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Asymptotic Completeness in Quantum Field Theory: Translation Invariant Nelson Type Models Restricted to the Vacuum and One-Particle Sectors

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Abstract

Time-dependent scattering theory for a large class of translation invariant models, including the Nelson and Polaron models, restricted to the vacuum and one-particle sectors is studied. Asymptotic completeness of these Hamiltonians is shown. The translation invariance imply that the Hamiltonian is fibered with respect to the total momentum. On the way to asymptotic completeness we determine the spectral structure of the fiber Hamiltonians, establish a Mourre estimate and derive a geometric asymptotic completeness statement as an intermediate step.

1 Introduction and motivation

In this paper, we study the spectral and scattering theory of a class of Hamiltonians that arise when one restricts e.g. the Nelson or Polaron model to the subspace of at most one field particle. As our results are valid for both models, we will use the term “field particles” rather than photons or phonons, and in the same spirit, we will use the term “matter particle” rather than electron or positron.

In [14], two of the authors prove a Mourre estimate and C^2 regularity for the full model, with respect to a suitably chosen conjugate operator. The estimate holds in the part of the energy-momentum spectrum lying between the bottom of the essential energy-momentum spectrum and either the two-body threshold, if there are no exited isolated mass shells, or the one-body threshold pertaining to the first exited isolated mass shell, if it exists. This is a natural first step for scattering theory.

As the full model in that energy-momentum regime is expected to resemble the model with at most one field particle in many aspects, the scattering theory of the cut-off model is of obvious interest. We note that in [10], the spectral and scattering theory of the massless Nelson model is studied, and that the stationary methods used there would to some extend also work on the class of models considered here. However, the scattering theory in [10] is obtained via a Kato-Birman argument, a method one cannot hope to work on the full model.

In recent years a lot of effort was put into investigating the spectral and scattering theory of various models of quantum field theory (see among many other papers [1], [3], [7], [8], [9], [11], [15], [19] and references therein). Substantial progress was made by applying methods originally developed in the study of N-particle Schrödinger operators namely the Mourre positive commutator method and the method of propagation observables to study the behavior of the unitary group e^{-itH} for large times. Up to now, the most complete results on the scattering theory for these models have only been available for models where the translation invariance is broken [1], [7], [11], [15], [19], or for small coupling constants [8]. In fact the only asymptotic completeness result valid for arbitrary coupling strength, in time-dependent scattering theory of translation invariant models known to us are variations of the N-body problem, where the dispersion relations are of the non-relativistic form $\frac{p^2}{M}$. Our results hold for a large class of dispersion relations, including a combination of the relativistic and non-relativistic choices.

In order to appreciate the difficulties associated with proving asymptotic completeness for translation invariant models of QFT, we explain the structure of scattering channels. If a system starts in a scattering state at total momentum ξ and energy E, it will emit field particles with momenta $k_1, \ldots, k_n$ until the remaining interacting system reaches a a total momentum ξ' and an eigenvalue $E'(\xi')$ for the Hamiltonian at total momentum ξ'. In order to conserve energy and momentum we must have $\xi = \xi' + k_1 + \cdots + k_n$ and $E = E'(\xi') + \omega(k_1) + \cdots + \omega(k_n)$, where ω is the dispersion relation for the field.

That is, the scattering channels are labeled by bound states at momenta ξ' and the number of emitted field particles n, under the constraint of conservation of energy and total momentum. The resulting bound particle will not be at rest but rather move according to a dispersion relation which is in fact the eigenvalue band, or mass shell, to which it belongs. This band may a priori be an isolated mass shell or an embedded one. If one wants to capture the behaviour of scattering states through a Mourre estimate, then one needs to build into a conjugate operator the dynamics of all the mass shells that appear in the available channels. This is a difficult task. The thresholds at total momentum ξ are energies E that has a scattering channel with the property that the bound state and the emitted field particles do not separate over time.

When introducing a number cutoff in the model, one simplifies the situation in that the scattering channels are now labeled by bound states of Hamiltonians with strictly fewer field particles. In particular in our case, we can label the scattering channels by mass shells of the Hamiltonian on the vacuum sector, which are easily understood. Indeed, there is in fact only one mass shell and it is identical to the matter dispersion relation Ω.

Finally, we will briefly outline the contents of this paper. In Section 2 we introduce the model in details and state our main result, the asymptotic completeness. In Section 3 we briefly go through the spectral theory for the fiber Hamiltonians, in particular we prove an HVZ theorem, a Mourre estimate, absence of singular continuous spectrum and a semi-continuity statement about the Mourre estimate. In Section 4 we prove the following propagation estimates: A large velocity estimate, a phase-space propagation estimate, an improved phase-space propagation estimate and a minimal velocity estimate. These form the technical foundation for Section 5, where we introduce the asymptotic observable, the spaces of asymptotically bound resp. free particles, the wave operators and prove asymptotic completeness via so-called geometric asymptotic completeness.

2 The model and the result

The Hilbert space for the Hamiltonian is

$$\mathcal{H} = L^2(\mathbb{R}^\nu, \mathrm{d}y) \otimes (\mathbb{C} \oplus L^2(\mathbb{R}^\nu, \mathrm{d}x)) = L^2(\mathbb{R}^\nu, \mathrm{d}y) \oplus L^2(\mathbb{R}^{2\nu}, \mathrm{d}x\, \mathrm{d}y),$$

where $\nu \in \mathbb{N}$. We write $D_x = -\mathrm{i}\nabla_x$, $D_y = -\mathrm{i}\nabla_y$ for the respective momentum operators. The Hamiltonian we wish to study the spectral and scattering theory of is given by

$$H = H_0 + V = \begin{pmatrix} \Omega(D_y) & 0 \\ 0 & \Omega(D_y) + \omega(D_x) \end{pmatrix} + \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix},$$

where

$$(vu_0)(x,y) = \rho(x-y)u_0(y) \quad \text{and} \quad (v^*u_1)(x) = \int \rho(x-y)u_1(x,y)dy$$

for some $\rho \in L^2(\mathbb{R}^\nu)$. Here Ω is the dispersion relation for the matter particle, ω the dispersion relation for the field particles and ρ a coupling function. One may view it as the translation invariant Nelson or Polaron model restricted to the subspace with at most one field particle, depending on the choice of dispersion relations.

The coupling function will be assumed to satisfy a short-range condition which implies a UV-cutoff (see Condition 2.3). We work with more general dispersion relations ω and Ω than $\omega(k) = \sqrt{k^2+m^2}$ or $\omega(k) = \omega_0 > 0$ and $\Omega(\eta) = \eta^2/2M$ respectively (see Conditions 2.1 and 2.2 for details). As the infrared problem is not present in this model due to the finite number of field particles, the mass of the field particle is not important. However, the singular behavior of the dispersion relation $\omega(k) = |k|$ at $k = 0$ makes this choice fall outside of what can be handled in this treatment, although it seems likely that one with minor adjustments may include this case in the same framework. For a treatment of the case where $\Omega(\eta) = \frac{1}{2}\eta^2$ and $\omega(k) = |k|$, see [10].

The operator H commutes with the operator of total momentum, $\mathrm{P} = \begin{pmatrix} D_y & 0 \\ 0 & D_x + D_y \end{pmatrix}$, and hence H is fibered, $H = U^{-1} \int_{\mathbb{R}^\nu}^{\oplus} H(P)\, \mathrm{d}P U$, where

$$U(u_0, u_1)(x,y) = \big(u_0(y), u_1(y, x+y)\big)$$

and

$$H(P) = H_0(P) + \tilde{V} = \begin{pmatrix} \Omega(P) & 0 \\ 0 & \Omega(P - D_x) + \omega(D_x) \end{pmatrix} + \begin{pmatrix} 0 & \langle\rho| \\ |\rho\rangle & 0 \end{pmatrix},$$

where $\langle\cdot|$ and $|\cdot\rangle$ denote the Dirac brackets. The fiber Hamiltonians are operators on the Hilbert space $\mathcal{K} = \mathbb{C} \oplus L^2(\mathbb{R}^\nu)$.

The precise assumptions on Ω, ω and ρ are given below.

Condition 2.1 (Matter particle dispersion relation). *Let $\Omega \in C^\infty(\mathbb{R}^\nu)$ be a non-negative, real-analytic and rotation invariant*[1] *function. There exists $s_\Omega \in [0,2]$ such that Ω satisfies:*

(i) There is a $C > 0$ such that $\Omega(\eta) \geq C^{-1}\langle\eta\rangle^{s_\Omega} - C$.

(ii) For any multi-index α there is a $C_\alpha > 0$ such that $|\partial^\alpha \Omega(\eta)| \leq C_\alpha \langle\eta\rangle^{s_\Omega - |\alpha|}$.

Note that this assumption is satisfied by the standard non-relativistic and relativistic choices, $\Omega(\eta) = \frac{\eta^2}{2M}$ and $\Omega(\eta) = \sqrt{\eta^2 + M^2}$.

Condition 2.2 (Field particle dispersion relation). *Let $\omega \in C^\infty(\mathbb{R}^\nu)$ be non-negative, real-analytic, rotation invariant and satisfy:*

(i) For any multi-index α with $|\alpha| \geq 1$, we have $\sup_{k\in\mathbb{R}^\nu} |\partial^\alpha \omega(k)| < \infty$.

(ii) If $s_\Omega = 0$, then $\omega(k) \to \infty$ as $|k| \to \infty$.

This is satisfied e.g. for $\omega(k) = \sqrt{k^2 + m^2}$, $m \neq 0$, and if $s_\Omega \neq 0$, also for the Polaron[2], $\omega(k) = \omega_0$.

Condition 2.3 (Coupling function). *Let $\rho \in L^2(\mathbb{R}^\nu)$ be rotation invariant and satisfy that*

(i) $\hat{\rho} \in C^2(\mathbb{R}^\nu)$.

