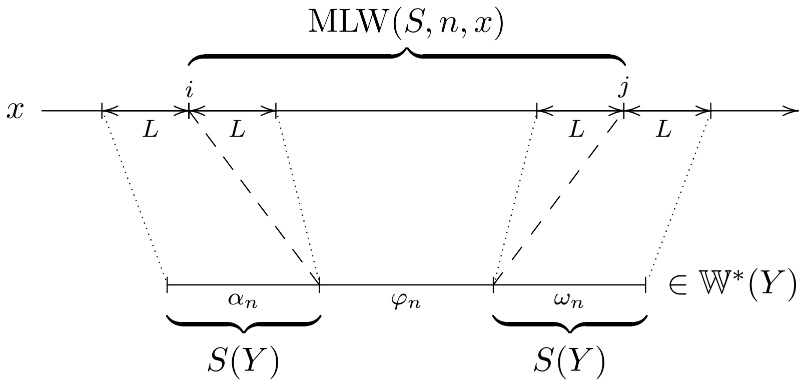
Morphisms Between Sofic Shift Spaces



PhD Thesis

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July 2010

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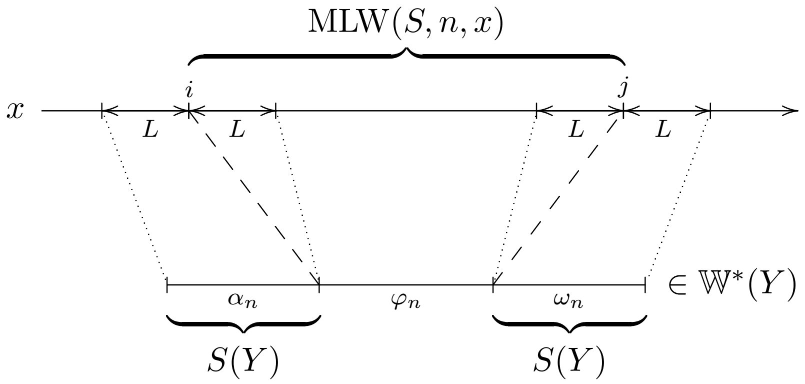

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Preface

This thesis is written as part of the requirements in the PhD programme in Mathematics at Aarhus University. It represents the research part of my work during my PhD studies which have been carried out under the supervision of Klaus Thomsen (Aarhus University).

About the Thesis

My general area of research is symbolic dynamics. If the reader is not familiar with symbolic dynamics, the second chapter contains a brief introduction to the basic concepts.

The ultimate goal of my research project is to solve the lower entropy factor problem for irreducible sofic shift spaces. In other words, I want to find a necessary and sufficient condition for one irreducible sofic shift to factor onto another irreducible sofic shift of strictly lower entropy. The condition should be effective in the sense that there should be a deterministic algorithm to check whether it is satisfied.

There are currently two known partial answers. One by Mike Boyle in [B] and one by Klaus Thomsen in [T]. Their conditions are formulated in terms of periodic points and both are proved using the marker strategy which was originally developed by Wolfgang Krieger in [K].

In [B] Mike Boyle solves the lower entropy factor problem for irreducible shifts of finite type by proving that the fact that a morphism respects the period of the periodic points, leads to a necessary and sufficient condition for one shift of finite type to factor onto another shift of finite type of strictly lower entropy. He then extends his result to the case when X is a sofic shift and Y is a mixing sofic shift with some special properties.

In [T] Klaus Thomsen shows that morphisms preserve more information about the periodic points than just their period. He proves that they also respect affiliation, a concept which involves the connection between individual periodic points in X and the top component of X. He shows that, when Y is mixing sofic and X is irreducible sofic with some special properties, that leads to a necessary and sufficient condition for a factor map to exist.

