\bigoplus_{0\leq i\leq(2r+1)pm-\delta}\mathcal{U}^{2i+1}\Big)=\begin{cases}(r+1)m, & p\mid r+1;\\ rm, & p\nmid r+1.\end{cases}$$

Turning to the target space of $\overline{h}_1[\frac{1}{u}]$, $H^{\mathrm{odd}}((\mathcal{F}_{pm})_{hS^1})[\frac{1}{u}]$, we use Theorem 5.9:

$$H^{\mathrm{odd}}((\mathcal{F}_{pm})_{hS^1})[\frac{1}{u}]\cong H^{\mathrm{odd}}(\mathcal{F}_m)\otimes\mathbb{F}_p[u,u^{-1}] \tag{61}$$

Consequently, by Corollary 5.8 we can calculate the rank as an $\mathbb{F}_p[u]$-module:

$$\operatorname{rank} H^{\mathrm{odd}}((\mathcal{F}_{pm})_{hS^1})[\frac{1}{u}] = \begin{cases} m(r+1), & p \mid r+1; \\ mr, & p \nmid r+1. \end{cases}$$

So $\overline{h}_1[\frac{1}{u}]$ is a surjective map between two free $\mathbb{F}_p[u,u^{-1}]$-modules of the same rank. Then $\overline{h}_1[\frac{1}{u}]$ must also be injective.

All that remains is to show that $h_1 \oplus h_2$ is injective. Actually it will be enough to show that $\overline{h}_1 \oplus h_2$ is injective for each m, since a given element will be in the domain of $\overline{h}_1 \oplus h_2$ for a large enough m. So consider an element $(a,t) \in \mathbb{F}_p[u] \otimes \bigoplus_{i \le (2r+1)pm-\delta} \mathcal{U}^{2i+1} \oplus \mathcal{T}^*$ in the kernel of $\overline{h}_1 \oplus h_2$. When localizing, t vanishes, so c localized must be in the kernel of $\overline{h}_1$ localized, which we have shown is injective. This means c localized is zero. But the localization map on $\mathbb{F}_p[u] \otimes \mathcal{U}^*$,

$$\mathbb{F}_p[u] \otimes \mathcal{U}^* \xrightarrow{\text{localization}} \mathbb{F}_p[u,u^{-1}] \otimes_{\mathbb{F}_p[u]} (\mathbb{F}_p[u] \otimes \mathcal{U}^*) \cong \mathbb{F}_p[u,u^{-1}] \otimes \mathcal{U}^*$$

is injective, so c is zero itself. This means t is in the kernel of h_1. And by Lemma 5.11, h_1 is injective, so t is zero.

The expression with generators follows directly from the isomorphism $H^{\mathrm{odd}}(L\mathbb{H}P^r{}_{hS^1}) \cong (\mathbb{F}_p[u] \otimes \mathcal{U}^*) \oplus \mathcal{T}^*$ together with the computation of the Poincaré series in Remarks 5.12 and 5.14. □

We can now prove the general Main Theorem, giving a complete description of $H^*(L\mathbb{H}P^r{}_{hS^1}; \mathbb{F}_p)$:

Theorem 5.17. *As a graded $\mathbb{F}_p[u]$-module, $H^*(L\mathbb{H}P^r{}_{hS^1}; \mathbb{F}_p)$ is isomorphic to*

$$\mathbb{F}_p[u] \oplus \bigoplus_{2k \in \mathcal{IF}} \mathbb{F}_p[u] f_{2k} \oplus \bigoplus_{2k \in \mathcal{IF}} \mathbb{F}_p[u] f_{2k-1} \oplus \bigoplus_{2k \in \mathcal{IT}} (\mathbb{F}_p[u]/\langle u \rangle)\, t_{2k-1}.$$

Here the lower index denotes the degree of the generator, and their names are meant to suggest free and torsion generators.

Proof. First, note that when taking the odd part, we have already proved this in Theorem 5.16. So it remains to show that $H^{\mathrm{even}}(L\mathbb{H}P^r; \mathbb{F}_p)$ is a free $\mathbb{F}_p[u]$-module with generators in the stated degrees.

First I argue why $H^{\mathrm{even}}(L\mathbb{H}P^r{}_{hS^1}; \mathbb{F}_p)$ is free, using the Morse spectral sequence, $E_s^{*,*} = E_s^{*,*}(\mathcal{M})(L\mathbb{H}P^r{}_{hS^1})$. By SF(1) and SF(2), E_1^{even} is a free $\mathbb{F}_p[u]$-module, which is concentrated in $E_1^{pm,*}$. Since by SF(3) all non-trivial differentials start in even degrees, E_∞^{even} is a submodule of E_1^{even}. Note that $E_1^{pm,*}$ is a finitely generated $\mathbb{F}_p[u]$-module. Since $\mathbb{F}_p[u]$ is a principal ideal domain, the submodule $E_\infty^{(pm,*)\text{even}}$ of the free $\mathbb{F}_p[u]$-module $E_1^{(pm,*)\text{even}}$ is also free. Since the spectral sequence E_s converges to $H^*(L\mathbb{H}P^r{}_{hS^1};\mathbb{F}_p)$, $H^{\text{even}}(L\mathbb{H}P^r{}_{hS^1};\mathbb{F}_p)$ is filtered by free $\mathbb{F}_p[u]$ modules and is thus free itself. The generators are the generators of E_∞^{even}.

Now we must find the degrees of the generators. We will compute E_∞^{even} in terms of Poincaré series, and deduce the generator degrees from this. The Morse spectral sequence alone does not provide enough information, so we compare with Serre's spectral sequence for the fibration

$$L\mathbb{H}P^r \longrightarrow L\mathbb{H}P^r{}_{hS^1} \longrightarrow BS^1,$$

that is,

$$H^*(BS^1;H^*(L\mathbb{H}P^r,\mathbb{F}_p)) \Rightarrow H^*(L\mathbb{H}P^r{}_{hS^1};\mathbb{F}_p).$$

Denote this spectral sequence by $E_s^{*,*}(\mathcal{S})$. Then $E_2^{*,*}(\mathcal{S}) = H^*(L\mathbb{H}P^r;\mathbb{F}_p) \otimes \mathbb{F}_p[u]$. According to (54) and (53), $H^*(L\mathbb{H}P^r;\mathbb{F}_p)$ has the following form: the non-trivial part is one-dimensional in each degree, and, apart from degree zero, sits in degrees that come in pairs of odd-even, with at least 2 zero-rows between the pairs. I have tried to diagram what this might look like below, a star indicating a non-trivial group.

$$E_2(\mathcal{S}) \quad \begin{array}{r|cccccc}
8 & * & & * & & * & \cdots \\
7 & * & & * & & * & \cdots \\
 & & & & & & \\
 & & & & & & \\
4 & * & & * & & * & \cdots \\
3 & * & & * & & * & \cdots \\
 & & & & & & \\
 & & & & & & \\
0 & * & & * & & * & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 \quad \ldots
\end{array}
\qquad
E_3(\mathcal{S}) \quad \begin{array}{r|cccccc}
8 & * & & * & & * & \cdots \\
7 & * & & * & & * & \cdots \\
 & & & & & & \\
 & & & & & & \\
4 & & & & & & \cdots \\
3 & * & & & & & \cdots \\
 & & & & & & \\
 & & & & & & \\
0 & * & & * & & * & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 \quad \ldots
\end{array}$$

We also see the only non-trivial d_2 differentials must be from the even to the odd row in the odd-even pairs. What happens when we pass to $E_3(\mathcal{S})$ depends on whether d_2 is zero or an isomorphism (the only possibilities). If d_2 is zero, the odd-even row pair will survive to E_3, and if d_2 is an isomorphism, only the odd group in filtration 0 will survive to E_3, as indicated above.

Here we can use a shortcut: The differential d_2 can be determined geometrically; it is actually given by the action differential. By Lemma 5.10 (i) we then see that $d_2^{0,2k} = 0$ if and only if $2k \in \mathcal{IF}$. Then we can write down the Poincaré series of the E_3 page:

$$P(E_3(\mathcal{S}))(t) = \frac{1}{1-t^2} + P(H^{\text{odd}}(L\mathbb{H}P^r))(t) + \frac{P_{\mathcal{IF}}(t)}{1-t^2} + \frac{tP_{\mathcal{IF}}(t)}{1-t^2}. \tag{62}$$

This might not look very helpful, but if we use (52) to calculate

$$\begin{aligned} P(E_3^{\text{even}}(\mathcal{S}))(t) - \frac{1}{t}P(E_3^{\text{odd}}(\mathcal{S}))(t) &= \frac{1}{1-t^2} + \frac{1}{t}P(H^{\text{odd}}(L\mathbb{H}P^r))(t) = \\ \frac{1}{1-t^2} - \frac{t^2(1-t^{4r})}{(1-t^4)(1-t^{4r+2})} &= \frac{1-t^{4r+4}}{(1-t^4)(1-t^{4r+2})} \end{aligned} \tag{63}$$

we get a quantity that does not depend on $P_{\mathcal{IF}}(t)$.