(ii) $\langle\cdot\rangle|\nabla\hat{\rho}|, \partial_j\hat{\rho}, \langle\cdot\rangle\|\nabla^2\hat{\rho}\| \in L^2(\mathbb{R}^\nu)$.

(iii) There exist constants $C, \mu > 0$ such that $|\rho(x)| \leq C\langle x\rangle^{-1-\frac{\nu}{2}-\mu}$.

Condition 2.3 (iii) is the so-called short-range condition. Note that it implies that for $J \in C^\infty(\mathbb{R}^\nu)$ with support away from 0, we have

$$|\rho(x)J(\tfrac{x}{t})| = O(t^{-1-\mu}). \tag{1}$$

For the rest of this paper, Conditions 2.1, 2.2 and 2.3 will tacitly be assumed to be fulfilled, and under this assumption, our main result will be the following

[1]By rotation invariance of a function f we mean that $f(\eta) = f(O\eta)$ a.e. for any $O \in O(\nu)$ where $O(\nu)$ denotes the ν-dimensional orthogonal group.

[2]In fact the Fröhlich Polaron has $\Omega(\eta) = \frac{\eta^2}{2M_{\mathrm{eff}}}$, so $s_\Omega = 2 \neq 0$.

Theorem 2.1 (Asymptotic completeness)**.** *The wave operator*

$$W^+ = \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} P^+(H_0)$$

exists, where $P^+(H_0)$ is the projection onto $\{0\} \oplus L^2(\mathbb{R}^{2\nu})$, and the system is asymptotically complete:

$$\operatorname{Ran} W^+ = \mathcal{H}_{\mathrm{bd}}^\perp,$$

where $\mathcal{H}_{\mathrm{bd}} = U^{-1} \int_{\mathbb{R}^\nu}^{\oplus} \mathbb{1}_{\mathrm{pp}}(H(P))\,\mathrm{d}P U\mathcal{H}$.

Remark 2.2. That $P \mapsto \mathbb{1}_{\mathrm{pp}}(H(P))$ is weakly – and hence strongly – measurable follows from an application of the RAGE theorem, [5, Theorem 5.8], see the proof of [5, Theorem 9.4] for details.

3 Spectral analysis

We begin by recalling the following well-known properties of the fibered Hamiltonian. The Hamiltonian $H_0(P)$ is essentially self-adjoint on $\mathbb{C} \oplus C_0^\infty(\mathbb{R}^\nu)$ and the domain $\mathcal{D} = \mathcal{D}(H_0(P))$ is independent of P. As $\tilde{V}$ is bounded, the Kato-Rellich theorem implies that the same is true for $H(P)$ and that $\mathcal{D}(H(P)) = \mathcal{D}$.

The following threshold set will play an important role in our analysis:

$$\vartheta(P) = \{\lambda \in \mathbb{R} \,|\, \exists k \in \mathbb{R}^\nu : \lambda = \Sigma(P-k) + \omega(k), \nabla\Omega(P-k) = \nabla\omega(k)\}.$$

By rotation invariance and analyticity it is easy to see that $\vartheta(P)$ is locally finite and closed.

The following results, Theorems 3.1 to 3.4, correspond to completely analogous statements for the full model, see [14].

Theorem 3.1. *Assume that the vector field $v_P \in C^\infty(\mathbb{R}^\nu;\mathbb{R}^\nu)$ satisfies that for any multi-index α, $|\alpha| \in \{0,1,2\}$, there is a constant $C_\alpha > 0$ such that $|\partial^\alpha v_P(\eta)| \leq C_\alpha \langle\eta\rangle^{1-|\alpha|}$. Then the operator $a_P = \frac{1}{2}(v_P(D_x)\cdot x + x \cdot v_P(D_x))$ is essentially self-adjoint on the Schwarz space $\mathcal{S}$ and $H(P)$ is of class $C^2(A_P)$, where $A_P = \left(\begin{smallmatrix} 0 & 0 \\ 0 & a_P \end{smallmatrix}\right)$ is self-adjoint on $\mathcal{D}(A_P)$. The first commutator is given by*

$$[H(P), \mathrm{i}A_P]^\circ = \begin{pmatrix} 0 & \langle \mathrm{i}a_P\rho| \\ |\mathrm{i}a_P\rho\rangle & v_P(D_x)\cdot\nabla(\omega(D_x) + \Omega(P - D_x)) \end{pmatrix}$$

as a form on $\mathcal{D}$.

This can be seen either by direct computations or by following [14].

We now introduce the extended space $\mathcal{K}^{\mathrm{ext}} = \mathcal{K} \oplus L^2(\mathbb{R}^\nu)$ to be able to make a geometric partition of unity in configuration space. The partition of unity is similar to what is done in the analysis of the N-body Schrödinger operator (see e.g. [6]) and in complete analogy with what is done in e.g. [7] and [13]. The partition of unity used here may actually be seen as the partition of unity introduced in [7] restricted to the subspace with at most 1 field particle.

Let $j_0, j_\infty \in C^\infty(\mathbb{R}^\nu)$ be real, non-negative functions satisfying $j_0 = 1$ on the set $\{x \,|\, |x| \leq 1\}$, $j_0 = 0$ on $\{x \,|\, |x| > 2\}$ and $j_0^2 + j_\infty^2 = 1$. We now define

$$
\begin{aligned}
&j^R \colon \mathcal{K} \to \mathcal{K}^{\mathrm{ext}} \\
&j^R(v_0, v_1) = (v_0, j_0(\tfrac{\cdot}{R})v_1) \oplus (j_\infty(\tfrac{\cdot}{R})v_1).
\end{aligned}
$$

Clearly, j^R is isometric.

We introduce two self-adjoint operators, the extended Hamiltonian, $H^{\mathrm{ext}}(P)$, and the extended conjugate operator, A_P^{ext}, acting in $\mathcal{K}^{\mathrm{ext}}$,

$$
\begin{aligned}
H^{\mathrm{ext}}(P) &= H(P) \oplus F_P(D_x) \quad \text{and} \\
A_P^{\mathrm{ext}} &= A_P \oplus a_P,
\end{aligned}
$$

where $F_P(D_x) = \omega(D_x) + \Omega(P - D_x)$, with the obvious domains denoted by $\mathcal{D}^{\mathrm{ext}}$ and $\mathcal{D}(A_p^{\mathrm{ext}})$. The extended Hamiltonian describes an interacting system and a system with a free field particle. It is easy to see that Theorem 3.1 holds true with $H(P)$ and A_P replaced by $H^{\mathrm{ext}}(P)$ and A_P^{ext}, respectively, and the commutator equal to

$$
[H^{\mathrm{ext}}(P), \mathrm{i}A_p^{\mathrm{ext}}]^\circ = [H(P), \mathrm{i}A_P]^\circ \oplus \big(v_P(D_x) \cdot \big(\nabla\omega(D_x) - \nabla\Omega(P - D_x)\big)\big).
$$

We have the following localisation error when applying j^R.

Lemma 3.2. *Let $f \in C_0^\infty(\mathbb{R})$. Then*

$$
\begin{aligned}
&j^R f(H(P)) = f(H^{\mathrm{ext}}(P)) j^R + o_R(1) \quad \textit{and} \\
&j^R f(H(P))[H(P), \mathrm{i}A_P]^\circ f(H(P)) \\
&\qquad = f(H^{\mathrm{ext}}(P))[H^{\mathrm{ext}}(P), \mathrm{i}A_P^{\mathrm{ext}}]^\circ f(H^{\mathrm{ext}}(P)) j^R + o_R(1),
\end{aligned}
$$

for $R \to \infty$.

This can be seen either by a direct computation or by applying [14, Corollary 5.3]. The following two results, an HVZ theorem and a Mourre estimate, are now almost immediate.

Theorem 3.3. *The spectrum of $H(P)$ below $\Sigma_{\mathrm{ess}}(P) = \inf_{k \in \mathbb{R}^\nu}\{\Omega(P - k) + \omega(k)\}$ consists at most of eigenvalues of finite multiplicity and can only accumulate at $\Sigma_{\mathrm{ess}}(P)$. The essential spectrum is given by $\sigma_{\mathrm{ess}}(H(P)) = [\Sigma_{\mathrm{ess}}(P), \infty)$.*

Proof. Using Lemma 3.2 for an $f \in C_0^\infty(\mathbb{R})$ supported in $(-\infty, \Sigma_{\mathrm{ess}}(P))$ and letting R tend to infinity shows that $f(H(P))$ is compact. This proves the first part.

To prove the last part, let $\lambda \in [\Sigma_{\mathrm{ess}}(P), \infty)$ and note that there exists a $k_0 \in \mathbb{R}^\nu$ such that $\lambda = \Omega(P - k_0) + \omega(k_0)$. Now choose $u_n = (0, u_{1n}) \in \mathbb{C} \oplus L^2(\mathbb{R}^\nu)$ with $\hat{u}_{1n}(\cdot) = n^{\frac{\nu}{2}} f(n(\cdot - k_0))$ for some $f \in C_0^\infty(\mathbb{R}^\nu)$ with $f \geq 0$ and $f(0) = 1$. One may now check that u_n is a Weyl sequence for the energy λ. □

Theorem 3.4. *Assume that $\lambda \notin \vartheta(P)$. Let A_P be given as in Theorem 3.1 with $v_P(D_x) = \nabla\omega(D_x) - \nabla\Omega(P - D_x))$. Then there exist constants $\kappa, c > 0$ and a compact operator K such that*

$$
E_{\lambda,\kappa}(H(P))[H(P), \mathrm{i}A_P]^\circ E_{\lambda,\kappa}(H(P)) \geq c E_{\lambda,\kappa}(H(P)) + K,
$$

where $E_{\lambda,\kappa}$ denotes the characteristic function of the interval $[\lambda - \kappa, \lambda + \kappa]$.