Like Boyle and Thomsen, I focus on the case when Y is mixing and I use a modified version of the marker strategy. But unlike them, I do not focus on periodic points. It turns out than when X is an irreducible sofic shift and Y is a mixing sofic shift, then all that hinders us from using Boyle's or Thomsen's result is a finite set of periodic points $E_{X,Y}$ in X. The necessary condition by Thomsen implies that $E_{X,Y}$ must be a subset of the derived shift ∂X of X, i.e. the set of points in X which do not contain any synchronizing words. I therefore focus on the problem of extending a morphism $\varphi\colon \partial X \to Y$ to a factor map from X onto Y. Because a necessary and sufficient condition for the existence of such an extension will lead to a necessary and sufficient condition for X to factor onto Y, since the existence of a morphism $\partial X \to Y$ is clearly necessary.

I find a non-trivial necessary and sufficient condition for such an extension to exist by proving:

Result 1. *Let X be an irreducible sofic shift and Y be a mixing sofic shift such that* $\mathrm{h}(X) > \mathrm{h}(Y)$. *Then a morphism $\varphi_{\partial X}\colon \partial X \to Y$ extends to a factor map $\varphi\colon X \twoheadrightarrow Y$ if and only if it is marked and $E_{X,Y} \subseteq \partial X$.*

In fact, I prove a more general result involving subshifts $S \subseteq X$.

Result 1 implies:

Result 2. *Let X be an irreducible sofic shift and Y be a mixing sofic shift such that* $\mathrm{h}(X) > \mathrm{h}(Y)$. *Then*

$$X \twoheadrightarrow Y \Leftrightarrow E_{X,Y} \subseteq \partial X, \exists \varphi\colon \partial X \to Y \textit{ marked.}$$

The marked property says that $\varphi(\partial X)$ has to be connected to the top component of Y in the same way that ∂X is connected to the top component of X. Since that is a necessary condition, it is clear that morphisms preserve a lot more structure than was previously known. In particular it shows that it is not enough to look at individual periodic points; one has to look at how several periodic points at the same time are connected to the top component. This new problem is investigated in Chapter 5, which can be read independently of the rest of the thesis.

The condition is not effective in general, but for a lot of X's and Y's it is. This is illustrated by the fact that I prove a result (Thm. 7.2.7), which generalises both Thomsen's and Boyle's results, in which the necessary and sufficient condition involves only finitely many periodic points (see section 7.2).

Even though the condition is generally hard to work with in praxis, it has theoretic value in that it for example allows me to show that a factor map exists if and only if an apparently weaker condition is satisfied, namely that there is a morphism, which hits a synchronizing word (Prop. 4.3.7).

So far, the lower entropy factor problem has turned out to be strongly linked to the higher entropy embedding problem, which is the problem: Given two sofic shifts of different entropy, when does one embed into the other? All results on one of the problems have led to similar results on the other. That is also the case with my result. With few, very natural changes the condition that works for the lower entropy factor problem becomes a necessary and sufficient condition for the higher entropy embedding problem.

In [B] Boyle asks if there is a tractable condition on the periodic points, which is both necessary and sufficient for a sofic shift to factor onto another. In an attempt to find such a condition, I investigate how much of the information about periodic points a morphism preserves. The result is a very strong but rather complicated necessary condition, which is still not sufficient. Hence my hopes for a positive answer to Boyle's question are very slim.

Chapter Overview

Chapter 1 contains the, to my knowledge, strongest currently known partial answers to the lower entropy factor problem and higher entropy embedding problem for sofic shifts; namely the ones by Krieger, Boyle and Thomsen. I have renamed the different periodic conditions in order to make it easier for the reader to remember what a given condition says, when I refer to it later on in the text. For example $\mathrm{Per\,X} \overset{\sigma}{\to} \mathrm{Per\,Y}$ is the condition that there is a shift-commuting map from Per X to Per Y. And $\mathrm{Per\,X} \overset{\sigma}{\hookrightarrow} \mathrm{Per\,Y}$ is the condition that there is an injective shift-commuting map.