Let us return to the Morse spectral sequence. Using Remark 2.10, we can compute the same quantity for the $E_1(\mathcal{M})$ page. For $p \nmid r+1$ this yields

$$\begin{aligned} & P(E_1^{\text{even}}(\mathcal{M}))(t) - \frac{1}{t}P(E_1^{\text{odd}}(\mathcal{M}))(t) \\ = \; & \frac{1-t^{4r+4}+K(t)(1-t^{4r})t^{4r+3}}{(1-t^2)(1-t^4)} \\ & - \frac{1-t^{4r}}{(1-t^2)(1-t^4)}\left(\frac{(1-t^{4r+4})t^2}{1-t^{4r+2}} + K(t)t^{4r+3}\right) \\ = \; & \frac{1-t^{4r+4}}{(1-t^2)(1-t^4)}\left(1-\frac{(1-t^{4r})t^2}{1-t^{4r+2}}\right) = \frac{1-t^{4r+4}}{(1-t^4)(1-t^{4r+2})}. \end{aligned} \tag{64}$$

Using the formulas for $p \mid r+1$, though slightly different, also give the same quantity. As we wanted to compute $E_\infty(\mathcal{M})$, we really want to know this quantity for $E_\infty(\mathcal{M})$. Since by SF(3), all non-trivial differentials in $E_*(\mathcal{M})$ goes from even to odd total degree, we have

$$\dim E_\infty^{2n+1} + \dim\left(\bigoplus_{k\geq 1; i+j=2n+1} \operatorname{Im}(d_k : E_k^{i-k,j-k+1} \longrightarrow E_k^{i,j})\right) = \dim E_1^{2n+1}.$$

From this we deduce

$$\begin{aligned} \dim E_\infty^{2n} &= \dim E_1^{2n} - \dim\left(\bigoplus_{k\geq 1; i+j=2n+1} \operatorname{Im}(d_k : E_k^{i-k,j-k+1} \longrightarrow E_k^{i,j})\right) \\ &= \dim E_1^{2n} - \dim E_1^{2n+1} + \dim E_\infty^{2n+1}. \end{aligned}$$

Expressing this by Poincaré series yields

$$P(E_\infty^{\text{even}})(\mathcal{M}) - \frac{1}{t}P(E_\infty^{\text{odd}})(\mathcal{M}) = P(E_1^{\text{even}})(\mathcal{M}) - \frac{1}{t}P(E_1^{\text{odd}})(\mathcal{M})$$

Now by (63) and (64) we can conclude

$$P(E_\infty^{\text{even}})(\mathcal{M}) - \frac{1}{t}P(E_\infty^{\text{odd}})(\mathcal{M}) = P(E_3^{\text{even}})(\mathcal{S}) - \frac{1}{t}P(E_3^{\text{odd}})(\mathcal{S})$$

To conclude $P(E_\infty^{\text{even}})(\mathcal{M}) = P(E_3^{\text{even}})(\mathcal{S})$, we must show $P(E_\infty^{\text{odd}})(\mathcal{M}) = P(E_3^{\text{odd}})(\mathcal{S})$. We can compute $P(E_\infty^{\text{odd}})(\mathcal{M})$ by Theorem 5.16:

$$\begin{aligned} P(E_\infty^{\text{odd}})(\mathcal{M}) &= P(H^{\text{odd}}(L\mathbb{H}P^r{}_{hS^1})) = P((\mathbb{F}_p[u]\otimes\mathcal{U}^*)\oplus\mathcal{T}^*) \\ &= \frac{1}{1-t^2}P_{\mathcal{U}^*}(t) + P_{\mathcal{T}^*}(t) = \frac{1}{t(1-t^2)}P_{\mathcal{IF}}(t) + \frac{1}{t}P_{\mathcal{IT}}(t), \end{aligned}$$

where I have used Remarks 5.12 and 5.14. Now by Lemma 5.10 (i),

$$\begin{aligned} P(E_3^{\text{odd}}(\mathcal{S}))(t) &= P(H^{\text{odd}}(L\mathbb{H}P^r))(t) + \frac{tP_{\mathcal{IF}}(t)}{1-t^2} \\ &= \frac{1}{t}P_{\mathcal{IA}}(t) + \frac{t}{1-t^2}P_{\mathcal{IF}}(t) = \frac{1}{t(1-t^2)}P_{\mathcal{IF}}(t) + \frac{1}{t}P_{\mathcal{IT}}(t). \end{aligned}$$

This allows us to conclude that $P(E_\infty^{\text{even}})(\mathcal{M}) = P(E_3^{\text{even}})(\mathcal{S})$, and we can compute by (62),

$$P(E_\infty^{\text{even}})(\mathcal{M}) = P(E_3^{\text{even}})(\mathcal{S}) = \frac{1}{1-t^2} + \frac{P_{\mathcal{IF}}(t)}{1-t^2},$$

as stated in the Theorem. $\square$

6 S^1-equivariant K-theory of $L\mathbb{C}P^r$

Recall that the Morse spectral sequence comes from the S^1-equivariant energy filtration

$$\mathbb{C}P^r = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_\infty = L\mathbb{C}P^r, \tag{65}$$

which consequently gives a filtration $\{(\mathcal{F}_n)_{hS^1}\}_n$ of $L\mathbb{C}P^r{}_{hS^1}$. The Morse spectral sequence $E_*(\mathcal{M})(L\mathbb{C}P^r{}_{hS^1})$ in K-theory has the following structure,

Theorem 6.1. *The Morse spectral sequence $E_r^{*,*}(\mathcal{M})(L\mathbb{C}P^r{}_{hS^1})$ converging to $K^*(L\mathbb{C}P^r{}_{hS^1})$ is a spectral sequence of $K^*(BS^1) = \mathbb{Z}[[t]]$-modules, and it has the following E_1 page, using the $\mathbb{Z}/2\mathbb{Z}$ grading of K-theory:*

$$\begin{aligned} E_1^{0,j} &= \begin{cases} \mathbb{Z}[[t]] \otimes_{\mathbb{Z}} \mathbb{Z}[h]/\langle h^r \rangle, & j \text{ even;} \\ 0, & j \text{ odd.} \end{cases} \\ E_1^{n,j} &= \begin{cases} 0, & j \text{ even;} \\ \mathbb{Z}[[t]]^{(n)} \otimes_R \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1} \rangle, & j \text{ odd.} \end{cases} \quad \text{for } n \geq 1. \end{aligned}$$

Here, $R = R(S^1) = \mathbb{Z}[U, U^{-1}]$, and $\mathbb{Z}[[t]]^{(n)}$ denotes the R-module structure $U \mapsto (t+1)^n$ on $\mathbb{Z}[[t]]$. The R-module structure on $\mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle$ is $U \mapsto (x-y)/(1+y) + 1$.

Proof. The method is exactly as in Theorem 5.1. The Morse spectral sequence is Theorem 4.4, and we use Theorem 3.7 which gives $K^*_{hS^1}(G(r)^{(n)})$, with the module structures stated just below the Theorem. Finally, using the $\mathbb{Z}/2\mathbb{Z}$-grading from Bott-periodicity, we suppress the Thom isomorphism, and simply get a shift from even to odd degree when $n \geq 1$. $\square$

Remark 6.2. Note that when $n = 1$, the S^1-action is free on $G(r)$, so $G(r)_{hS^1} \simeq \Delta(r)$. So $E^{1,\mathrm{odd}} \cong K^0(\Delta(r)) \cong \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle$, with $\mathbb{Z}[[t]]$-module structure $t \mapsto (x-y)/(1+y)$.

We can depict the Morse spectral sequence schematically as follows, where an empty space denotes zero, and a $*$ denotes a non-trivial module:

$$\begin{array}{r|cccccc} 3 & & & * & & * & \\ 2 & * & * & & * & & * \\ 1 & & & * & & * & \\ 0 & * & * & & * & & * \\ \hline -1 & & & * & & * & \\ -2 & * & * & & * & & * \\ -3 & & & * & & * & \\ -4 & * & * & & * & & * \end{array}$$

From the configuration of this spectral sequence, we can immediately establish a number of structural facts. Recall the notation $K^*_{hS^1}(X) = K^*(X_{hS^1})$, when X is an S^1-space.

Proposition 6.3. *The Morse spectral sequence converging to $K^*_{hS^1}(L\mathbb{C}P^r)$ has the following properties:*

(i) The only possible non-trivial differentials start from column 0.

(ii) $K^0_{hS^1}(L\mathbb{C}P^r)$ is a submodule of $K^0_{hS^1}(\mathcal{F}_0) = K^0(BS^1) \otimes_{\mathbb{Z}} K^0(\mathbb{C}P^r)$, and in particular it is a free abelian group.

(iii) The spectral sequence for the filtration $\{\mathcal{F}_i/\mathcal{F}_0\}_i$ has $K^*(\text{point})$ in column 0, and thus it collapses. So $\tilde{K}^0_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) = 0$, and $K^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0)$ is free abelian.

We will also need the twisted case, i.e the Morse spectral sequence for the (n)-twisted filtration $\mathcal{F}_0 = \mathcal{F}_0^{(n)} \subseteq \mathcal{F}_1^{(n)} \subseteq \cdots \subseteq (L\mathbb{C}P^r)^{(n)}$, where we have

Lemma 6.4. *For the (n)-twisted filtration $\mathcal{F}_0^{(n)} \subseteq \mathcal{F}_1^{(n)} \subseteq \cdots \subseteq (L\mathbb{C}P^r)^{(n)}$, the following holds: $\tilde{K}^0_{hS^1}(\mathcal{F}_1^{(n)}/\mathcal{F}_0) = 0$, and*

$$\tilde{K}^1_{hS^1}(\mathcal{F}_1^{(n)}/\mathcal{F}_0) \cong \mathbb{Z}[[t]]^{(n)} \otimes_R \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle .$$

Proof. Morse theory says that $\mathcal{F}_1/\mathcal{F}_0 \simeq Th(\mu_1^-)$ as S^1-spaces, since the filtration is S^1-equivariant. As a consequence,

$$\mathcal{F}_1^{(n)}/\mathcal{F}_0 = (\mathcal{F}_1/\mathcal{F}_0)^{(n)} \simeq (Th(\mu_1^-))^{(n)} = Th((\mu_1^-)^{(n)}),$$

where the last equality is clear from the definition $Th(\xi) = D(\xi)/S(\xi)$. So by Thom isomorphism, $\tilde{K}^1_{hS^1}(\mathcal{F}_1^{(n)}/\mathcal{F}_0) \cong K^0_{hS^1}(G(r)^{(n)})$, which by Theorem 3.7 is isomorphic to $\mathbb{Z}[[t]]^{(n)} \otimes_R \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle$. Likewise for $\tilde{K}^0_{hS^1}$. □

6.1 The first differential

We want to determine the first differential $d_1 : E_1^{0,*} \longrightarrow E_1^{1,*}$ in the Morse spectral sequence converging to $K^*_{hS^1}(L\mathbb{C}P^r)$. Using Remark 6.2, we have a concrete description of the E_1 term, and we get the following explicit formula for d_1:

Theorem 6.5. *The first differential d_1 in $E_*(\mathcal{M})(L\mathbb{C}P^r{}_{hS^1})$ is the $\mathbb{Z}[[t]]$-module homomorphism*

$$d_1 : \mathbb{Z}[[t]] \otimes \mathbb{Z}[h]/h^{r+1} \longrightarrow \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle$$

given by $d_1(h^j) = x^j - y^j$ for $j = 0, 1, \ldots, r$.