Proof. We may find a κ such that $[\lambda-2\kappa,\lambda+2\kappa]\cap\vartheta(P)=\emptyset$. Choose $f\in C_0^\infty(\mathbb{R})$ with support in $[\lambda-2\kappa,\lambda+2\kappa]$ and equal to 1 on $[\lambda-\kappa,\lambda+\kappa]$. Note that

$$\begin{aligned}
&f(H(P))[H(P),\mathrm{i}A_P]^\circ f(H(P))\\
&\quad= j^{R*}j^R f(H(P))[H(P),\mathrm{i}A_P]^\circ f(H(P))\\
&\quad= j^{R*}f(H^{\mathrm{ext}}(P))[H^{\mathrm{ext}}(P),\mathrm{i}A_P^{\mathrm{ext}}]^\circ f(H^{\mathrm{ext}}(P))j^R+o_R(1),
\end{aligned}$$

by Lemma 3.2. Note that

$$\begin{aligned}
&f(H^{\mathrm{ext}}(P))[H^{\mathrm{ext}}(P),\mathrm{i}A_P^{\mathrm{ext}}]^\circ f(H^{\mathrm{ext}}(P))j^R\\
&\quad= f(H(P))[H(P),\mathrm{i}A_P]^\circ f(H(P))\begin{pmatrix}1\\ j_0(\frac{\cdot}{R})\end{pmatrix} \qquad (2)\\
&\qquad\oplus f(F_P(D_x))|\nabla\omega(D_x)-\nabla\Omega(P-D_x)|^2 f(F_P(D_x))j_\infty(\tfrac{\cdot}{R}).
\end{aligned}$$

Taking the support of f into account, one finds that

$$f(F_P(D_x))|\nabla\omega(D_x)-\nabla\Omega(P-D_x)|^2 f(F_P(D_x))\geq 2cf^2(F_P(D_x))$$

for some positive constant $c>0$. It is easy to see that $K(R)=f(H(P))\left(\begin{smallmatrix}1\\ j_0(\frac{\cdot}{R})\end{smallmatrix}\right)$ is compact. Let $g\in C_0^\infty(\mathbb{R})$ equal 1 on the support of f. Then

$$B=f(H(P))[H(P),\mathrm{i}A_P]^\circ g(H(P))$$

is bounded and (2) equals $BK(R)$. Hence by Lemma 3.2

$$\begin{aligned}
&f(H(P))[H(P),\mathrm{i}A_P]^\circ f(H(P))\\
&\quad\geq j^{R*}2cf^2(H(P))\begin{pmatrix}1\\ j_0(\frac{\cdot}{R})\end{pmatrix}\oplus 2cf^2(F_P(D_x))j_\infty(\tfrac{\cdot}{R})\\
&\qquad+j^{R*}(B-2cf(H(P)))K(R)\oplus 0+o_R(1)\\
&\quad=2cf^2(H(P))+K_R+o_R(1),
\end{aligned}$$

for some compact operator K_R depending on R. One may now choose R so large that $\|o_R(1)\|\leq c$ and sandwich the inequality with $E_{\lambda,\kappa}(H(P))$ on both sides to arrive at the desired result. □

We infer the following corollary of Theorems 3.1 and 3.4 by standard arguments of regular Mourre theory.

Corollary 3.5. *The essential spectrum of the fiber Hamiltonians is non-singular:*

$$\sigma_{\mathrm{sing}}(H(P))=\emptyset.$$

Theorem 3.6. *Let $(P_0,\lambda_0)\in\mathbb{R}^{\nu+1}$. Assume that $\lambda_0\notin\vartheta(P_0)\cup\sigma_{\mathrm{pp}}(P_0)$. Then there exists a constant $C>0$, a neighbourhood $\mathcal{O}$ of P_0 and a function $f\in C_0^\infty(\mathbb{R})$ with $f=1$ in a neighbourhood of λ_0 such that for all $P\in\mathcal{O}$,*

$$f(H(P))[H(P),\mathrm{i}A_{P_0}]^\circ f(H(P))\geq Cf^2(H(P))$$

where A_{P_0} is given as in Theorem 3.4.

Proof. We begin by noting that the object $[H(P), \mathrm{i}A_{P_0}]^\circ$ is well-defined by Theorem 3.1. By standard arguments using the fact that $\lambda_0 \notin \sigma_{\mathrm{pp}}(P_0)$ and Theorem 3.4, there exist a function $\tilde{f} \in C_0^\infty(\mathbb{R})$ and a constant $\tilde{C}$ such that

$$\tilde{f}(H(P_0))[H(P_0), \mathrm{i}A_{P_0}]^\circ \tilde{f}(H(P_0)) \geq \tilde{C}\tilde{f}^2(H(P_0)),$$

with $\tilde{f} = 1$ on a neighbourhood of λ_0. It is easy to see that $(H(P) - z)^{-1}(H_0(0) - \mathrm{i})$ and $(H_0(0) - \mathrm{i})^{-1}[H(P), \mathrm{i}A_{P_0}]^\circ(H_0(0) - \mathrm{i})^{-1}$ are norm continuous as functions of P, and hence it follows by an application of the functional calculus of almost analytic extensions that $\tilde{f}^2(H(P))$ and $\tilde{f}(H(P))[H(P), \mathrm{i}A_{P_0}]^\circ \tilde{f}(H(P))$ are norm continuous as functions of P.

Let $\mathcal{O} \ni P_0$ be a neighbourhood such that

$$\|\tilde{f}^2(H(P)) - \tilde{f}^2(H(P_0))\| \leq \frac{\tilde{C}}{3} \quad \text{and}$$
$$\big\|\tilde{f}(H(P))[H(P), \mathrm{i}A_{P_0}]^\circ \tilde{f}(H(P)) - \tilde{f}(H(P_0))[H(P_0), \mathrm{i}A_{P_0}]^\circ \tilde{f}(H(P_0))\big\| \leq \frac{\tilde{C}}{3}$$

for all $P \in \mathcal{O}$. Then

$$\tilde{f}(H(P))[H(P), \mathrm{i}A_{P_0}]^\circ \tilde{f}(H(P)) \geq -\frac{2\tilde{C}}{3}I + \tilde{C}\tilde{f}^2(H(P)). \tag{3}$$

Choose now $C = \frac{\tilde{C}}{3}$ and $f \in C_0^\infty(\mathbb{R})$ such that $f = 1$ on a neighbourhood of λ_0 and $f = f\tilde{f}$. The result is then obtained by multiplying (3) from both sides with $f(H(P))$. □

4 Propagation estimates

We will write $\mathbf{D} = [H, \mathrm{i}\,\cdot]$ and $\mathbf{d}_0 = [\Omega(D_x + D_y) + \omega(D_x), \mathrm{i}\,\cdot]$ for the Heisenberg derivatives. The following abbreviation will be used to ease the notation:

$$[B] := \left(\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right).$$

Theorem 4.1 (Large velocity estimate)**.** *Let $\chi \in C_0^\infty(\mathbb{R})$. There exists a constant C_1 such that for $R' > R > C_1$, one has*

$$\int_1^\infty \big\|\big[\mathbb{1}_{[R,R']}\big(\tfrac{|x-y|}{t}\big)\big]e^{-\mathrm{i}tH}\chi(H)u\big\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2$$

Proof. Let C_1 be a constant to be specified later and $R' > R > C_1$. Let $F \in C^\infty(\mathbb{R})$ equal 0 near the origin and 1 near infinity such that $F'(s) \geq c\mathbb{1}_{[R,R']}(s)$ for some positive constant $c > 0$. Let

$$\Phi(t) = -\chi(H)\big[F\big(\tfrac{|x-y|}{t}\big)\big]\chi(H),$$
$$b(t) = -\mathbf{d}_0 F\big(\tfrac{|x-y|}{t}\big).$$

By using e.g. Theorem B.3 or pseudo-differential calculus one sees that

$$b(t) = \frac{1}{t}\Big(\tfrac{|x-y|}{t} - (\nabla\Omega(D_y) - \nabla\omega(D_x))\tfrac{x-y}{|x-y|}\Big)F'\big(\tfrac{|x-y|}{t}\big) + O(t^{-2}).$$

Hence for any $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that $\chi = \chi\tilde{\chi}$ one finds that

$$
\begin{aligned}
&-\chi(H)[b(t)]\chi(H) \\
&= \tfrac{1}{t}\chi(H)\Big(\tfrac{|x-y|}{t} - (\nabla\Omega(D_y) - \nabla\omega(D_x))\tfrac{x-y}{|x-y|}\Big)F'\big(\tfrac{|x-y|}{t}\big)\chi(H) + O(t^{-2}) \\
&= \tfrac{1}{t}\chi(H)\Big(\tfrac{|x-y|}{t} - \tilde{\chi}(H)(\nabla\Omega(D_y) - \nabla\omega(D_x))\tfrac{x-y}{|x-y|}\Big)\mathbb{1}_{[C_1,\infty)}\big(\tfrac{|x-y|}{t}\big) \\
&\qquad \times F'\big(\tfrac{|x-y|}{t}\big)\chi(H) + O(t^{-2}) \\
&\geq \frac{C_0}{t}\chi(H)F'\big(\tfrac{|x-y|}{t}\big)\chi + O(t^{-2})
\end{aligned}
$$

for some $C_0 > 0$ if one chooses $C_1 > \|\tilde{\chi}(H)(\nabla\Omega(D_y) - \nabla\omega(D_x))\frac{x-y}{|x-y|}\|$.