In chapter 2 I have gathered the necessary definitions and results necessary to be able to work on the factor problem and the embedding problem for sofic shifts. Far most of it is well known and can be found in [LM], [M], [T] or [B]. Besides a change in notation, the only new things are the slightly stronger affiliation concept and its properties in section 2.7 and the modified marker strategy described in Section 2.10, which is critical for understanding the proofs of my main results. If the reader is already familiar with symbolic dynamics, he can probably skim through the rest of the chapter and use the notation index to look up any new notation, when needed.

The research part of the report begins in chapter 3. I introduce the concept of local words, which leads to the definition of 'the marked condition'. The condition is shown to be necessary for a morphism to exist between two shift spaces under mild assumptions on the shifts involved. The marked condition is then compared to the previously known necessary conditions.

In Chapter 4 I prove my main result, Theorem 4.1.1, which says that under suitable assumptions the following holds: If S is a subshift of an irreducible sofic shift X and Y is a mixing sofic shift of strictly lower entropy than X, then a morphism $S \to Y$ extends to a factor map $X \twoheadrightarrow Y$ if and only if it is marked.

In chapter 5 I introduce a generalized affiliation concept called simultaneous affiliation and show that it behaves much like the affiliation concept from Chapter 2 under morphisms. That leads to a very strong necessary condition on the periodic points, which sadly is not sufficient.

In Chapter 6 I use the ideas from Chapter 4 to prove a higher entropy embedding theorem, which is stronger than the higher entropy embedding results from Chapter 1.

Chapter 7 is dedicated to investigating what the marked property looks like under different simplifying assumptions. It turns out that in many cases the marked property is a lot simpler, decidable even, and in some cases, including those handled by Thomsen and Boyle, it concerns only periodic points. That leads to a solution of the following extension problem under suitable assumptions on the shifts involved: Let $S \subseteq X$ and let $\varphi\colon S \to Y$ be morphism. Does φ extend to a factor map from X to Y?

Chapter 8 contains my first (slightly) interesting research result after being accepted into the PhD programme. Namely a new proof of the fact that a mixing sofic shift is always an eventual factor of another mixing sofic shift of strictly higher entropy. The new thing is that my proof does not use Krieger's marker lemma.

The notation index doubles as a traditional index. I hope the reader will find it useful.

Chapter 1

History

In this chapter I have gathered the currently known partial answers to the lower entropy factor and higher entropy embedding problem for sofic shifts.

1.1 Krieger

In 1982 Krieger published [K] in which he proved the marker lemma:

Lemma 1.1.1. *Let X be a shift space and $k, T \in \mathbb{N}$, such that $k > T > 1$. Then there exists a clopen set $F \subseteq X$, which satisfies:*

1. *The sets $\sigma^i(F)$, $0 \le i < T$, are disjoint.*
2. *If a point $x \in X$ satisfies*

$$\sigma^i(x) \notin \bigcup_{-T<j<T} \sigma^j(F),$$

for some $i \in \mathbb{Z}$, then $x_{[i-k,i+k]}$ is periodic with period less than T.

He then used the marker lemmas ability to place markers in points in a shift space to prove the following embedding theorem:

Theorem 1.1.2. *Let X be a shift space and Y be a mixing SFT such that $\mathrm{h}(X) < \mathrm{h}(Y)$. Then*

$$X \hookrightarrow Y \Leftrightarrow \mathrm{Per\,X} \overset{\sigma}{\hookrightarrow} \mathrm{Per\,Y}.$$

Where $\mathrm{Per\,X} \overset{\sigma}{\hookrightarrow} \mathrm{Per\,Y}$ means that there is a shift-commuting injective map from $\mathrm{Per}\,X$ to $\mathrm{Per}\,Y$.

In other words: A shift space X embeds into a mixing SFT Y of strictly higher entropy if and only if Y contains at least as many periodic points of each period as X.