Proof. The first differential is induced by the boundary map δ below:

$$(\mathcal{F}_0)_{hS^1} \longrightarrow (\mathcal{F}_1)_{hS^1} \longrightarrow (\mathcal{F}_1)_{hS^1}/(\mathcal{F}_0)_{hS^1} \xrightarrow{\delta} \Sigma((\mathcal{F}_0)_{hS^1})$$

where Σ denotes the (reduced) suspension. From Morse theory (40) we have $(\mathcal{F}_1)_{hS^1}/(\mathcal{F}_0)_{hS^1} \simeq Th((\mu_1^-)_{hS^1})$, where μ_1^- is the negative bundle over $X = G(r)$, and we have the diagram

$$\begin{tikzcd}
S((\mu_1^-)_{hS^1}) \arrow[r] \arrow[d] & D((\mu_1^-)_{hS^1}) \arrow[r] \arrow[d] & Th((\mu_1^-)_{hS^1}) \arrow[r] \arrow[d, "\cong"] & \Sigma S((\mu_1^-)_{hS^1}) \arrow[d] \\
(\mathcal{F}_0)_{hS^1} \arrow[r] & (\mathcal{F}_1)_{hS^1} \arrow[r] & (\mathcal{F}_1)_{hS^1}/(\mathcal{F}_0)_{hS^1} \arrow[r, "\delta"] & \Sigma((\mathcal{F}_0)_{hS^1})
\end{tikzcd}$$

The vertical maps from the sphere- and disc bundles are given by the flow of the energy functional; we return to them later. First, since μ_1^- is an S^1-vector bundle, we can assume that the Riemmanian metric on it is S^1-invariant, so that $S((\mu_1^-)_{hS^1}) = ES^1 \times_{S^1} S(\mu_1^-)$, and $D((\mu_1^-)_{hS^1}) = ES^1 \times_{S^1} D(\mu_1^-)$. Then $Th((\mu_1^-)_{hS^1}) \cong ES^1_+ \wedge_{S^1} Th(\mu_1^-)$, see [Bökstedt-Ottosen] Lemma 5.2, and we get the diagram

$$\begin{tikzcd}
ES^1 \times_{S^1} S(\mu_1^-) \arrow[r] \arrow[d, "\mathrm{id}\times(f_+\sqcup f_-)"] & ES^1 \times_{S^1} D(\mu_1^-) \arrow[r] \arrow[d] & ES^1_+ \wedge_{S^1} Th(\mu_1^-) \arrow[d, "\cong"] \\
ES^1 \times_{S^1} \mathcal{F}_0 \arrow[r] & ES^1 \times_{S^1} \mathcal{F}_1 \arrow[r] & ES^1_+ \wedge_{S^1} \mathcal{F}_1/\mathcal{F}_0
\end{tikzcd}$$

This means we can simply ignore the ES^1-factor, and consider the diagram

$$\begin{tikzcd}
S(\mu_1^-) \arrow[r] \arrow[d, "f_+\sqcup f_-"] & D(\mu_1^-) \arrow[r] \arrow[d] & Th(\mu_1^-) \arrow[r] \arrow[d, "\cong"] & \Sigma S(\mu_1^-) \arrow[d, "\Sigma f_+\vee\Sigma f_-"] \\
\mathcal{F}_0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_1/\mathcal{F}_0 \arrow[r, "\delta"] & \Sigma\mathcal{F}_0
\end{tikzcd}$$

By the proof of Prop. 4.2, μ_1^- is a trivial real line bundle, and over a geodesic $\gamma \in X$, we can parametrize μ_1^- as $\mathbb{R}i\gamma'$. Therefore the sphere bundle $S(\mu_1^-) = X_+ \sqcup X_-$ is a disjoint union of two copies of the base space X, where the fiber is $(X_+)_\gamma = +i\gamma'$ and $(X_-)_\gamma = -i\gamma'$. The map $f_\pm : X_\pm \longrightarrow \mathcal{F}_0$ is given by the flow of the energy functional: For a geodesic $\gamma \in X$, $f_\pm(\gamma)$ gives the endpoint in $\mathcal{F}_0 = \mathbb{C}P^r$ for the flowlines in direction $\pm i\gamma'$. Since μ_1^- is 1-dimensional, the Thom space $Th(\mu_1^-)$ is just the suspension ΣX of the base space X, and $\Sigma S(\mu_1^-) = \Sigma X_+ \vee \Sigma X_-$. The map $\delta : \mathcal{F}_1/\mathcal{F}_0 \longrightarrow \Sigma\mathcal{F}_0$ is now the composition

$$\delta :\ \mathcal{F}_1/\mathcal{F}_0 \xrightarrow{\cong} \Sigma(X) \longrightarrow \Sigma X_+ \vee \Sigma X_- \xrightarrow{\Sigma f_+ \vee \Sigma f_-} \Sigma\mathcal{F}_0. \qquad (66)$$

Here, the last map folds the two summands in the wedge.

We now investigate the maps $f_\pm : G(r) \longrightarrow \mathbb{C}P^r$. Recall from (4) that the simple closed geodesic γ in $\mathbb{C}P^r$ determined by $[v,w] \in PV_2$ is given by the map

$$PV_2 \longrightarrow G(r), \quad [v,w] \mapsto q \circ c(x,v),$$

where $c(x,v)(t) = \cos(\pi t)x + \sin(\pi t)v$ for $t \in [0,1]$, and $q : S^{2r+1} \longrightarrow \mathbb{C}P^r$ is the projection. Such a γ is a geodesic on a $\mathbb{C}P^1 = \mathbb{P}\{v,w\} \subseteq \mathbb{C}P^r$, and we can give $\mathbb{P}\{v,w\}$ homogeneous coordinates, $[a_v, a_w] = q(a_v v + a_w w)$, and map

$$\mathbb{P}\{v,w\} \longrightarrow \mathbb{C} \cup \{\infty\}, \quad [a_v, a_w] \mapsto \frac{a_v}{a_w}.$$

We see that γ under this map is the curve $t \mapsto \frac{\cos(\pi t)}{\sin(\pi t)} = \frac{1}{\tan(\pi t)} \in \mathbb{C} \cup \{\infty\}$ for $t \in [0,1]$, i.e. the real line traversed in the "negative" direction, from $+\infty$ to $-\infty$. It is now clear that the flow in direction $+i\gamma'$ will end in $-i \in \mathbb{C} \cup \{\infty\}$, or homogeneous coordinates $\frac{1}{\sqrt{2}}[1,i] \in \mathbb{P}\{v,w\}$, so $f_+(\gamma) = \frac{1}{\sqrt{2}}[1,i] \in \mathbb{P}\{v,w\}$. The flow in direction $-i\gamma'$ ends in $i \in \mathbb{C} \cup \{\infty\}$, so $f_-(\gamma) = \frac{1}{\sqrt{2}}[1,-i] \in \mathbb{P}\{v,w\}$.

Having determined $f_\pm$, we can now calculate the induced map $f_\pm^*$ on $K^0(\mathbb{C}P^r) \cong \mathbb{Z}[h]/\langle h^r \rangle$, so we need only determine $f_\pm^*(h)$, where $h = [H] - 1$ and $H \searrow \mathbb{C}P^r$ is the standard line bundle. We do this by determining the pullback $f_\pm^*(H)$. From the preceding paragraph we see that the fiber of $f_+^*(H)$ over a simple closed geodesic γ determined by $[v,w] \in PV_2$ is exactly all the points on the line given by $\frac{1}{\sqrt{2}}(v + iw)$. Recall that the line bundle X was defined as the pullback of the standard bundle $\gamma_1 \searrow \mathbb{P}(\gamma_2)$ under the composite

$$\begin{array}{ccccccc} G(r) & \longrightarrow & PV_2 & \longrightarrow & \widetilde{PV}_2 & \longrightarrow & \mathbb{P}(\gamma_2), \\ \gamma & \mapsto & [v,w] & \mapsto & \frac{1}{\sqrt{2}}[v+iw, v-iw] & \mapsto & \mathbb{C}(v+iv) \subseteq \mathbb{C}v \oplus \mathbb{C}w \end{array}$$

It follows that $f_+^*(H) = X$, so $f_+^*(h) = x$. Likewise we get $f_-^*(h) = y$, because Y is the pullback of the complement of γ_1 in γ_2. Since $f_\pm^*$ is a ring homomorphism, we get $f_+^*(h^j) = x^j$, and $f_-^*(h^j) = y^j$. From (66), we can now compute $d_1(h^j)$. When folding the maps, the second suspension in the wedge $\Sigma X_+ \vee \Sigma X_-$ has the orientation reversed, so we obtain $d_1(h^j) = x^j - y^j$. $\square$

In the Morse spectral sequence $E_*(\mathcal{M})((L\mathbb{C}P^r)^{(n)}_{hS^1})$ for the (n)-twisted filtration, the first differential is a map $d_1^{(n)} : K^*_{hS^1}(\Sigma\mathcal{F}_0) \longrightarrow \tilde{K}^*_{hS^1}(\mathcal{F}_1^{(n)}/\mathcal{F}_0)$, cf. Lemma 6.4.