It follows from Condition 2.3 (iii) that

$$
\big[V, \mathrm{i}[F\big(\tfrac{|x-y|}{t}\big)]\big] = O(t^{-1-\mu}),
$$

cf. (1). Putting this together, we get

$$
\mathbf{D}\Phi(t) \geq \tfrac{C_0}{t}\chi(H)[F'(\tfrac{|x-y|}{t})]\chi(H) + O(t^{-1-\mu}),
$$

which combined with Lemma A.1 implies the result. □

Theorem 4.2 (Phase-space propagation estimate)**.** *Let $\chi \in C_0^\infty(\mathbb{R})$, $0 < c_0 < c_1$. Write*

$$
\begin{aligned}
&\Theta_{[c_0,c_1]}(t) = \\
&\quad \big[\big\langle \tfrac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y), \mathbb{1}_{[c_0,c_1]}\big(\tfrac{|x-y|}{t}\big)\big(\tfrac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)\big)\big\rangle\big].
\end{aligned}
$$

Then

$$
\int_1^\infty \|\Theta_{[c_0,c_1]}(t)^{\frac{1}{2}} e^{-\mathrm{i}tH}\chi(H)u\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2. \tag{4}
$$

Proof. The following construction is taken from [7] but ultimately goes back to a construction of Graf, see e.g. [12]. There exists a function $R_0 \in C^\infty(\mathbb{R}^\nu)$ such that

$$
\begin{aligned}
R_0(x) &= 0 && \text{for } |x| \leq \tfrac{c_0}{2}, \\
R_0(x) &= \tfrac{1}{2}x^2 + c && \text{for } |x| \geq 2c_1, \\
\nabla^2 R_0(x) &\geq \mathbb{1}_{[c_0,c_1]}(|x|).
\end{aligned}
$$

Without loss of generality, we may assume that $c_1 > C_1 + 1$, where C_1 is the constant whose existence is ensured by Theorem 4.1. Choose a constant $c_2 > c_1 + 1$ and a smooth function F such that $F(s) = 1$ for $s < c_1$ and $F(s) = 0$ for $s \geq c_2$. Let

$$
R(x) = F(|x|)R_0(x).
$$

Then R satisfies

$$
\begin{aligned}
\nabla^2 R(x) &\geq \mathbb{1}_{[c_0,c_1]}(|x|) - C\mathbb{1}_{[C_1+1,c_2]}(|x|), \\
|\partial^\alpha R(x)| &\leq C_\alpha.
\end{aligned} \tag{5}
$$

Write $X = \frac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)$ and let

$$\Phi(t) = \chi(H)[b(t)]\chi(H),$$

where

$$b(t) = R(\tfrac{x-y}{t}) - \tfrac{1}{2}\big(\langle \nabla R(\tfrac{x-y}{t}), X\rangle + \mathrm{h.c.}\big).$$

By using Condition 2.3 (iii) and pseudo-differential calculus, one sees that

$$\left\| \chi(H) \begin{pmatrix} 0 & 0 \\ -\mathrm{i}b(t)\rho(x-\cdot) & 0 \end{pmatrix} \chi(H) \right\| \in O(t^{-1-\mu})$$

and hence

$$\chi(H)[V, \mathrm{i}[b(t)]]\chi(H) \in O(t^{-1-\mu}).$$

Compute

$$\begin{aligned} \tfrac{d}{dt} b(t) = & -\tfrac{1}{t}\langle \tfrac{x-y}{t}, \nabla R(\tfrac{x-y}{t})\rangle \\ & + \tfrac{1}{2}\tfrac{1}{t}\big(\langle \tfrac{x-y}{t}, \nabla^2 R(\tfrac{x-y}{t}) X\rangle + \mathrm{h.c.}\big) \\ & + \tfrac{1}{t}\langle \nabla R(\tfrac{x-y}{t}), \tfrac{x-y}{t}\rangle \\ = & \, \tfrac{1}{2}\tfrac{1}{t}\big(\langle \tfrac{x-y}{t}, \nabla^2 R(\tfrac{x-y}{t}) X\rangle + \mathrm{h.c.}\big), \end{aligned}$$

and by pseudo-differential calculus one sees that

$$\begin{aligned} [\omega(D_x) + \Omega(D_y), \mathrm{i}b(t)] = & \, \tfrac{1}{2}\tfrac{1}{t}\big(\langle \nabla\omega(D_x) - \nabla\Omega(D_y), \nabla R(\tfrac{x-y}{t})\rangle + \mathrm{h.c.}\big) \\ & - \tfrac{1}{2}\tfrac{1}{t}\big(\langle \nabla\omega(D_x) - \nabla\Omega(D_y), \nabla^2 R(\tfrac{x-y}{t}) X\rangle + \mathrm{h.c.}\big) \\ & - \tfrac{1}{2}\tfrac{1}{t}\big(\langle \nabla R(\tfrac{x-y}{t}), \nabla\omega(D_x) - \nabla\Omega(D_y)\rangle + \mathrm{h.c.}\big) \\ & + O(t^{-2}) \\ = & -\tfrac{1}{2}\tfrac{1}{t}\big(\langle \nabla\omega(D_x) - \nabla\Omega(D_y), \nabla^2 R(\tfrac{x-y}{t}) X\rangle + \mathrm{h.c.}\big) \\ & + O(t^{-2}), \end{aligned}$$

hence by using (5), it follows that

$$\begin{aligned} & \chi(H)[\mathbf{d}_0 b(t)]\chi(H) \\ & \quad = \tfrac{1}{t}\chi(H)[\langle X, \nabla^2 R(\tfrac{x-y}{t}) X\rangle]\chi(H) + O(t^{-2}) \\ & \quad \geq \tfrac{1}{t}\chi(H)\big[\big\langle X, \mathbb{1}_{[c_0,c_1]}\big(\tfrac{|x-y|}{t}\big) X\big\rangle\big]\chi(H) \\ & \qquad - \tfrac{C}{t}\chi(H)\big[\big\langle X, \mathbb{1}_{[C_1+1,c_2]}\big(\tfrac{|x-y|}{t}\big) X\big\rangle\big]\chi(H) + O(t^{-2}) \end{aligned}$$

By introducing $J \in C_0^\infty(\mathbb{R}; [0,1])$ supported above C_1 with $J\mathbb{1}_{[C_1+1,c_2]} = \mathbb{1}_{[C_1+1,c_2]}$ and $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\tilde{\chi}\chi = \chi$ and using pseudo-differential calculus, the functional calculus of almost analytic extensions and Condition 2.3 (iii) again, one gets that

$$\begin{aligned} & \tfrac{C}{t}\chi(H)\big[X_i \mathbb{1}_{[C_1+1,c_2]}\big(\tfrac{|x-y|}{t}\big) X_i\big]\chi(H) \\ & \quad \leq \tfrac{C}{t}\chi\tilde{\chi}(H)\big[X_i J^3\big(\tfrac{|x-y|}{t}\big) X_i\big]\tilde{\chi}\chi(H) \\ & \quad = \tfrac{C}{t}\chi(H)\big[J\big(\tfrac{|x-y|}{t}\big)\big]\tilde{\chi}(H)\big[X_i J\big(\tfrac{|x-y|}{t}\big) X_i\big]\tilde{\chi}(H)\big[J\big(\tfrac{|x-y|}{t}\big)\big]\chi(H) + O(t^{-2}) \\ & \quad \leq \tfrac{C'}{t}\chi(H)\big[J^2\big(\tfrac{|x-y|}{t}\big)\big]\chi(H) + Ct^{-2}, \end{aligned}$$

where we estimated $\tilde{\chi}(H)\big[X_i J\big(\frac{|x-y|}{t}\big)X_i\big]\tilde{\chi}(H)$ by a constant. Putting it all together yields

$$\mathbf{D}\Phi(t) \geq \tfrac{1}{t}\chi(H)\Theta_{[c_0,c_1]}(t)\chi(H) - \tfrac{C}{t}\chi(H)[J^2(\tfrac{|x-y|}{t})]\chi(H) + O(t^{-1-\mu}),$$

where the second term is integrable along the evolution by Theorem 4.1, so the result now follows from Lemma A.1. □

Theorem 4.3 (Improved phase-space propagation estimate)**.** *Let* $0 < c_0 < c_1$, $J \in C_0^\infty(c_0 < |x| < c_1)$, $\chi \in C_0^\infty(\mathbb{R})$. *Then for* $1 \leq i \leq \nu$

$$\int_1^\infty \big\|\big[\big|J(\tfrac{x-y}{t})\big(\tfrac{x_i-y_i}{t} - \partial_i\omega(D_x) + \partial_i\Omega(D_y)\big) + \mathrm{h.\,c.}\big|\big]^{\frac{1}{2}} e^{-\mathrm{i}tH}\chi(H)u\big\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2$$

Proof. For brevity, we write $X = \frac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)$ and $R_0 = (H_0 - \lambda)^{-1}$ for some real $\lambda \in \rho(H_0)$. Let

$$A = X^2 + t^{-\delta},$$

$\delta > 0$. Note that $[J(\frac{x-y}{t})A^{\frac{1}{2}}]R_0$ is uniformly bounded in $t \geq 1$.