Thus by knowing only the periodic points in X and Y, it is possible to tell whether or not X embeds into Y.

1.2 Boyle

Shortly after, Boyle published [B] in which he used Krieger's idea of placing markers in points in a shift space to prove the following factor theorem:

Theorem 1.2.1. *Suppose X and Y are irreducible SFTs, such that* $\mathrm{h}(X) > \mathrm{h}(Y)$. *Then*
$$X \twoheadrightarrow Y \Leftrightarrow \mathrm{Per\,X} \stackrel{\sigma}{\to} \mathrm{Per\,Y}\,.$$

Where $\mathrm{Per\,X} \stackrel{\sigma}{\to} \mathrm{Per\,Y}$ means that there is a shift commuting map from $\mathrm{Per}\,X$ to $\mathrm{Per}\,Y$.

Equivalently: An irreducible SFT X factors onto another irreducible SFT Y of strictly lower entropy if and only if Y contains a periodic point whose period divides n, whenever X contains a periodic point of period n.

Note the similarity between Boyle's and Krieger's conditions in that both conditions involve only periodic points.

Later in [B] Boyle extends both his and Krieger's result to some sofic shifts by proving the following:

Theorem 1.2.2. *Suppose X and Y are shift spaces satisfying the following properties:*

1. Y is mixing sofic,

2. X contains an SFT $\overline{X}$ such that $\mathrm{h}(\overline{X}) > \mathrm{h}(Y)$, (*for example, X is sofic and* $\mathrm{h}(X) > \mathrm{h}(Y)$)

3. Whenever X contains a periodic point of period n, Y contains a periodic point whose period divides n and which is 1-affilitated to the top component.

Then $X \twoheadrightarrow Y$.

Where affiliation is defined in the following way: Let x be a periodic point of least period n in a shift space X. x is said to be d-affiliated to the top component of X, denoted by $x \in X_0^{(d)}$, if there are synchronizing words a, b in $\mathbb{W}(X)$ such that $ax_{[0,n[}^{dk}b \in \mathbb{W}(X)$ for all $k \in \mathbb{N}$.

To be fair Boyle did not formulate his result in terms of affiliation, a concept introduced 20 years later by Thomsen in [T], but this equivalent formulation makes the comparison between Boyle's and Thomsen's results easier.

Corollary 1.2.3. *Suppose X and Y are sofic shift spaces, Y mixing receptive, such that* $\mathrm{h}(X) > \mathrm{h}(Y)$, *then*

$$X \twoheadrightarrow Y \Leftrightarrow \mathrm{Per\,X} \stackrel{\sigma}{\to} \mathrm{Per\,Y}\,.$$

Where receptive means that whenever Y contains a periodic point of period n, then there is a periodic point also in Y whose period divides n and which is 1-affilitated to the top component of Y.

Theorem 1.2.4. *Suppose X and Y are shift spaces satisfying the following properties:*

1. Y *is mixing sofic,*
2. $\mathrm{h}(Y) > \mathrm{h}(X)$
3. $\forall n \in \mathbb{N}\colon Q_n(X) \leq Q_n(Y_0^{(1)})$.
Then $X \hookrightarrow Y$.

Where $Q_n(X)$ is the number of periodic points in X with minimal period n.

Corollary 1.2.5. *Suppose* X *and* Y *are sofic shift spaces,* Y *mixing inclusive, such that* $\mathrm{h}(Y) > \mathrm{h}(X)$, *then*

$$X \hookrightarrow Y \Leftrightarrow \mathrm{Per\,X} \overset{\sigma}{\hookrightarrow} \mathrm{Per\,Y}\,.$$

Where inclusive means that all periodic points in Y are 1-affiliated to the top component of Y.

In fact Boyle proved that the inclusive and receptive sofic shifts are exactly those that behave like mixing SFTs with regards to embeddings and factors.