Lemma 6.6. *The first differential in $E_*(\mathcal{M})((L\mathbb{C}P^r)^{(n)}_{hS^1})$ is the map of $\mathbb{Z}[[t]]$-modules given by*

$$\begin{aligned} d_1^{(n)} : \mathbb{Z}[[t]] \otimes \mathbb{Z}[h]/\left\langle h^{r+1} \right\rangle &\longrightarrow \mathbb{Z}[[t]]^{(n)} \otimes_R \mathbb{Z}[x,y]/\left\langle Q_r, Q_{r+1} \right\rangle, \\ d_1^{(n)}(h^j) &= x^j - y^j, \quad \text{for } j = 0, 1, \ldots, r. \end{aligned}$$

Proof. Using the same diagram as in the proof of Theorem 6.5 above, we see that the geometry of this situation is exactly the same, so the flow map is identical to the one computed before. □

From (44), the power map $\mathcal{P}_j$ gives a map of the following exact sequences, giving a commutative diagram:

$$\begin{tikzcd} \mathcal{F}_0 \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{F}_1^{(j)} \arrow[r] \arrow[d, "\mathcal{P}_j"] & \mathcal{F}_1^{(j)}/\mathcal{F}_0 \arrow[r, "\delta_1^{(j)}"] \arrow[d, "\mathcal{P}_j"] & \Sigma\mathcal{F}_0 \arrow[d, "\mathrm{id}"] \\ \mathcal{F}_0 \arrow[r] & \mathcal{F}_\infty \arrow[r] & \mathcal{F}_\infty/\mathcal{F}_0 \arrow[r, "\delta"] & \Sigma\mathcal{F}_0 \end{tikzcd}$$

where $\delta_1^{(j)}$ denotes the boundary map which induces the first differential $d_1^{(j)}$ in the Morse spectral sequence $E_*(\mathcal{M})((L\mathbb{C}P^r)^{(j)}_{hS^1})$. So the differential $d_1^{(j)}$ determined in Lemma 6.6 can also be written as the composite map

$$d_1^{(j)} : K^1_{hS^1}(\Sigma\mathcal{F}_0) \xrightarrow{\delta} \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \xrightarrow{\mathcal{P}_j^*} \tilde{K}^1_{hS^1}(\mathcal{F}_1^{(j)}/\mathcal{F}_0). \tag{67}$$

6.2 The Main Theorem for $r > 1$

Again recall the notation $K^*_{hS^1}(X) = K^*(X_{hS^1})$. Now we introduce some more notation: For an S^1-space X with a connected set F of fixed points for the S^1-action, let $x \in F$ be some fixed point. The inclusion of x in X gives an S^1-equivariant map $i = i_x : * \longrightarrow X$. (Since F is connected, any two such inclusions i_x and i_y, $x, y \in F$, are homotopic.) Since i is S^1-equivariant, we obtain a map

$$BS^1 = ES^1 \times_{S^1} * \longrightarrow ES^1 \times_{S^1} X = X_{hS^1}.$$

Thus we can consider the relative group $K^*(X_{hS^1}, BS^1)$, and we use the notation $K^*_{hS^1}(X, *) := K^*(X_{hS^1}, BS^1)$. Note that since the composition $* \xrightarrow{i} X \longrightarrow *$ is the identity, we get

$$K^*_{hS^1}(*) \longrightarrow K^*_{hS^1}(X) \xrightarrow{i^*} K^*_{hS^1}(*)$$

is the identity. This gives a canonical splitting $K^*_{hS^1}(X) = K^*(BS^1) \oplus \mathrm{Ker}(i^*)$, and we see that $K^*_{hS^1}(X, *) = \mathrm{Ker}(i^*)$.

In this section, we will investigate $K^*_{hS^1}(L\mathbb{C}P^r)$. The idea is to twist the filtration with an integer. First we need a technical lemma:

Lemma 6.7. *Let $f \in \mathbb{Z}[[t]]$, and let $q_i : \mathbb{Z}[[t]] \longrightarrow \mathbb{Z}[[t]]^{(i)} \otimes_R \mathbb{Z}$ be the natural map, where $R = \mathbb{Z}[U, U^{-1}]$, $\mathbb{Z}[[t]]^{(i)}$ is $\mathbb{Z}[[t]]$ with the R-module structure $U \mapsto (t+1)^i$, and $\mathbb{Z}$ has the module structure $U \mapsto 1$. Then:*

(*i*) *If $q_i(f) \in n \cdot \mathbb{Z}[[t]]^{(i)} \otimes_R \mathbb{Z}$ for all $i \in \mathbb{N}$, then $f \in n \cdot \mathbb{Z}[[t]]$.*

(*ii*) *If $q_i(f) = 0$ for all $i \in \mathbb{N}$, then $f = 0$ in $\mathbb{Z}[[t]]$.*

Proof. First note that (*ii*) follows from (*i*): If $q_i(f) = 0$ for all $i \in \mathbb{N}$, then $q_i(f) \in n \cdot \mathbb{Z}[[t]]^{(i)} \otimes_R \mathbb{Z}$ for all i and all n. By (*i*) we get $f \in n \cdot \mathbb{Z}[[t]]$ for all $n \in \mathbb{N}$, and since only 0 in $\mathbb{Z}[[t]]$ is divisible by any n, this implies that $f = 0$ in $\mathbb{Z}[[t]]$.

So we must prove (*i*). By prime factoring n, we can assume $n = p^s$ where p is a prime number. Assume $q_i(f) \in n \cdot \mathbb{Z}[[t]]^{(i)} \otimes_R \mathbb{Z}$ for all $i \in \mathbb{N}$.

We have an injective map $i_p : \mathbb{Z}[[t]] \hookrightarrow \hat{\mathbb{Z}}_p[[t]]$, and we claim: If $i_p(f) \in p^s \hat{\mathbb{Z}}_p[[t]]$, then $f \in p^s \mathbb{Z}[[t]]$. Writing $f = \sum_j c_j t^j$ we have $f \in p^s \mathbb{Z}[[t]]$ if and only if $p^s \mid c_j$ for all j. By assumption we know $p^s \mid i_p(c_j)$ for all j. This means that the image of c_j under the composition

$$\mathbb{Z} \xrightarrow{i_p} \hat{\mathbb{Z}}_p = \varprojlim_m \mathbb{Z}/p^m \longrightarrow \mathbb{Z}/p^s,$$

is zero. But the composition is clearly the natural map $\mathbb{Z} \longrightarrow \mathbb{Z}/p^s$, so $p^s \mid c_j$ for any j. This proves the claim.

Knowing this, it suffices to show that $i_p(f) \in p^s \hat{\mathbb{Z}}_p[[t]]$. We apply the isomorphism

$$\varepsilon : \hat{\mathbb{Z}}_p[[t]] \xrightarrow{\cong} \varprojlim_m \hat{\mathbb{Z}}_p[C_{p^m}]$$

cf [Lang], Thm. 1.1, where C_k denotes the kth roots of unity, to make the following diagram for any $i \in \mathbb{N}$:

$$\begin{tikzcd}
\mathbb{Z}[[t]] \arrow[r, "q_{p^i}"] \arrow[d, "i_p"] & \mathbb{Z}[[t]]^{(p^i)} \otimes_R \mathbb{Z} \arrow[d, "\varphi"', "\cong"] \\
\hat{\mathbb{Z}}_p[[t]] \arrow[d, "\cong"', "\varepsilon"] & \mathbb{Z} \oplus \hat{\mathbb{Z}}_p\{V\} \oplus \cdots \oplus \hat{\mathbb{Z}}_p\{V^{p^i-1}\} \arrow[d, hook] \\
\varprojlim_m \hat{\mathbb{Z}}_p[C_{p^m}] \arrow[r, "\mathrm{pr}_i"] & \hat{\mathbb{Z}}_p[C_{p^i}]
\end{tikzcd} \tag{68}$$

Here the map pr_i denotes the natural projection on the ith term in the inverse limit, and the isomorphism φ is Lemma 3.9 and 3.5. This diagram is commutative by the definitions of the maps. Let $g = \varepsilon(i_p(f)) \in \lim \hat{\mathbb{Z}}_p[C_{p^m}]$. It is clear that if g satisfies $\mathrm{pr}_i(g) \in p^s \cdot \hat{\mathbb{Z}}_p[C_{p^i}]$ for all i, then g is divisible by p^s. Together with the commutativity of (68), this proves that $i_p(f) \in p^s \hat{\mathbb{Z}}_p[[t]]$, and we are done. □

We will prove the following

Theorem 6.8. *The map*

$$\delta : K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \longrightarrow \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0)$$

is injective.