The following identities hold as forms on $C_0^\infty(\mathbb{R}^\nu)$.

$$e^{\mathrm{i}t(\omega(D_x)+\Omega(D_y))}Xe^{-\mathrm{i}t(\omega(D_x)+\Omega(D_y))} = \tfrac{x-y}{t},$$

$$e^{\mathrm{i}t(\omega(D_x)+\Omega(D_y))}A^{\frac{1}{2}}e^{-\mathrm{i}t(\omega(D_x)+\Omega(D_y))} = ((\tfrac{x-y}{t})^2 + t^{-\delta})^{\frac{1}{2}} := A_0^{\frac{1}{2}} \tag{6}$$

and

$$e^{\mathrm{i}t(\omega(D_x)+\Omega(D_y))}J(X)e^{-\mathrm{i}t(\omega(D_x)+\Omega(D_y))} = J(\tfrac{x-y}{t}). \tag{7}$$

That the following commutator, viewed as a form on $C_0^\infty(\mathbb{R}^\nu)$, extends by continuity to a bounded form on $L^2(\mathbb{R}^\nu)$ can be seen using pseudo-differential calculus:

$$[X, A_0^{\frac{1}{2}}] = [\nabla\omega(D_x), A_0^{\frac{1}{2}}] - [\nabla\Omega(D_y), A_0^{\frac{1}{2}}] = O(t^{-\min\{1,2-\frac{\delta}{2}\}}).$$

Together with the functional calculus of almost analytic extensions this implies that

$$[J(X), A_0^{\frac{1}{2}}] = O(t^{-\min\{1,2-\frac{\delta}{2}\}}),$$

and hence using (6) and (7) that

$$[J(\tfrac{x-y}{t}), A^{\frac{1}{2}}] = O(t^{-\varepsilon}), \tag{8}$$

where $\varepsilon = \min\{1, 2 - \frac{\delta}{2}\}$. Write $h = \Omega(D_y) + \omega(D_x)$. Note that

$$\begin{aligned} e^{\mathrm{i}th}\mathbf{d}_0 A^{\frac{1}{2}}e^{-\mathrm{i}th} &= e^{\mathrm{i}th}[h, \mathrm{i}A^{\frac{1}{2}}]e^{-\mathrm{i}th} + e^{\mathrm{i}th}(\tfrac{d}{dt}A^{\frac{1}{2}})e^{-\mathrm{i}th} \\ &= \tfrac{d}{dt}\big(e^{\mathrm{i}th}A^{\frac{1}{2}}e^{-\mathrm{i}th}\big) = \tfrac{d}{dt}A_0^{\frac{1}{2}} \\ &= -\tfrac{1}{t}A_0^{\frac{1}{2}} - \tfrac{(2-\delta)t^{-\delta-1}}{2((\frac{x-y}{t})^2+t^{-\delta})^{\frac{1}{2}}}, \end{aligned}$$

so

$$\mathbf{d}_0 A^{\frac{1}{2}} = -\tfrac{1}{t}A^{\frac{1}{2}} + O(t^{-1-\frac{\delta}{2}}). \tag{9}$$

In addition

$$[R_0, [X_i]] = R_0^{\frac{1}{2}+\rho_1} O(t^{-1}) R_0^{1-\rho_1} \tag{10}$$

for any ρ_1, $0 < \rho_1 < \frac{1}{2}$ and that

$$[R_0, [A^{\frac{1}{2}}]] = R_0^{\rho_2} O(t^{\frac{\delta}{2}-1}) R_0^{1-\rho_2} \tag{11}$$

for any ρ_2, $0 < \rho_2 < 1$. The identity (11) can be seen e.g. by using (10) and the representation formula

$$s^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty (s+y)^{-1} y^{-\frac{1}{2}} \,\mathrm{d}y,$$

which can be verified for $t > 0$ by direct computations.

Let $J_1, J_2 \in C_0^\infty(c_0 < |x| < c_1)$ such that $JJ_1 = J$ and $J_1 J_2 = J_1$ and write for $i = 1, \dots, \nu$:

$$B_{0,i} = R_0[J(\tfrac{x-y}{t}) X_i] R_0 + \mathrm{h.\,c.}$$

and

$$B_1 = R_0[J_1(\tfrac{x-y}{t}) A^{\frac{1}{2}} J_1(\tfrac{x-y}{t})] R_0. \tag{12}$$

We compute using (8), (10) and (11):

$$\begin{aligned}
B_{0,i}^2 &= 4R_0[X_i J(\tfrac{x-y}{t})] R_0^2 [J(\tfrac{x-y}{t}) X_i] R_0 + O(t^{-1}) \\
&= 4R_0^2 [X_i J^2(\tfrac{x-y}{t}) X_i] R_0^2 + O(t^{-1}) \\
&\le C R_0^2 [X_i J_1^4(\tfrac{x-y}{t}) X_i] R_0^2 + C t^{-1} \\
&= C R_0^2 [J_1^2(\tfrac{x-y}{t}) X_i^2 J_1^2(\tfrac{x-y}{t})] R_0^2 + O(t^{-1}) \\
&\le C R_0^2 [J_1^2(\tfrac{x-y}{t}) A J_1^2(\tfrac{x-y}{t})] R_0^2 + O(t^{-\delta}) \\
&= C R_0 [J_1^2(\tfrac{x-y}{t}) A^{\frac{1}{2}}] R_0^2 [A^{\frac{1}{2}} J_1^2(\tfrac{x-y}{t})] R_0 + O(t^{-\min\{1-\frac{\delta}{2},\delta\}}) \\
&= C R_0 [J_1(\tfrac{x-y}{t}) A^{\frac{1}{2}} J_1(\tfrac{x-y}{t})] R_0^2 [J_1(\tfrac{x-y}{t}) A^{\frac{1}{2}} J_1(\tfrac{x-y}{t})] R_0 + O(t^{-\min\{1-\frac{\delta}{2},\delta\}}) \\
&= C B_1^2 + O(t^{-\kappa}),
\end{aligned}$$

where $\kappa = \min\{1 - \frac{\delta}{2}, \delta\}$. By the matrix monotonicity of $\lambda \mapsto \lambda^{\frac{1}{2}}$ [4, Sec. 2.2.2], we deduce that

$$|B_{0,i}| \le C B_1 + C t^{-\frac{\kappa}{2}}. \tag{13}$$

Now let

$$\Phi(t) = -\chi(H) \big[J(\tfrac{x-y}{t}) A^{\frac{1}{2}} J(\tfrac{x-y}{t}) \big] \chi(H) \tag{14}$$

It follows from (8) that

$$\Phi(t) = -\chi(H) \big[J(\tfrac{x-y}{t})^2 A^{\frac{1}{2}} \big] \chi(H) + O(t^{-\varepsilon}) \tag{15}$$

is uniformly bounded for $t > 1$.

We compute

$$-\mathbf{D}\Phi(t) = \chi(H)[V, \mathrm{i}[J(\tfrac{x-y}{t}) A^{\frac{1}{2}} J(\tfrac{x-y}{t})]] \chi(H) + \chi(H) \big[\mathbf{d}_0 \big(J(\tfrac{x-y}{t}) A^{\frac{1}{2}} J(\tfrac{x-y}{t}) \big) \big] \chi(H) \tag{16}$$

Using Condition 2.3(iii) we see that

$$\chi(H)[V, \mathrm{i}\big[J(\tfrac{x-y}{t})A^{\frac12}J(\tfrac{x-y}{t})]\big]\chi(H) = O(t^{-1-\mu}).$$

Indeed,

$$\begin{aligned}
&\chi(H)\big[V, \mathrm{i}[J(\tfrac{x-y}{t})A^{\frac12}J(\tfrac{x-y}{t})]\big]\chi(H) \\
&\quad = \chi(H)\Big(\begin{smallmatrix} 0 & 0 \\ -\mathrm{i}J(\frac{x-y}{t})A^{\frac12}J(\frac{x-y}{t})v & 0\end{smallmatrix}\Big)\chi(H) + \mathrm{h.\,c.} \\
&\quad = \chi(H)(H_0-\lambda)R_0\Big(\begin{smallmatrix} 0 & 0 \\ -\mathrm{i}\big(A^{\frac12}J(\frac{x-y}{t})+O(t^{-\varepsilon})\big)J(\frac{x-y}{t})v & 0\end{smallmatrix}\Big)\chi(H) + \mathrm{h.\,c.}
\end{aligned}$$

Now by Condition 2.3 (iii) we have that $\|J(\frac{x-y}{t})v\| = O(t^{-1-\mu})$ and hence we also have that $R_0\Big(\begin{smallmatrix} 0 & 0 \\ -\mathrm{i}\big(A^{\frac12}J(\frac{x-y}{t})+O(t^{-\varepsilon})\big)J(\frac{x-y}{t})v & 0\end{smallmatrix}\Big) = O(t^{-1-\mu})$.

Note that

$$\mathbf{d}_0 J(\tfrac{x-y}{t}) = -\tfrac{1}{t}\nabla J(\tfrac{x-y}{t})\cdot v + O(t^{-2}) \tag{17}$$

and using (9) and (13) (cf. (12)),

$$\begin{aligned}
&-\chi(H)[J(\tfrac{x-y}{t})(\mathbf{d}_0 A^{\frac12})J(\tfrac{x-y}{t})]\chi(H) \\
&\qquad \geq \tfrac{C_0}{t}\chi(H)[|J(\tfrac{x-y}{t})X_i + \mathrm{h.\,c.}|]\chi(H) - Ct^{-1-\frac{\kappa}{2}}.
\end{aligned}$$

Again we compute using (8):

$$\begin{aligned}
&R_0[\nabla J(\tfrac{x-y}{t})\cdot XA^{\frac12}J(\tfrac{x-y}{t})]R_0 + \mathrm{h.\,c.} \\
&\quad = R_0[J_2(\tfrac{x-y}{t})X\cdot\nabla J(\tfrac{x-y}{t})J(\tfrac{x-y}{t})A^{\frac12}J_2(\tfrac{x-y}{t})]R_0 + \mathrm{h.\,c.} + O(t^{-1}) \\
&\quad = \sum_{i=1}^{\nu} R_0[J_2(\tfrac{x-y}{t})A^{\frac12}X_iA^{-\frac12}\partial_i J(\tfrac{x-y}{t})J(\tfrac{x-y}{t})A^{\frac12}J_2(\tfrac{x-y}{t})]R_0 + \mathrm{h.\,c.} + O(t^{-1}) \\
&\quad \leq CR_0[J_2(\tfrac{x-y}{t})AJ_2(\tfrac{x-y}{t})]R_0 + Ct^{-1} \\
&\quad \leq CR_0[J_2(\tfrac{x-y}{t})X^2J_2(\tfrac{x-y}{t})]R_0 + O(t^{-\min\{1,\delta\}}) \\
&\quad \leq CR_0[\langle X, J_2^2(\tfrac{x-y}{t})X\rangle]R_0 + Ct^{-\min\{1,\varepsilon\}}.
\end{aligned}$$