1.3 Thomsen

In [T], Klaus Thomsen investigates the structure of sofic shifts, which leads to the definition of irreducible components in sofic shifts, which generalize the very useful irreducible components in shifts of finite type.

He shows that the structure of irreducible components in a sofic shift space is a conjugacy invariant, and that it is respected by morphisms in the following sense:

Theorem 1.3.1. *Let* X *and* Y *be sofic shift spaces, and* $\varphi\colon X \to Y$ *a morphism of shift spaces. It follows that there is a map* $X_c \to Y_{\varphi(c)}$ *from the set of irreducible components in* X *to the irreducible components of* Y, *such that*

$$x \in X_c^{(d)} \Rightarrow \varphi(x) \in Y_{\varphi(c)}^{\left(\frac{\mathrm{period}(x)d}{\mathrm{period}(\varphi(x))}\right)}.$$

In particular, $\varphi(\overline{X_c}) \subseteq \overline{Y_{\varphi(c)}}$.

Using that and the same marker strategy employed by Boyle and Krieger, he proves the following extensions of Boyle's factor and embedding theorems:

Theorem 1.3.2. *Let* X *and* Y *be irreducible sofic shift spaces. Then*

$$X \twoheadrightarrow Y \Rightarrow \mathrm{Per\,X} \overset{\mathrm{(d)}}{\twoheadrightarrow} \mathrm{Per\,Y}$$

And if furthermore Y *is mixing,* $\mathrm{h}(X) > \mathrm{h}(Y)$ *and* X *has transparent affiliation pattern, then*

$$\mathrm{Per\,X} \overset{\mathrm{(d)}}{\twoheadrightarrow} \mathrm{Per\,Y} \Rightarrow X \twoheadrightarrow Y$$

Theorem 1.3.3. *Let X and Y be sofic shift spaces, X mixing with transparent affiliation pattern. If X embeds into Y, there is an irreducible component Y_c in Y such that $X \subseteq \overline{Y_c}$,*

a) $\operatorname{Per} X \overset{(d)}{\hookrightarrow} \operatorname{Per} \overline{Y_c}$, *and*

b) $\mathrm{h}(X) \leq \mathrm{h}(\overline{Y_c})$.

Conversely, if there is an irreducible component Y_c in Y such that a) holds and

b') $\mathrm{h}(X) < \mathrm{h}(\overline{Y_c})$,

then $X \subseteq \overline{Y_c} \subseteq Y$.

Where $\operatorname{Per X} \overset{(\mathrm{d})}{\twoheadrightarrow} \operatorname{Per Y}$ is defined by

$$\forall F \subseteq \mathbb{N}, n \in \mathbb{N}\colon \bigcap_{d \in F} Q_n(X_0^{(d)}) \neq \emptyset \Rightarrow \bigcap_{d \in F} \bigcup_{m|n} Q_m(Y_0^{(\frac{n}{m}d)}) \neq \emptyset,$$

and $\operatorname{Per X} \overset{(\mathrm{d})}{\hookrightarrow} \operatorname{Per Y}$ means that there is a shift-commuting injective map $\lambda\colon \operatorname{Per} X \to \operatorname{Per} Y$ such that

$$x \in X_0^{(d)} \Rightarrow \lambda(x) \in Y_0^{(d)}$$

for all $d \in \mathbb{N}$.

And transparent affiliation pattern means that all but finitely many periodic points are 1-affiliated to the top component of X and the rest have marked entries and exits as defined below.

Definition 1.3.4 (Entry and Exit)**.** Let X be a shift space, $z \in \operatorname{Per} X$ and $x \in X$. A z-entry of length n is an interval $[i, j[$ of length n, such that $x_{[i,j[}$ occurs in z, but $x_{[i-1,j[}$ does not. A z-exit of length n is an interval $[i, j[$ of length n, such that $x_{[i,j[}$ occurs in z, but $x_{[i,j+1[}$ does not.