Proof. We restrict the differential $d_1^{(j)} : K^1_{hS^1}(\Sigma\mathcal{F}_0) \longrightarrow \tilde{K}^1_{hS^1}(\mathcal{F}_1^{(j)}/\mathcal{F}_0)$ to the summand $K^1_{hS^1}(\Sigma\mathcal{F}_0, *)$; it is zero on $K^0_{hS^1}(*)$. By (67) this differential is the composition,

$$d_1^{(j)} : K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \xrightarrow{\delta} \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \xrightarrow{\mathcal{P}_j^*} \tilde{K}^1_{hS^1}(\mathcal{F}_1^{(j)}/\mathcal{F}_0).$$

Thus we can make a combined map, call it d,

$$d : K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \xrightarrow{\delta} \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \longrightarrow \coprod_j \tilde{K}^1_{hS^1}(\mathcal{F}_1^{(j)}/\mathcal{F}_0).$$

To prove that δ is injective, it suffices to show that d is injective. So let $a \in K^1_{hS^1}(\Sigma\mathcal{F}_0, *)$ with $d_1^{(i)}(a) = 0$ for all i. We must prove $a = 0$. Recall by Lemma 6.4,

$$\tilde{K}^1_{hS^1}(\mathcal{F}^{(i)}/\mathcal{F}_0) = \mathbb{Z}[[t]]^{(i)} \otimes_R M,$$

where $M = K^0(\Delta(r)) = \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle$. Let $M_j \subseteq M$ be the filtration from Remark 3.2. Then $\mathbb{Z}[[t]]^{(i)} \otimes_R M_j$ gives a filtration of $\mathbb{Z}[[t]]^{(i)} \otimes_R M$. Similarly, let $L_j \subseteq \mathbb{Z}[h]/\langle h^{r+1}\rangle$ be generated by $\{h^j, \ldots, h^r\}$. Then $K^1_{hS^1}(\Sigma\mathcal{F}_0, *) = \mathbb{Z}[[t]] \otimes_{\mathbb{Z}} L_1$.

Write $a = f_1(t)h + f_2(t)h^2 + \ldots + f_r(t)h^r$, where $f_j(t) \in \mathbb{Z}[[t]]$. For the purpose of induction, consider $a_j = f_j(t)h^j + f_{j+1}(t)h^{j+1} + \ldots + f_r(t)h^r$, and assume $d_1^{(i)}(a_j) = 0$ for all i. This holds for $j = 1$. Then $a_j \in \mathbb{Z}[[t]] \otimes L_j$, and we consider the image of under $d_1^{(i)}$, see Lemma 6.6:

$$\begin{array}{ccc} \mathbb{Z}[[t]] \otimes L_j & \xrightarrow{d_1^{(i)}} & \mathbb{Z}[[t]]^{(i)} \otimes_R M_j, \\ f_j h^j + \ldots + f_r h^r & \mapsto & f_j(x^j - y^j) + \ldots + f_r(x^r - y^r). \end{array}$$

By assumption, $0 = d_1^{(i)}(a_j) = f_j(x^j - y^j) + \ldots + f_r(x^r - y^r)$ for all i. Now we use the projection $\pi_j : M_j \longrightarrow M_j/M_{j+1}$, which induces a map

$$\mathbb{Z}[[t]]^{(i)} \otimes_R M_j \xrightarrow{\pi_j} \mathbb{Z}[[t]]^{(i)} \otimes_R M_j/M_{j+1}.$$

Then $0 = \pi_j(d_1^{(i)}(a_j)) = f_j(x^j - y^j)$ in $\mathbb{Z}[[t]]^{(i)} \otimes_R M_j/M_{j+1}$ for all i. Note that $M_j/M_{j+1} = \mathbb{Z}x^j \oplus \mathbb{Z}x^{j-1}y \oplus \cdots \oplus \mathbb{Z}y^j$. Construct a map $q : M_j/M_{j+1} \longrightarrow \mathbb{Z}$, by

$$q(x^j) = 1, \quad q(x^{j-1}y) = -1, \quad q(x^{j-k}y^k) = 0, \text{ for } k > 1. \tag{69}$$

This is well-defined: If $j < r$ the monomials are independent, and if $j = r$ we have in M_j/M_{j+1} the relation $Q_r = 0$, and the map satisfies $q(Q_r) = 0$. So we get a map

$$q : \mathbb{Z}[[t]]^{(i)} \otimes_R M_j/M_{j+1} \longrightarrow \mathbb{Z}[[t]]^{(i)} \otimes_R \mathbb{Z}. \tag{70}$$

If $j > 1$ we get $q(f_j(x^j - y^j)) = f_j$, and if $j = 1$ we get $q(f_1(x - y)) = 2f_1$, but we also have $q(f_j(x^j - y^j)) = q(\pi_j(d_1^{(i)}(a_j))) = 0$. The conclusion is in both cases that $f_j(t) = 0$ in $\mathbb{Z}[[t]]^{(i)} \otimes_R \mathbb{Z}$ for all i. By Lemma 6.7 this implies $f_j(t) = 0$ in $\mathbb{Z}[[t]]$. Since $a_j = f_j(t)x^j + a_{j+1}$, inductively we get $d_1^{(i)}(a_{j+1}) = 0$ for all i. This finishes the induction step. This induction shows that $a = 0$ in $K^1_{hS^1}(\Sigma\mathcal{F}_0, *)$. □

As a corollary, we obtain

Main Theorem 6.9. *As $K^*(BS^1)$-modules,*

$$K^0_{hS^1}(L\mathbb{C}P^r) = K^0(BS^1) = \mathbb{Z}[[t]]\,.$$

Proof. It suffices to show that $K^0_{hS^1}(L\mathbb{C}P^r, *) = 0$. We use the long exact sequence for $\mathcal{F}_0 \longrightarrow \mathcal{F}_\infty \longrightarrow \mathcal{F}_\infty/\mathcal{F}_0 \longrightarrow \Sigma\mathcal{F}_0$,

$$\begin{array}{rcl} 0 & \longrightarrow & \tilde{K}^0_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \longrightarrow K^0_{hS^1}(\mathcal{F}_\infty) \longrightarrow \\ K^1_{hS^1}(\Sigma\mathcal{F}_0) & \xrightarrow{\delta} & \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \longrightarrow K^1_{hS^1}(\mathcal{F}_\infty) \longrightarrow 0 \end{array} \tag{71}$$

By the Morse spectral sequence, we know that $\tilde{K}^0_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) = 0$, see Prop. 6.3. We can write part of (71) as follows:

$$0 \longrightarrow K^0_{hS^1}(\mathcal{F}_\infty, *) \oplus K_{hS^1}(*) \longrightarrow K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \oplus K_{hS^1}(*) \xrightarrow{\delta} \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0)$$

Theorem 6.8 tells us that $\delta : K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \longrightarrow K^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0)$ is injective, so when we split off the summand $K_{hS^1}(*)$, we get that $K^0_{hS^1}(\mathcal{F}_\infty, *) = 0$. □

Having determined $K^0_{hS^1}(L\mathbb{C}P^r)$, we now move on to $K^1_{hS^1}(L\mathbb{C}P^r)$. Regrettably, we are only able to determine this as an abelian group, not a $K^*(BS^1)$-module.

Main Theorem 6.10. *$K^1_{hS^1}(L\mathbb{C}P^r)$ is a free abelian group.*

In this section we prove the Theorem in all cases except one:

Theorem 6.11. *If $(r,n) \neq (1,2)$, then $K^1_{hS^1}(L\mathbb{C}P^r)$ has no n-torsion.*

The essential part of the proof is the following proposition:

Proposition 6.12. *Let $a \in K^1_{hS^1}(\Sigma\mathcal{F}_0, *)$, and assume that for all $i \geq 1$, $d_1^{(i)}(a) \in n\tilde{K}^1_{hS^1}(\mathcal{F}_1^{(i)}/\mathcal{F}_0)$. Then,*

(*i*) *If $r > 1$, then $a \in nK^1_{hS^1}(\Sigma\mathcal{F}_0, *)$.*

(*ii*) *If $r = 1$ and $n > 2$, then $2a \in nK^1_{hS^1}(\Sigma\mathcal{F}_0, *)$.*

Proof that Theorem 6.11 follows from Prop. 6.12. Assume $b \in K^1_{hS^1}(\mathcal{F}_\infty)$ with $nb = 0$ for some $n \in \mathbb{Z}$. We will show b is not n-torsion. By the exact sequence

$$K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \xrightarrow{\delta} \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \longrightarrow K^1_{hS^1}(\mathcal{F}_\infty) \longrightarrow 0,$$

we can lift b to $\bar{b} \in \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0)$, and there is $a \in K^1_{hS^1}(\Sigma\mathcal{F}_0, *)$ with image $\delta(a) = n\bar{b}$. Since $d_1^{(i)}$ is the composition,

$$d_1^{(i)} : K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \xrightarrow{\delta} \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \longrightarrow \tilde{K}^1_{hS^1}(\mathcal{F}^{(i)}/\mathcal{F}_0),$$

and $\delta(a) = n\bar{b}$, we see that $d_1^{(i)}(a) \in n\tilde{K}^1_{hS^1}(\mathcal{F}_1^{(i)}/\mathcal{F}_0)$ for all i. So we can apply the proposition. In case (i) we get $a \in nK^1_{hS^1}(\Sigma\mathcal{F}_0, *)$, so $a = na'$. Then in $\tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0)$, $n\delta(a') = n\bar{b}$. But $\tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0)$ is torsion-free by Prop. 6.3, so $\delta(a') = \bar{b}$, which implies $b = 0$. This proves the claim in case (i). In case (ii), we get $2a = na'$, so $n\delta(a') = 2n\bar{b}$ in $\tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0)$ which is torsion-free, so $\delta(a') = 2\bar{b}$, i.e. $2b = 0$. Since $n > 2$, b is not n-torsion. □

Proof of Proposition 6.12. Let $a \in K^1_{hS^1}(\Sigma\mathcal{F}_0, *)$, and assume $n \mid d_1^{(i)}(a)$ for all i. This proof is similar to the proof of Theorem 6.8.