Hence (cf. (17))

$$\begin{aligned}
&-\chi(H)\big[\mathbf{d}_0\big(J(\tfrac{x-y}{t})A^{\frac12}J(\tfrac{x-y}{t})\big)\big]\chi(H) \\
&\qquad = \chi(H)[(\mathbf{d}_0 J(\tfrac{x-y}{t}))A^{\frac12}J(\tfrac{x-y}{t})]\chi(H) + \mathrm{h.\,c.} \\
&\qquad\quad + \chi(H)[J(\tfrac{x-y}{t})(\mathbf{d}_0 A^{\frac12})J(\tfrac{x-y}{t})]\chi(H) \\
&\qquad \geq \tfrac{C_0}{t}\chi(H)[|J(\tfrac{x-y}{t})X_i + \mathrm{h.\,c.}|]\chi(H) \\
&\qquad\quad - \tfrac{C}{t}\chi(H)[\langle X, J_2^2(\tfrac{x-y}{t})X\rangle]\chi(H) + O(t^{-1-\gamma})
\end{aligned} \tag{18}$$

for some $\gamma > 0$. Since by Theorem 4.2 the second term in the r.h.s. of (18) is integrable along the evolution, the theorem follows from Lemma A.1. □

Theorem 4.4 (Minimal velocity estimate)**.** *Assume that* $(P_0,\lambda_0)\in\mathbb{R}^{\nu+1}$ *satisfies that* $\lambda_0\in\mathbb{R}\setminus(\vartheta(P_0)\cup\sigma_{\mathrm{pp}}(P_0))$*. Then there exists an* $\varepsilon>0$*, a neighbourhood* N *of* (P_0,λ_0) *and a function* $\chi\in C_0^\infty(\mathbb{R}^{\nu+1})$ *such that* $\chi=1$ *on* N *and*

$$\int_1^\infty \Big\| [\mathbb{1}_{[0,\varepsilon]}](\tfrac{|x|}{t})\int^\oplus e^{-\mathrm{i}tH(P)}\chi(P,H(P))\,\mathrm{d}Pu\Big\|^2\frac{\mathrm{d}t}{t}\le C\|u\|^2$$

Proof. By Theorem 3.6, it follows that there exists a neighbourhood $\mathcal{O}$ of P_0 and a function f with $f=1$ in a neighbourhood of λ_0 such that

$$f(H(P))[H(P),\mathrm{i}A_{P_0}]f(H(P))\ge Cf^2(H(P)) \tag{19}$$

for all P in $\mathcal{O}$. Let $\chi\in C_0^\infty(\mathbb{R}^{\nu+1};[0,1])$ be supported in $\mathcal{O}\times\{\lambda\,|\,f(\lambda)=1\}$ and $\chi=1$ in a neighbourhood N of (P_0,λ_0). It follows that

$$\chi(P,H(P))[H(P),\mathrm{i}A_{P_0}]\chi(P,H(P))\ge\tfrac{C}{2}\chi^2(P,H(P)). \tag{20}$$

Let $q\in C_0^\infty(\{|x|\le 2\varepsilon\})$ satisfy $0\le q\le 1$, $q=1$ in a neighbourhood of $\{|x|\le\varepsilon\}$ for some $\varepsilon>0$ to be specified later on. Write

$$Q(t)=\begin{pmatrix}1 & 0\\ 0 & q(\frac{x}{t}).\end{pmatrix}$$

Let

$$\Phi(t)=\int^\oplus\chi(P,H(P))Q(t)\frac{A_{P_0}}{t}Q(t)\chi(P,H(P))\,\mathrm{d}P.$$

Taking into account the support of q and that v_{P_0} is ω-bounded, and using pseudo-differential calculus, it is easy to see that $\Phi(t)$ is uniformly bounded.

We compute the Heisenberg derivative:

$$\begin{aligned}\mathbf{D}\Phi(t)&=\int^\oplus\chi(P,H(P))[\mathbf{d}_0q(\tfrac{x}{t})]\frac{A_{P_0}}{t}Q(t)\chi(P,H(P))\,\mathrm{d}P+\mathrm{h.\,c.}\\&\quad+\int^\oplus\chi(P,H(P))[V,\mathrm{i}Q(t)]\frac{A_{P_0}}{t}Q(t)\chi(P,H(P))\,\mathrm{d}P+\mathrm{h.\,c.}\\&\quad+\frac1t\int^\oplus\chi(P,H(P))Q(t)[H(P),\mathrm{i}A_{P_0}]Q(t)\chi(P,H(P))\,\mathrm{d}P\\&\quad-\frac1t\int^\oplus\chi(P,H(P))Q(t)\frac{A_{P_0}}{t}Q(t)\chi(P,H(P))\,\mathrm{d}P\\&=R_1+R_2+R_3+R_4.\end{aligned}$$

By the same arguments as before it follows that $\frac{A_{P_0}}{t}Q(t)\chi(P,H(P))$ is uniformly bounded. Using pseudo-differential calculus gives

$$\begin{aligned}R_1=\frac1t\int^\oplus&\chi(P,H(P))\big[\langle\tfrac{x}{t}-\nabla\omega(D_x)+\nabla\Omega(D_y),\nabla q(\tfrac{x}{t})\rangle\big]\frac{A_{P_0}}{t}Q(t)\chi(P,H(P))\,\mathrm{d}P\\&+\mathrm{h.\,c.}+O(t^{-2}).\end{aligned}$$

Let

$$B_1=\int^\oplus\chi(P,H(P))[\langle\tfrac{x}{t}-\nabla\omega(D_x)+\nabla\Omega(D_y),\nabla q(\tfrac{x}{t})\rangle]\,\mathrm{d}P$$

and

$$B_2 = \int^{\oplus} \chi(P, H(P)) Q(t) \frac{A_{P_0}}{t} \,\mathrm{d}P.$$

Then

$$R_1 = \tfrac{1}{t} B_1 B_2^* + \tfrac{1}{t} B_2 B_1^* \geq -\varepsilon_0^{-1} \tfrac{1}{t} B_1 B_1^* - \varepsilon_0 \tfrac{1}{t} B_2 B_2^*.$$

Now by Theorem 4.2, we get that $\frac{1}{t} B_1 B_1^*$ is integrable along the evolution. Using pseudo-differential calculus and functional calculus of almost analytic extensions one can verify that

$$[\chi(P, H(P)), Q(t)] = (H_0(P) - R)^{-1+\rho} O(t^{-1}) (H_0(P) - R)^{-\frac{1}{2}-\rho} \tag{21}$$

for any $R \in \mathbb{R} \setminus \sigma(H_0(P))$ and any ρ, $0 \leq \rho \leq \frac{1}{2}$. Hence it follows by introducing cutoff functions $\tilde{\chi} \in C_0^\infty(\mathbb{R}^{\nu+1})$ and $\tilde{q} \in C_0^\infty(\mathbb{R}^\nu)$ with $\tilde{\chi}\chi = \chi$ and $\tilde{q} q = q$ that

$$\begin{aligned}
-\tfrac{1}{t} B_2 B_2^* = &-\frac{1}{t} \int^{\oplus} Q(t) \chi \tilde{\chi}(P, H(P)) [\tilde{q}(\tfrac{x}{t})] \frac{A_{P_0}^2}{t^2} [\tilde{q}(\tfrac{x}{t})] \tilde{\chi} \chi(P, H(P)) Q(t) \,\mathrm{d}P \\
&+ O(t^{-2}) \\
\geq &-\frac{C_1}{t} \int^{\oplus} Q(t) \chi^2(P, H(P)) Q(t) \,\mathrm{d}P + O(t^{-2}) \\
= &-\frac{C_1}{t} \int^{\oplus} \chi(P, H(P)) Q^2(t) \chi(P, H(P)) \,\mathrm{d}P + O(t^{-2})
\end{aligned} \tag{22}$$

By Condition 2.3(iii) it follows that $\begin{pmatrix} 0 & 0 \\ \mathrm{i}(1-q(\frac{x}{t}))|\rho\rangle & 0 \end{pmatrix} \in O(t^{-1-\mu})$ and hence

$$R_2 \in O(t^{-1-\mu}) \tag{23}$$

Using (20) and (21) twice, we see that

$$\begin{aligned}
R_3 &= \frac{1}{t} \int^{\oplus} Q(t) \chi(P, H(P)) [H(P), \mathrm{i} A_{P_0}] \chi(P, H(P)) Q(t) \,\mathrm{d}P + O(t^{-2}) \\
&\geq \frac{C_2}{t} \int^{\oplus} Q(t) \chi^2(P, H(P)) Q(t) \,\mathrm{d}P + O(t^{-2}) \\
&\geq \frac{C_2}{t} \int^{\oplus} \chi(P, H(P)) Q(t)^2 \chi(P, H(P)) \,\mathrm{d}P + O(t^{-2}).
\end{aligned} \tag{24}$$

Again using the cutoff functions and pseudo-differential calculus and taking into account the support of q, we see that

$$\begin{aligned}
&\pm \chi(P, H(P)) Q(t) \frac{A_{P_0}}{t} Q(t) \chi(P, H(P)) \\
&\quad = \pm Q(t) \chi \tilde{\chi}(P, H(P)) [\tilde{q}(\tfrac{x}{t})] \frac{A_{P_0}}{t} [\tilde{q}(\tfrac{x}{t})] \tilde{\chi} \chi(P, H(P)) Q(t) \pm O(t^{-1}) \\
&\quad \leq \varepsilon C_3 Q(t) \chi^2(P, H(P)) Q(t) + O(t^{-1}) \\
&\quad = \varepsilon C_3 \chi(P, H(P)) Q(t)^2 \chi(P, H(P)) + O(t^{-1})
\end{aligned}$$

so

$$R_4 \geq -\frac{C_3\varepsilon}{t}\int^{\oplus}\chi(P,H(P))Q(t)^2\chi(P,H(P))\,\mathrm{d}P + O(t^{-2}). \tag{25}$$