Definition 1.3.5 (Marked entries and Exits)**.** Let X be a shift space and $z \in \operatorname{Per} X$. z is said to have marked entries if there is an $N \in \mathbb{N}$ such that when $x \in X$ and $x_{[i,j[}$ is a z-entry of length at least $2N$, then there is a unique $y \in \pi^{-1}(z)$ such that $\omega_{i+N-1} = y_{i+N-1}$ for all $\omega_{[i-N,j+N[} \in \pi^{-1}(x_{[i-N,j+N[})$.

z is said to have marked exits if there is an $N \in \mathbb{N}$ such that when $x \in X$ and $x_{[i,j[}$ is a z-exit of length at least $2N$, then there is a unique $y \in \pi^{-1}(z)$ such that $\omega_{i-N} = y_{i-N}$ for all $\omega_{[i-N,j+N[} \in \pi^{-1}(x_{[i-N,j+N[})$.

If one is willing to assume something about both X and Y, and not just X, it is possible to get a stronger result from Thomsen's proof. A careful reading of the proof of Theorem 9.13 in [T] shows that the assumption about 1-affiliation is used only to ensure that the set

$$E_{X,Y} = \{x \in \operatorname{Per} X \mid \bigcup_{m||p_x|} Q_m(Y_0^{(\frac{|p_x|}{m})}) = \emptyset\}.$$

is finite. i.e. that all but finitely many periodic points in X have a partner in $\operatorname{Per} Y$ which is almost 1-affiliated to the top component in the sense that

$\bigcup_{m|n} Y_0^{(\frac{n}{m})} \neq \emptyset$. But that is automatic when Y is mixing sofic. Thus with almost the same proof we get the following result:

Theorem 1.3.6. *Let X and Y be irreducible sofic. Then*

$$X \twoheadrightarrow Y \Rightarrow \mathrm{Per\,X} \overset{(\mathrm{d})}{\twoheadrightarrow} \mathrm{Per\,Y}$$

And if furthermore Y is mixing, $\mathrm{h}(X) > \mathrm{h}(Y)$ and all points in $E_{X,Y}$ have marked entries and exits, then

$$\mathrm{Per\,X} \overset{(\mathrm{d})}{\twoheadrightarrow} \mathrm{Per\,Y} \Rightarrow X \twoheadrightarrow Y$$

Which contains Boyle's result (Thm. 1.2.2), when X is an irreducible sofic shift space with higher entropy than Y. Because under the assumptions of Theorem 1.2.2, $E_{X,Y}$ is empty and $\mathrm{Per\,X} \overset{(\mathrm{d})}{\twoheadrightarrow} \mathrm{Per\,Y}$ is satisfied.

Note that Theorem 1.3.6 implies that $E_{X,Y} \subseteq \partial X$ is necessary for X to factor onto Y. Because $\mathrm{Per}\, X_0 \subseteq X_0^{(1)}$, as shown by Thomsen. So if $\mathrm{Per\,X} \overset{(\mathrm{d})}{\twoheadrightarrow} \mathrm{Per\,Y}$, then $E_{X,Y} \cap X_0 = \emptyset$, which implies that $E_{X,Y} \subseteq \partial X$, since $\mathrm{Per}\, X = \mathrm{Per}\, X_0 \cup \mathrm{Per}\, \partial X$.

In section 7.2 I show that it is possible to weaken the assumption on the points in $E_{X,Y}$, if the condition on the periodic points is strengthened even further.

Chapter 2

Introduction

This chapter is a brief introduction to the basic concepts of symbolic dynamics. For a more thorough exposition see for example [LM].

2.1 Shift Spaces

In the following Σ denotes a finite set, called an alphabet. The elements of Σ are called symbols. We will use the notation $x_I = \{x_i\}_{i\in I}$ for sequences of elements in Σ. The objects we will be studying are subsets of the full Σ-shift, which is the set of doubly infinite sequences of symbols from Σ.