Let $M = \mathbb{Z}[x,y]/\langle Q_r, Q_{r+1}\rangle$, and let $M_j \subseteq M$ be the filtration from Remark 3.2. Then $\mathbb{Z}[[t]]^{(i)} \otimes_R M_j$ gives a filtration of $\mathbb{Z}[[t]]^{(i)} \otimes_R M$. Similarly, let $L_j \subseteq \mathbb{Z}[x]/\langle x^{r+1}\rangle$ be generated by $\{x^j, \ldots, x^r\}$. Then $K^1_{hS^1}(\Sigma\mathcal{F}_0, *) = \mathbb{Z}[[t]] \otimes L_1$. Write $a = f_1(t)x + f_2(t)x^2 + \ldots + f_r(t)x^r$, where $f_j(t) \in \mathbb{Z}[[t]]$. For the purpose of induction, consider $a_j = f_j(t)x^j + f_2(t)x^2 + \ldots + f_r(t)x^r$, and assume $n \mid d_1^{(i)}(a_j)$ for all i. This holds for $j = 1$. Then $a_j \in \mathbb{Z}[[t]] \otimes L_j$, and we consider the image of under $d_1^{(i)}$:

$$\begin{array}{rcl} \mathbb{Z}[[t]] \otimes_R L_j & \xrightarrow{d_1^{(i)}} & \mathbb{Z}[[t]]^{(i)} \otimes_R M_j, \\ f_j x^j + \ldots + f_r x^r & \mapsto & f_j(x^j - y^j) + \ldots + f_r(x^r - y^r). \end{array}$$

By assumption, $f_j(x^j - y^j) + \ldots + f_r(x^r - y^r) = nb$ for some b. Now we use the projection $\pi_j : M_j \longrightarrow M_j/M_{j+1}$, which induces a map

$$\begin{array}{rcl} \mathbb{Z}[[t]]^{(i)} \otimes_R M_j & \xrightarrow{\pi_j} & \mathbb{Z}[[t]]^{(i)} \otimes_R M_j/M_{j+1}, \\ f_j(x^j - y^j) + \ldots + f_r(x^r - y^r) = nb & \mapsto & f_j(x^j - y^j) = n \cdot \pi_j(b). \end{array}$$

We wish to map $M_j/M_{j+1} \longrightarrow \mathbb{Z}$. For now, assume $r > 1$. If $j > 1$, we use the map q from (69), (70). Since $q(x^j - y^j) = 1$ for $j > 1$, we get

$$\begin{array}{rcl} \mathbb{Z}[[t]]^{(i)} \otimes_R M_j/M_{j+1} & \xrightarrow{q} & \mathbb{Z}[[t]]^{(i)} \otimes_R \mathbb{Z}, \\ f_j(x^j - y^j) = n \cdot \pi_j(b) & \mapsto & f_j = n \cdot q\pi_j(b). \end{array} \tag{72}$$

If $j = 1$, we use the well-defined map $q_1(x) = 1$, $q_1(y) = 0$, and get the same result. The conclusion is that $f_j(t) \in n \cdot \mathbb{Z}[[t]]^{(i)} \otimes_R \mathbb{Z}$ for all i. By Lemma 6.7 this implies $f_j(t) \in n\mathbb{Z}[[t]]$. Since $a_j = f_j(t)x^j + a_{j+1}$, inductively we get $n \mid d_1^{(i)}(a_{j+1})$ for all i. This finishes the induction step. This induction shows that $n \mid f_j(t)$ for all $j = 1, \ldots, r$, so $a \in nK^1_{hS^1}(\Sigma\mathcal{F}_0, *)$.

Now take $r = 1$. Then $j = 1$. We use the map $q : M_1/M_2 \longrightarrow \mathbb{Z}$ from (69). Then in (72), we get instead $2f_1(t) \in n \cdot \mathbb{Z}[[t]] \otimes_R \mathbb{Z}$. By Lemma 6.7, $2f_1(t) \in n\mathbb{Z}[[t]]$, and $2a \in nK^1_{hS^1}(\Sigma\mathcal{F}_0, *)$. □

6.3 The Main Theorem for $r = 1$

In this section we show the result of Main Theorem 6.10 in the case $r = 1$:

Theorem 6.13. *$K^1_{hS^1}(L\mathbb{C}P^1)$ has no 2-torsion.*

First recall by Theorem 6.1 and Lemma 6.4 that when $r = 1$,

$$\tilde{K}^1_{hS^1}(\mathcal{F}_1^{(k)}/\mathcal{F}_0) \cong \tilde{K}^1_{hS^1}(\mathcal{F}_k/\mathcal{F}_0) \cong \mathbb{Z}[[t]]^{(k)} \otimes_R M, \quad \text{where } M = \mathbb{Z}[x]/x^2.$$

This is because $M = \mathbb{Z}[x,y]/\langle Q_1, Q_2\rangle$, and $Q_1 = x + y$, so $y = -x$, which when substituting in $Q_2 = x^2 + xy + y^2$ gives $x^2 = 0$.

In the proof we will need the S^1 transfer map on K-theory:

Lemma 6.14. *There is an S^1 transfer map τ on K-theory, which fits into the following exact sequence,*

$$\longrightarrow K^0(X) \xrightarrow{\tau} K^1_{hS^1}(X) \xrightarrow{\varphi} K^1_{hS^1}(X) \xrightarrow{q} K^1(X) \xrightarrow{\tau} K^0_{hS^1}(X) \longrightarrow$$

where $K^*(BS^1) = \mathbb{Z}[[t]]$, and the map φ is multiplication by $-t$.

Proof. Let $T \longrightarrow BS^1$ denote the standard complex line bundle, as usual. Let $p : ES^1 \times_{S^1} X \longrightarrow BS^1$, be projection on the first factor, and let $\xi = p^*T$ denote the pullback. As in (48), we use the cofiber sequence,

$$S(\xi) \longrightarrow D(\xi) \longrightarrow Th(\xi).$$

As shown in (49), $S(\xi) \cong ES^1 \times X \simeq X$. The long exact sequence on K-theory becomes, using the Thom isomorphism, cf. [Atiyah] Cor. 2.7.3,

$$K^{*-1}(X) \xrightarrow{\delta} K^*(ES^1 \times_{S^1} X) \xrightarrow{\varphi} K^*(ES^1 \times_{S^1} X) \longrightarrow K^*(X) \xrightarrow{\delta}$$

The map φ is given by multiplication with $\Lambda_{-1}(T) = 1 - T = -t$, since T is a line bundle. We define the S^1 transfer map τ to be the boundary map δ in the long exact sequence. □

By exactness, $\mathrm{Im}(\tau) = \mathrm{Ker}(\varphi)$, and so we will need the kernel of t:

Lemma 6.15. *The kernel of the map given by multiplication by t,*

$$t : \mathbb{Z}[[t]]^{(k)} \otimes_R \mathbb{Z}[x]/x^2 \longrightarrow \mathbb{Z}[[t]]^{(k)} \otimes_R \mathbb{Z}[x]/x^2$$

is $\mathbb{Z}p_{k-1}(t)x$, where $(t+1)^k - 1 = tp_{k-1}(t)$.

Proof. First we relate the kernel of t to the kernel of $u : M \longrightarrow M$ (this part holds for all r). Recall $R = R(S^1) = \mathbb{Z}[U, U^{-1}]$, and let $u = U - 1$. Then M is an R-module by $u \mapsto (x - y)/(1 + y)$, and $\mathbb{Z}[[t]]^{(k)}$ is an R-module by $u \mapsto (t + 1)^k - 1$. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}[[t]] \xrightarrow{t} \mathbb{Z}[[t]] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Tensoring with M over R yields the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(\mathbb{Z}, M) \longrightarrow \mathbb{Z}[[t]]^{(k)} \otimes_R M \xrightarrow{t} \mathbb{Z}[[t]]^{(k)} \otimes_R M$$

To compute $\mathrm{Ker}(t) \cong \mathrm{Tor}_1^R(\mathbb{Z}, M)$, we use the following free resolution of $\mathbb{Z}$ over R:

$$0 \longrightarrow R \xrightarrow{u} R \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Again, we tensor over R with M and find

$$0 \longrightarrow \mathrm{Tor}_1^R(\mathbb{Z}, M) \longrightarrow R \otimes_R M \xrightarrow{u} R \otimes_R M \longrightarrow \mathbb{Z}[[t]]^{(k)} \otimes_R \mathbb{Z} \longrightarrow 0.$$

so $\mathrm{Tor}_1^R(\mathbb{Z}, M) \cong \mathrm{Ker}(u)$. All we need to know is how to translate from $\mathrm{Ker}(u)$ to $\mathrm{Ker}(t)$. The following diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(u) \arrow[r] \arrow[d] & R \otimes_R M \arrow[r, "u"] \arrow[d, "p_{k-1}(t)\otimes \mathrm{id}"] & R \otimes_R M \arrow[r] \arrow[d] & \mathbb{Z}[[t]]^{(k)} \otimes_R \mathbb{Z} \arrow[d, "\mathrm{id}"] \\
0 \arrow[r] & \mathrm{Ker}(t) \arrow[r] & \mathbb{Z}[[t]]^{(k)} \otimes_R M \arrow[r, "t"] & \mathbb{Z}[[t]]^{(k)} \otimes_R M \arrow[r] & \mathbb{Z}[[t]]^{(k)} \otimes_R \mathbb{Z}
\end{tikzcd}$$

is commutative, since $tp_{k-1}(t) = (t+1)^k - 1 = u$. From this diagram, we see that $\mathrm{Ker}(t) = p_{k-1}(t)\mathrm{Ker}(u)$.