Putting (22), (23), (24) and (25) together, we see that

$$\begin{aligned}\mathbf{D}\Phi(t) \geq\ & \frac{-\varepsilon_0 C_1 + C_2 - \varepsilon C_3}{t}\int^{\oplus}\chi(P,H(P))Q(t)^2\chi(P,H(P))\,\mathrm{d}P \\ & - \frac{1}{\varepsilon t}B_1B_1^* + O(t^{-1-\mu}).\end{aligned}$$

Now choosing ε and ε_0 so small that $-\varepsilon_0 C_1 + C_2 - \varepsilon C_3 > 0$ together with Lemma A.1 yields the result. □

5 The asymptotic observable and asymptotic completeness

Theorem 5.1 (Asymptotic observable)**.** *Let $p \in C^\infty(\mathbb{R}^\nu)$ satisfy that $p(x) \leq p(y)$ for $|x| \leq |y|$, $p(x) = 0$ for $|x| \leq \frac{1}{2}$ and $p(x) = 1$ for $|x| \geq 1$. Define $p_\delta(x) = p(\frac{x}{\delta})$. Then the limits*

$$P_\delta^+(H) = \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH}, \tag{26}$$

$$P_0^+(H) = \operatorname*{s-lim}_{\delta\to 0} P_\delta^+(H), \tag{27}$$

$$P_0^+(H_0,H) = \operatorname*{s-lim}_{\delta\to 0}\operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH_0},$$

$$P_0^+(H,H_0) = \operatorname*{s-lim}_{\delta\to 0}\operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH_0}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH}$$

exist and $P_0^+(H)$ is a projection.

Remark 5.2. Note that $\delta \mapsto P_\delta^+(H)$ is increasing in the sense that $P_\delta^+(H) \leq P_{\delta'}^+(H)$ for $0 < \delta' < \delta$. We leave it to the reader to verify that the definition of $P_0^+(H)$ is independent of the choice of p, and that one in fact could have chosen any family of functions $\{p_\delta\}$ satisfying $p_\delta(x) \leq p_\delta(y)$ for $|x| \leq |y|$, $p_\delta(x) = 0$ for $|x| \leq \frac{\delta}{2}$ and $p_\delta(x) = 1$ for $|x| \geq \delta$.

Proof. We will prove the statements about $P_\delta^+(H)$ and $P_0^+(H)$. The statements about $P_0^+(H_0,H)$ and $P_0^+(H,H_0)$ are proved completely analogously.

Let

$$\Phi(t) = -\chi(H)[p_\delta(\tfrac{x-y}{t})]\chi(H),$$

and calculate using pseudo-differential calculus

$$\mathbf{d}_0 p_\delta(\tfrac{x-y}{t}) = -\tfrac{1}{2}\tfrac{1}{t}\Big(\big(\tfrac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)\big)\cdot\nabla p_\delta(\tfrac{x-y}{t}) + \text{h. c.}\Big) + O(t^{-2}).$$

This in combination with Condition 2.3(iii) gives

$$\mathbf{D}\Phi(t) = \tfrac{1}{t}\chi(H)\big[\tfrac{1}{2}X\cdot\nabla p_\delta(\tfrac{x-y}{t}) + \text{h. c.}\big]\chi(H) + O(t^{-\min\{1+\mu,2\}}),$$

where $X = \frac{x-y}{t} - \nabla\omega(D_x) + \nabla\Omega(D_y)$, so Theorem 4.3 in combination with Lemma A.2 gives the existence of the limit (26).

The existence of the weak limit $\text{w-}P_0^+(H) = \text{w-lim}_{\delta\to 0} P_\delta^+(H)$ is obvious. Moreover, for every $\delta > 0$, it is clear from Lemma A.3 that the strong limit $\operatorname*{s-lim}_{n\to\infty} P^+_{\frac{\delta}{2^n}}(H)$ exists, is a projection and equals $\text{w-}P_0^+(H)$. The inequality $P_\delta^+(H)^2 \leq P_\delta^+(H)$ implies

$$\begin{aligned}\lim_{\delta\to 0}\|(\text{w-}P_0^+(H) - P_\delta^+(H))u\|^2 &= \lim_{\delta\to 0}\langle(\text{w-}P_0^+(H) + P_\delta^+(H)^2 - 2P_\delta^+(H))u, u\rangle \\ &\leq \lim_{\delta\to 0}\langle(\text{w-}P_0^+(H) - P_\delta^+(H))u, u\rangle = 0.\end{aligned}$$

This finishes the argument. □

Proposition 5.3. *Let $\Sigma = \{(P,\lambda) \in \mathbb{R}^{\nu+1} \,|\, \lambda \in \sigma_{\mathrm{pp}}(H(P))\}$ denote the set in energy-momentum space consisting of eigenvalues for the fibered Hamiltonian and $\Theta = \{(P,\lambda) \in \mathbb{R}^{\nu+1} \,|\, \lambda \in \vartheta(P)\}$ the corresponding set of thresholds. Then $\Sigma\cup\Theta$ is a closed set of Lebesgue measure 0. Moreover, $(\Sigma\cup\Theta)(P) = \sigma_{\mathrm{pp}}(P)\cup\vartheta(P)$ is at most countable.*

Proof. By the usual arguments, Theorems 3.1 and 3.4 imply that eigenvalues of $H(P)$ can only accumulate at thresholds (see e.g. [2] for details), and by analyticity, the threshold set $\vartheta(P)$ is at most countable. Hence, if $\Sigma\cup\Theta$ is closed, it is in particular of measure 0.

Let $(P_0,\lambda_0) \notin \Sigma\cup\Theta$. Then by Theorem 3.6, there are neighbourhoods $\mathcal{O}$ of P_0 and I of λ_0 such that for all $P \in \mathcal{O}$, a strict Mourre estimate holds for $H(P)$ on the energy interval I with conjugate operator A_{P_0} given as in Theorem 3.4 and $H(P)$ is of class $C^2(A_{P_0})$ by Theorem 3.1, which by the Virial Theorem implies that there are no eigenvalues for $H(P)$ in I for any $P \in \mathcal{O}$. Clearly,

$$\Theta = \{(P,\lambda) \in \mathbb{R}^{\nu+1} \,|\, \exists k \in \mathbb{R}^\nu \colon \lambda = \Omega(P-k) + \omega(k), \nabla\omega(k) - \nabla\Omega(P-k) = 0\}$$

is a closed set. Hence, possibly after chosing smaller $\mathcal{O}$ and I, $\mathcal{O}\times I$ is a neighbourhood of (P_0,λ_0) which does not intersect $\Sigma\cup\Theta$. □

Let $\mathcal{H}_{\mathrm{bd}} = E_{\Sigma\cup\Theta}((\mathrm{P},H))\mathcal{H}$ and similarly $\mathcal{H}_{0,\mathrm{bd}} = E_{\Sigma_0\cup\Theta}((\mathrm{P},H_0))\mathcal{H}$. We remark that if we for a fixed P take the fiber $(\Sigma\cup\Theta)(P) = \{\lambda \,|\, (\lambda,P) \in \Sigma\cup\Theta\}$, then we have $E_{(\Sigma\cup\Theta)(P)}(H(P)) = \mathbb{1}_{\mathrm{pp}}(H(P))$.

Theorem 5.4. *With $\mathcal{H}_{\mathrm{bd}}$ and $P_0^+(H)$ given as above, we have $\mathcal{H}_{\mathrm{bd}} = (1-P_0^+(H))\mathcal{H}$.*

Proof. Let $(\lambda_0,P_0) \in \mathbb{R}^{\nu+1}\setminus(\Sigma\cup\Theta)$. Let the neighbourhood N and $\varepsilon > 0$ be those of Theorem 4.4 corresponding to the point (λ_0,P_0). Let $\psi \in E_N(\mathrm{P},H)\mathcal{H}$. Then by Theorem 4.4, there exists a sequence $t_n \to \infty$ such that

$$\psi = e^{\mathrm{i}t_nH}p_\varepsilon(\tfrac{x-y}{t_n})e^{-\mathrm{i}t_nH}\psi + e^{\mathrm{i}t_nH}(1-p_\varepsilon(\tfrac{x-y}{t_n}))e^{-\mathrm{i}t_nH}\psi \to P_\varepsilon^+(H)\psi + 0,$$

which implies that $\psi \in P_0^+(H)\mathcal{H}$. As the span of such ψ is dense in $\mathcal{H}_{\mathrm{bd}}^\perp$ and $P_0^+(H)\mathcal{H}$ is closed, this implies that $\mathcal{H}_{\mathrm{bd}} \supset (1-P_0^+(H))\mathcal{H}$.

By Proposition 5.3, $\Sigma\cup\Theta$ may be written as an at most countable union of graphs Σ_i of Borel functions from (subsets of) $\mathbb{R}^\nu$ to $\mathbb{R}$ (see [17, Théorème 21, p. 226]).

Let $\varphi = U\int^{\oplus}\varphi_P\,\mathrm{d}P \in \mathcal{H}$. Then $\psi = E_{\Sigma_j}(\mathrm{P}, H)\varphi = U\int^{\oplus} E_{\Sigma_j(P)}(H)\varphi_P\,\mathrm{d}P$. This implies that ψ can be written as

$$\psi = U\int^{\oplus}\psi_P\,\mathrm{d}P,$$

where ψ_P is an eigenvector for $H(P)$ with eigenvalue $\Sigma_j(P)$. Note that this ensures that ψ_P is Borel as a function of P. Now

$$\begin{aligned}
P_\delta^+(H)\psi &= \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH}\psi \\
&= \operatorname*{s-lim}_{t\to\infty} U\int^{\oplus} e^{\mathrm{i}tH(P)}[p_\delta(\tfrac{x}{t})]e^{-\mathrm{i}tH(P)}\psi_P\,\mathrm{d}P \\
&= \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH}U\int^{\oplus}[p_\delta(\tfrac{x}{t})]e^{-\mathrm{i}t\Sigma_j(P)}\psi_P\,\mathrm{d}P,
\end{aligned}$$

where the last integrand goes pointwise to 0 and hence by the dominated convergence theorem, the limit is 0. As δ was arbitrary, this shows that $P_0^+(H)\psi = 0$.