So all that remains is to determine $\mathrm{Ker}(u)$. This can be done for any r, but it is especially easy when $r = 1$, and $M = \mathbb{Z}[x]/x^2$, where $u1 = 2x$ and $ux = 0$. Clearly $\mathrm{Ker}(u) = \mathbb{Z}x$, and so $\mathrm{Ker}(t) = \mathbb{Z}p_{k-1}(t)x$. □

We can now prove the Main Theorem in case $r = 1$:

Proof of Theorem 6.13. By the exact sequence

$$K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \xrightarrow{\delta} \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \longrightarrow K^1_{hS^1}(\mathcal{F}_\infty) \longrightarrow 0,$$

we see that $K^1_{hS^1}(L\mathbb{C}P^1) = K^1_{hS^1}(\mathcal{F}_\infty)$ is isomorphic to the cokernel $\mathrm{Cok}(\delta)$ of δ. Since $r = 1$, $K^1_{hS^1}(\Sigma\mathcal{F}_0, *) = \mathbb{Z}[[t]] \cdot h$, so let $f(t) \in \mathbb{Z}[[t]]$ be given, and assume that $\delta(f(t)h)$ is divisible by 2. We will show that this implies $f(t)$ is divisible by 2, meaning that there is no 2-torsion in $\mathrm{Cok}(\delta)$.

For contradiction, assume that $f(t)$ is not divisible by 2. Then, without loss of generality, $f(t)$ has the form $t^l g(t)$, where $g(t) = 1 + tp(t)$ for some $p(t) \in \mathbb{Z}[[t]]$. Here l is the first exponent in $f(t)$ with an odd coefficient, and so $2 \mid \delta(f(t)h)$ if and only if $2 \mid \delta(t^l g(t)h)$. Then $g(t)$ is a unit in $\mathbb{Z}[[t]]$, so since δ is a $\mathbb{Z}[[t]]$-module homomorphism, $2 \mid \delta(t^l g(t)h)$ if and only if $2 \mid \delta(t^l h)$. We have shown that if $\delta(f(t)h)$ is divisible by 2, but $f(t)$ is not divisible by 2, then $\delta(t^{N-1}h)$ is also divisible by 2 for all $N > l$.

We will now show that this leads to a contradiction if $N = 2^n > l$. Consider the composite map, which we call $d_2^{(N)}$,

$$K^1_{hS^1}(\Sigma\mathcal{F}_0, *) \xrightarrow{\delta} \tilde{K}^1_{hS^1}(\mathcal{F}_\infty/\mathcal{F}_0) \longrightarrow \tilde{K}^1_{hS^1}(\mathcal{F}_{2N}/\mathcal{F}_0) \xrightarrow{\mathcal{P}_N{}^*} \tilde{K}^1_{hS^1}(\mathcal{F}_2^{(N)}/\mathcal{F}_0)$$

Then $d_2^{(N)}(t^{N-1}h)$ is divisible by 2, since $\delta(t^{N-1}h)$ is. We will investigate $d_2^{(N)}(t^{N-1}h)$ via the following diagram:

$$\begin{tikzcd} \Sigma\mathcal{F}_0 & & (\mathcal{F}_1/\mathcal{F}_0)^{(2N)} \arrow[ll] \arrow[d] \arrow[dr] & \\ (\mathcal{F}_1/\mathcal{F}_0)^{(N)} \arrow[u] \arrow[rr] & & (\mathcal{F}_2/\mathcal{F}_0)^{(N)} \arrow[ull] \arrow[r] & (\mathcal{F}_2/\mathcal{F}_1)^{(N)} \end{tikzcd}$$

The maps into $\Sigma\mathcal{F}_0$ are the ones inducing the various differentials in the Morse spectral sequences. The map $(\mathcal{F}_1/\mathcal{F}_0)^{(2N)} \longrightarrow (\mathcal{F}_2/\mathcal{F}_1)^{(N)}$ is simply the composite of the two other maps in the triangle

$$(\mathcal{F}_1/\mathcal{F}_0)^{(2N)} \xrightarrow{\mathcal{P}_2^{(N)}} (\mathcal{F}_2/\mathcal{F}_0)^{(N)} \longrightarrow (\mathcal{F}_2/\mathcal{F}_1)^{(N)} .$$

On S^1-equivariant K-theory this becomes

$$\begin{tikzcd} K^1_{hS^1}(\Sigma\mathcal{F}_0) \arrow[r, "d_1^{(2N)}"] \arrow[d, "d_1^{(N)}"] \arrow[dr, "d_2^{(N)}"] & \tilde{K}^1_{hS^1}((\mathcal{F}_1/\mathcal{F}_0)^{(2N)}) & \\ \tilde{K}^1_{hS^1}((\mathcal{F}_1/\mathcal{F}_0)^{(N)}) & \tilde{K}^1_{hS^1}((\mathcal{F}_2/\mathcal{F}_0)^{(N)}) \arrow[l, "i"'] \arrow[u, "k"] & \tilde{K}^1_{hS^1}((\mathcal{F}_2/\mathcal{F}_1)^{(N)}) \arrow[l, "j"'] \arrow[ul, "E_N"'] \end{tikzcd} \tag{73}$$

with the lower row short exact (i surjective and j injective). When $N = 2^n$, we have

$$p_{N-1}(t) = t^{-1}((t+1)^{2^n} - 1) = t^{N-1} + 2q(t),$$

for some polynomial $q(t)$, since all binomial coefficients $\binom{2^n}{j}$ are divisible by 2 for $j \neq 0, 2^n$. Since we have deduced that $d_2^{(N)}(t^{N-1}h)$ is divisible by 2, we therefore get $d_2^{(N)}(p_{N-1}(t)h)$ is also divisible by 2, say $d_2^{(N)}(p_{N-1}(t)h) = 2a$ for some a in $\tilde{K}^1_{hS^1}((\mathcal{F}_2/\mathcal{F}_0)^{(N)})$. By Lemma 5.3 we see that $d_1^{(N)}(p_{N-1}(t)h) = 2p_{N-1}(t)x$. Since the diagram (73) is commutative, we get $i(a) = p_{N-1}(t)x$, since the group $\tilde{K}^1_{hS^1}((\mathcal{F}_1/\mathcal{F}_0)^{(N)})$ is torsion-free, see Lemma 6.4.

We now use the S^1 transfer, see Lemma 6.14. We can choose a transfer class $e \in K^1(\mathcal{F}_1/\mathcal{F}_0)$, such that $\tau(e) = p_{N-1}(t)x$ by Lemma 6.15. We can lift this transfer class to $\bar{e} \in K^1(\mathcal{F}_2/\mathcal{F}_0)$, so $i(\tau(\bar{e})) = \tau(e) = p_{N-1}(t)x$. Thus we have an element $w = a - \tau(\bar{e}) \in \tilde{K}^1_{hS^1}((\mathcal{F}_2/\mathcal{F}_0)^{(N)})$ with $i(w) = 0$. By exactness of the lower row in (73), there is an element $z \in \tilde{K}^1_{hS^1}((\mathcal{F}_2/\mathcal{F}_1)^{(N)})$ with $j(z) = w$. By commutativity of (73), we get

$$E_N(z) = k(w) = k(a - \tau(\bar{e})) = k(a) - k(\tau(\bar{e})),$$

so let us compute this. Since $2a = d_2^{(N)}(p_{N-1}h)$, we see that $k(2a) = d_1^{(2N)}(p_{N-1}h) = 2p_{N-1}x$, and since $\tilde{K}^1_{hS^1}((\mathcal{F}_1/\mathcal{F}_0)^{(2N)})$ is torsion-free, $k(a) =$

$p_{N-1}x$. But $k(\tau(\bar{e}))$ is in the image of the transfer map, so by Lemma 6.15, $k(\tau(\bar{e})) = mp_{2N-1}(t)x$ for some $m \in \mathbb{Z}$. In conclusion,

$$E_N(z) = k(w) = (p_{N-1}(t) - mp_{2N-1}(t))x. \tag{74}$$

To investigate this equality, we will need to use $\mathbb{F}_2$-coefficients, and to determine the map E_N. This is done in the following lemmas:

Lemma 6.16. *As $K^*(BS^1) = \mathbb{Z}[[t]]$-modules,*

$$\tilde{K}^1_{hS^1}((\mathcal{F}_1/\mathcal{F}_0)^{(2^k)}; \mathbb{F}_2) \cong (\mathbb{F}_2[t]/t^{2^k})1 \oplus (\mathbb{F}_2[t]/t^{2^k})x.$$

Proof. As explained in the beginning,

$$\tilde{K}^1_{hS^1}((\mathcal{F}_1/\mathcal{F}_0)^{(2^k)}; \mathbb{F}_2) \cong \mathbb{Z}[[t]]^{(2^k)} \otimes_R M \otimes_{\mathbb{Z}} \mathbb{F}_2,$$

where $M = \mathbb{Z}[x]/x^2$, and $u1 = 2x$, $ux = 0$. So we see that $M \otimes_{\mathbb{Z}} \mathbb{F}_2 = \mathbb{F}_2 \oplus \mathbb{F}_2$ is trivial as an $R = \mathbb{Z}[U, U^{-1}]$-module. So

$$\mathbb{Z}[[t]]^{(2^k)} \otimes_R M \otimes_{\mathbb{Z}} \mathbb{F}_2 = (\mathbb{Z}[[t]]^{(2^k)} \otimes_R \mathbb{F}_2)1 \oplus (\mathbb{Z}[[t]]^{(2^k)} \otimes_R \mathbb{F}_2)x.$$

On $\mathbb{Z}[[t]]^{(2^k)}$, u acts as $(t+1)^{2^k} - 1 \equiv t^{2^k} (\text{mod } 2)$. Therefore, $\mathbb{Z}[[t]]^{(2^k)} \otimes_R \mathbb{F}_2 = \mathbb{F}_2[t]/t^{2^k}$. This shows the Lemma. □

Lemma 6.17. *The map E_N is multiplication by $1 - (t+1)^N$.*

Proof. We must determine the map induced by $(\mathcal{F}_1/\mathcal{F}_0)^{(2N)} \longrightarrow (\mathcal{F}_2/\mathcal{F}_1)^{(N)}$, which is the (N)-twisting of the composite map

$$(\mathcal{F}_1/\mathcal{F}_0)^{(2)} \xrightarrow{\mathcal{P}_2} \mathcal{F}_2/\mathcal{F}_0 \longrightarrow \mathcal{F}_2/\mathcal{F}_1.$$

We will first study this untwisted case. The induced map, call it E, is given as follows:

$$\begin{tikzcd}
\tilde{K}^1_{hS^1}(\mathcal{F}_2/\mathcal{F}_1) \arrow[r, "\cong"] \arrow[d] & \tilde{K}^1_{hS^1}(Th(\mu_2^-)) \arrow[r, "\Phi_2"] \arrow[d] & K^0_{hS^1}(G_2(r)) \arrow[d, "E"] \\
\tilde{K}^1_{hS^1}((\mathcal{F}_2/\mathcal{F}_1)^{(2)}) \arrow[r, "\cong"] & \tilde{K}^1_{hS^1}(Th((\mu_1^-)^{(2)})) \arrow[r, "\Phi_1"] & K^0_{hS^1}(G(r)^{(2)})
\end{tikzcd}$$

where the first isomorphisms are Morse theory, and the Φ_j denote the Thom isomorphisms (the index indicates which negative bundle). Also, $G_2(r)$ is the geodesics of length 2, which as an S^1-space is isomorphic to $G(r)^{(2)}$, the (2)-twisted space of simple closed geodesics of length 1.

This is a special case of the following general situation: For a bundle and a subbundle, $\xi \subseteq \eta$, over a space X, the following diagram commutes

$$\begin{tikzcd} \tilde{K}^*(Th(\eta)) \arrow[r] & \tilde{K}^*(Th(\xi)) \\ K^*(X) \arrow[u, "\Phi_\eta", "\cong"'] \arrow[r, "\Lambda"] & K^*(X) \arrow[u, "\Phi_\xi", "\cong"'] \end{tikzcd}$$

The vertical maps are the Thom isomorphisms. Then the induced map on K-theory of the base space is given by multiplication by the Euler class $\Lambda = \Lambda_{-1}(\eta - \xi)$ of the bundle $\eta - \xi$, i.e. the (orthogonal) complement of ξ inside η.

So we need the negative bundle $\mu_2^- = \varepsilon_2 \oplus \nu_2$ over $G_2(r)$, see Proposition 4.2. I have given ε and ν an index, so one can distinguish between them for μ_2^- and μ_1^-. Now $(\mu_1^-)^{(2)}$ is not a priori a subbundle of μ_2^-, but since $\mu_1^- = \varepsilon_1$ where the S^1 action is trivial on the fibers, we see that $(\mu_1^-)^{(2)} = \varepsilon_2$ as bundles over $G(r)^{(2)} \cong G_2(r)$, so that $\mu_2^- - (\mu_1^-)^{(2)} = \nu_2 = \nu$, where ν is the complex bundle found in the proof of Proposition 4.2. From here, we know that for a geodesic f of length 2, parametrized as $f(t)$ for $t \in [0,1]$, the fiber of ν over f is given by $g(t)if'(t)$ for $t \in [0,1]$, where $g \in \operatorname{span}_{\mathbb{R}} \{\cos(2\pi t), \sin(2\pi t)\}$. The rotation action of S^1 is given by, for $\theta \in [0,1]$:

$$\theta * (f(t), \cos(2\pi t)if'(t)) = (f(t-\theta), \cos(2\pi t - 2\pi\theta)if'(t-\theta))$$

and similarly for $\sin(2\pi t)$. The complex structure J found in the proof of Proposition 4.2 is $J(\cos(2\pi t)) = \sin(2\pi t)$.

Now let us compare this to the bundle T, i.e. the bundle coming from the standard representation of S^1. Ignoring the S^1 action, T is just a product bundle $G_2(r) \times \mathbb{C}$. The S^1 action of $\theta \in [0,1]$ is given by

$$\theta * (f(t), c) = (f(t-\theta), e^{2\pi i\theta}c), \quad \text{for } t \in [0,1].$$

We will now construct a map $\varphi : T \longrightarrow \nu$, given by

$$\varphi(f,c)(t) = (f(t), c\cos(2\pi t)if'(t)).$$

We check that this is S^1-equivariant, i.e. that the following diagram commutes (it suffices to check $c = 1$):

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, "\theta*"'] & \nu \arrow[d, "\theta*"] \\ T \arrow[r, "\varphi"] & \nu \end{tikzcd} \qquad \begin{tikzcd} (f(t),1) \arrow[r, mapsto, "\varphi"] \arrow[d, mapsto, "\theta*"'] & (f(t), \cos(2\pi t)if'(t)) \arrow[d, mapsto, dashed, "\theta*?"] \\ (f(t-\theta), e^{2\pi i\theta}) \arrow[r, mapsto, "\varphi"] & (f(t-\theta), e^{2\pi i\theta}\cos(2\pi t)if'(t-\theta)) \end{tikzcd}$$

This commutes, since $e^{2\pi i\theta} = \cos(2\pi\theta) + i\sin(2\pi\theta)$ is multiplied on $\cos(2\pi t)$ as

$$\begin{aligned} e^{2\pi i\theta}\cos(2\pi t) &= \cos(2\pi\theta)\cos(2\pi t) + \sin(2\pi\theta)J(\cos(2\pi t)) \\ &= \cos(2\pi\theta)\cos(2\pi t) + \sin(2\pi\theta)\sin(2\pi t) \\ &= \cos(2\pi(t-\theta)) \end{aligned}$$

by the trigonometric formula. So φ is S^1-equivariant. Then φ defines an isomorphism of S^1 bundles, since it is clearly an isomorphism on the fibers. We have shown $\mu_2^- - (\mu_1^-)^{(2)} = \nu \cong T$.

Now let us look at the (N)-twisted case. We get again $(\mu_1^-)^{(2N)} = \varepsilon_{2N}$, and so $(\mu_2^-)^{(N)} - (\mu_1^-)^{(2N)} = \nu^{(N)} \cong T^{(N)}$, by the above isomorphism. Now, $T^{(N)}$ is the bundle with S^1 action of $\theta \in [0,1]$ given by

$$\theta * (f(t), c) = (f(t-\theta), (e^{2\pi i t})^N c), \quad \text{for } t \in [0,1].$$

This shows that this is the same bundle as T^N, so the map E_N is multiplication by the Euler class of T^N, and since this is a line bundle, we get $\Lambda_{-1}(T^N) = 1 - T^N = 1 - (t+1)^N$. □

Using the previous two lemmas, we can now investigate equation (74) in $\tilde{K}^1_{hS^1}((\mathcal{F}_1/\mathcal{F}_0)^{(2N)}; \mathbb{F}_2)$, where $N = 2^n$. As already noted, $p_{N-1}(t) \equiv t^{N-1} (\text{mod } 2)$, and so the left-hand side of (74) is $(t^{N-1} - mt^{2N-1})x$ modulo 2. The right-hand side is $E_N(z) = (1 - (t+1)^{2^n})z \equiv -t^{2^n} z (\text{mod } 2)$. So

$$(t^{N-1} - mt^{2N-1})x = -t^N z \in (\mathbb{Z}[t]/t^{2N})1 \oplus (\mathbb{Z}[t]/t^{2N})x$$

Clearly, this is impossible, since the term $t^{N-1}x$ cannot be cancelled by $-t^N z$ in $(\mathbb{Z}[t]/t^{2N})1 \oplus (\mathbb{Z}[t]/t^{2N})x$. This gives a contradiction, so the given f we started with must be divisible by 2. This proves the Theorem. □

Notation

In this table can be found some of the frequently used notation in this paper:

$\simeq$	(between topological spaces): homotopy equivalent.
$\mathbb{F}$	$\mathbb{C}$ or $\mathbb{H}$.
$G(r)$	The space of simple parametrized closed geodesics on $\mathbb{F}P^r$. Sometimes written $G(\mathbb{H}P^r)$ or $G(\mathbb{C}P^r)$ to be specific.
$G_n(r)$	The space of parametrized closed geodesics of length n, can be obtained by iterating n times the elements of $G(r)$.
$\Delta(r)$	The quotient $S^1 \setminus G(r)$ under the rotation action of S^1.
EG	A contractible space with a free action of the group G; unique up to homotopy.
BG	EG/G, the classifying space of G.
X_{hS^1}	$ES^1 \times_{S^1} X$, where X is an S^1-space.
$K^*_{hS^1}(X)$	$K^*(X_{hS^1})$.
$K^*_{hS^1}(X,*)$	The relative group $K^*(X_{hS^1}, BS^1)$.
T	The standard complex line bundle over $BS^1 = \mathbb{C}P^\infty$, or its pullback to X_{hS^1} under the map $\mathrm{pr}_1 : ES^1 \times_{S^1} X \longrightarrow BS^1$. Also used for the class of this bundle in K-theory.
t	the class $T - 1$, see T.
$\mathcal{F}_n$	$E^{-1}(]-\infty, n^2])$, the nth term in the Morse filtration.
μ_n^-	the negative bundle for the critical manifold $G_n(r)$.

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