Since the span of the set of ψ we have covered is dense in $\mathcal{H}_{\mathrm{bd}}$ and $P_0^+(H)$ is closed, we conclude that $\mathcal{H}_{\mathrm{bd}} \subset (1 - P_0^+(H))\mathcal{H}$. □

Theorem 5.5 (Existence of wave operators)**.** *The wave operator $W^+\colon \mathcal{H}\mapsto\mathcal{H}$ given by*

$$W^+u = \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0}P_0^+(H_0)u,$$

where $P_0^+(H_0)$ is the projection onto $\{0\}\oplus L^2(\mathbb{R}^{2\nu}) = \mathcal{H}_{0,\mathrm{bd}}^{\perp}$, exists.

Proof. From Theorem 5.1 and Theorem 5.4 with $H = H_0$ it follows that $P_0^+(H_0)$ can be given as in Theorem 5.1, and by passing to the fibered representation, it is easy to see that the assumptions on Ω and ω imply that $\mathcal{H}_{0,\mathrm{bd}} = L^2(\mathbb{R}^\nu)\oplus\{0\}$.

By Theorem 5.1,

$$e^{\mathrm{i}tH}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH_0} = e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0}e^{\mathrm{i}tH_0}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH_0}$$

tends strongly to $P_0^+(H_0, H)$ when $t\to\infty$ and $\delta\to 0$ (in that order). On the other hand,

$$e^{\mathrm{i}tH_0}[p_\delta(\tfrac{x-y}{t})]e^{-\mathrm{i}tH_0}$$

tends strongly to $P_0^+(H_0)$ in the same limit. This implies that

$$P_0^+(H_0, H) = \operatorname*{s-lim}_{t\to\infty}(e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0})P_0^+(H_0)$$

exists. □

Theorem 5.6 (Geometric asymptotic completeness)**.** *With W^+ as in Theorem 5.5,* $\operatorname{Ran} W^+ = P_0^+(H)\mathcal{H}$.

Proof. Consider

$$\begin{aligned}
W^+ &= \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} P_0^+(H_0)^2 \\
&= P_0^+(H_0, H) P_0^+(H_0) \\
&= \operatorname*{s-lim}_{\delta\to 0} \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH} [p_\delta(\tfrac{x-y}{t})] e^{-\mathrm{i}tH_0} P^+(H_0) \\
&= \operatorname*{s-lim}_{\delta\to 0} \operatorname*{s-lim}_{t\to\infty} \big(e^{\mathrm{i}tH} [p_\delta(\tfrac{x-y}{t})] e^{-\mathrm{i}tH} \big) \operatorname*{s-lim}_{\delta\to 0} \operatorname*{s-lim}_{t\to\infty} \big(e^{-\mathrm{i}tH} e^{-\mathrm{i}tH_0} \big) P_0^+(H_0) \\
&= P_0^+(H) W^+,
\end{aligned}$$

which proves that $\operatorname{Ran} W^+ \subset P_0^+(H)\mathcal{H}$. For the other inclusion, we similarly calculate

$$\begin{aligned}
P_0^+(H) &= \operatorname*{s-lim}_{\delta\to 0} \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH} [p_\delta(\tfrac{x-y}{t})] e^{-\mathrm{i}tH} P_0^+(H) \\
&= \operatorname*{s-lim}_{\delta\to 0} \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} e^{\mathrm{i}tH_0} [p_\delta(\tfrac{x-y}{t})] e^{-\mathrm{i}tH} P_0^+(H) \\
&= \operatorname*{s-lim}_{\delta\to 0} \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} P_0^+(H, H_0) P_0^+(H) \\
&= \operatorname*{s-lim}_{\delta\to 0} \operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH} e^{-\mathrm{i}tH_0} P_0^+(H_0) P_0^+(H, H_0) \\
&= W^+ P_0^+(H, H_0),
\end{aligned}$$

which proves $\operatorname{Ran} P_0^+(H) \subset \operatorname{Ran} W^+$. □

Theorem 2.1 now follows from Proposition 5.3, Theorem 5.4 and Theorem 5.6.

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Appendix A

For easy reference, we list the following lemmata, which are taken from the appendix of [DG]. The first lemma which is used to prove the propagation estimates, is a version of the Putnam-Kato theorem developed by Sigal–Soffer [18].

Lemma A.1. *Let H be a self-adjoint operator and $\mathbf{D}$ the corresponding Heisenberg derivative*

$$\mathbf{D} = \frac{\mathrm{d}}{\mathrm{d}t} + [H, \mathrm{i}\,\cdot\,].$$

Suppose that $\Phi(t)$ is a uniformly bounded family of self-adjoint operators. Suppose that there exist $C_0 > 0$ and operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \ldots, n$, such that

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t) B(t) - \sum_{i=1}^{n} B_i^*(t) B_i(t),$$

$$\int_1^\infty \|B_i(t)e^{-\mathrm{i}tH}\varphi\|^2\,\mathrm{d}t \le C\|\varphi\|^2, \quad i = 1, \ldots, n.$$

Then there exists C_1 such that

$$\int_1^\infty \|B(t)e^{-\mathrm{i}tH}\varphi\|^2\,\mathrm{d}t \le C_1\|\varphi\|^2.$$

The next lemma shows how to use propagation estimates to prove the existence of asymptotic observables and is a version of Cook's method due to Kato.

Lemma A.2. *Let H_1 and H_2 be two self-adjoint operators. Let ${}_2\mathbf{D}_1$ be the corresponding asymmetric Heisenberg derivative:*

$${}_2\mathbf{D}_1\Phi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + \mathrm{i}H_2\Phi(t) - \mathrm{i}\Phi(t)H_1.$$

Suppose that $\Phi(t)$ is a uniformly bounded function with values in self-adjoint operators. Let $\mathcal{D}_1 \subset \mathcal{H}$ be a dense subspace. Assume that

$$|\langle\psi_2, {}_2\mathbf{D}_1\Phi(t)\psi_1\rangle| \le \sum_{i=1}^n \|B_{2i}(t)\psi_2\|\|B_{1i}(t)\psi_1\|,$$

$$\int_1^\infty \|B_{2i}(t)e^{-\mathrm{i}tH_2}\varphi\|^2\,\mathrm{d}t \le \|\varphi\|^2, \quad \varphi \in \mathcal{H},\ i = 1, \ldots, n,$$

$$\int_1^\infty \|B_{1i}(t)e^{-\mathrm{i}tH_1}\varphi\|^2\,\mathrm{d}t \le C\|\varphi\|^2, \quad \varphi \in \mathcal{D}_1,\ i = 1, \ldots, n.$$

Then the limit

$$\operatorname*{s-lim}_{t\to\infty} e^{\mathrm{i}tH_2}\Phi(t)e^{-\mathrm{i}tH_1}$$

exists.

The final lemma gives us the actual asymptotic observable.

Lemma A.3. *Let Q_n be a commuting sequence of self-adjoint operators such that:*

$$0 \le Q_n \le 1, \quad Q_n \le Q_{n+1}, \quad Q_{n+1}Q_n = Q_n.$$

Then the limit

$$Q = \operatorname*{s-lim}_{n\to\infty} Q_n$$

exists and is a projection.

Appendix B

In this section, we recall a result from [16].

In the following, $A = (A_1, \ldots, A_\nu)$ is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space $\mathcal{H}$, and $B \in \mathcal{B}(\mathcal{H})$ is a bounded operator on $\mathcal{H}$. We shall use the notion of B being of class $C^{n_0}(A)$ introduced in [2]. For notational convenience, we adopt the following convention: If $0 \le j \le \nu$, then δ_j denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the j'th entry.

Definition B.1. Let $n_0 \in \mathbb{N} \cup \{\infty\}$. Assume that the multi-commutator form defined iteratively by $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^\alpha(B) = [\mathrm{ad}_A^{\alpha-\delta_j}(B), A_j]$ as a form on $\mathcal{D}(A_j)$, where $\alpha \geq \delta_j$ is a multi-index and $1 \leq j \leq \nu$, can be represented by a bounded operator also denoted by $\mathrm{ad}_A^\alpha(B)$, for all multi-indices α, $|\alpha| < n_0 + 1$. Then B is said to be of class $C^{n_0}(A)$ and we write $B \in C^{n_0}(A)$.

Remark B.2. The definition of $\mathrm{ad}_A^\alpha(B)$ does not depend on the order of the iteration since the A_j are pairwise commuting. We call $|\alpha|$ the *degree* of $\mathrm{ad}_A^\alpha(B)$.

In the following, $\mathcal{H}_A^s := D(|A|^s)$ for $s \geq 0$ will be used to denote the scale of spaces associated to A. For negative s, we define $\mathcal{H}_A^s := \mathcal{H}_A^{-s*}$.

Theorem B.3. *Assume that $B \in C^{n_0}(A)$ for some $n_0 \geq n + 1 \geq 1$, $0 \leq t_1, t_2$, $t_1 + t_2 \leq n + 2$ and that $\{f_\lambda\}_{\lambda \in I}$ satisfies*

$$\forall \alpha \, \exists C_\alpha \colon |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}$$

uniformly in λ for some $s \in \mathbb{R}$ such that $t_1 + t_2 + s < n + 1$. Then

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!} \partial^\alpha f_\lambda(A) \; \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A, B)$$

as an identity on $\mathcal{D}(\langle A \rangle^s)$, where $R_{\lambda,n}(A,B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})$ and there exist a constant C independent of A, B and λ such that

$$\|R_{\lambda,n}(A,B)\|_{\mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})} \leq C \sum_{|\alpha|=n+1} \|\mathrm{ad}_A^\alpha(B)\|.$$

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