Definition 2.1.1 (Full shift)**.** The full Σ-shift $\Sigma^{\mathbb{Z}}$ is defined by:

$$\Sigma^{\mathbb{Z}} = \{x_{\mathbb{Z}} \mid \forall i \in \mathbb{Z} \colon x_i \in \Sigma\}.$$

We turn the full shift into a topological space by equipping it with the topology induced by the metric $d\colon \Sigma^{\mathbb{Z}} \times \Sigma^{\mathbb{Z}} \to \mathbb{R}$ defined by

$$d(x, y) = \inf\{2^{-k} \mid x_{[-k,k]} = y_{[-k,k]}\}.$$

Lemma 2.1.2. *$\Sigma^{\mathbb{Z}}$ is compact.*

Proof. Let $\{x^i\}_{i\in\mathbb{N}}$ be a sequence of elements in $\Sigma^{\mathbb{Z}}$. We will construct a convergent subsequence of $\{x^i\}_{i\in\mathbb{N}}$. Choose first an $i_0 \in \mathbb{N}$ such that $x_0^j = x_0^{i_0}$ for infinitely many $j \geq i_0$. Then choose inductively an increasing sequence i_j such that $x^{i_j}_{[-j+1,j-1]} = x^{i_{j-1}}_{[-j+1,j-1]}$ and such that $x^{i_j}_{[-j,j]} = x^k_{[-j,j]}$ for infinitely many $k \geq i_j$. Then the subsequence $\{x^{i_j}\}_{j\in\mathbb{N}}$ converges to the element $\{x_k^{i_{|k|}}\}_{k\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$. □

Definition 2.1.3 (Shift map)**.** The map $\sigma\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ is called the shift map.

i.e. σ shifts an element $x \in \Sigma^{\mathbb{Z}}$ one step to the left.

$$\begin{array}{c} \cdots x_{-1}\, x_0\; x_1\; x_2\; x_3 \cdots \\ \sigma \big\downarrow \\ \cdots x_0\; x_1\; x_2\; x_3\; x_4 \cdots \end{array}$$

Definition 2.1.4 (Shift invariant)**.** A subset $X \subseteq \Sigma^{\mathbb{Z}}$ is called shift invariant if $\sigma(X) = X$.

This enables us to define what it means for a subset of $\Sigma^{\mathbb{Z}}$ to be a shift space.

Definition 2.1.5 (Shift Space)**.** A subset $X \subseteq \Sigma^{\mathbb{Z}}$ is called a shift space if it is closed and shift invariant.

Note that shift spaces are compact since they are closed subsets of the compact set $\Sigma^{\mathbb{Z}}$. Note also that the set of shift spaces is closed under arbitrary intersections and finite unions.

Example 2.1.6. $\Sigma^{\mathbb{Z}}$ is clearly a shift space. A nontrivial example of a shift space is the set $\{\sigma^n(\ldots abcabc \ldots)\}_{n\in\mathbb{N}}$. This is obviously shift invariant. It is closed because any convergent sequence of elements in it must be constant from some point on. The subset of $\{0,1\}^{\mathbb{Z}}$ consisting of the elements, which contain exactly one '1' symbol is not a shift space, as it is not closed, since the sequence of elements whose ith symbol is a 1, converges to 0^{∞}, which is not in the subset.

This topological way of thinking about shift spaces is however not always the most useful when working with them. We will often use an equivalent definition in terms of words and languages as defined below.

Definition 2.1.7 (Word)**.** A word over an alphabet Σ is a possibly infinite sequence of symbols in Σ. Let w and x be words over Σ. The length of w, denoted by $|w|$, is the number of symbols in it. w is said to occur in, or be a subword of x if there exists an interval $[i,j] \subseteq \mathbb{Z}$, such that $x_{[i,j]} = w$. This is denoted by $w \subseteq x$. The set of finite subwords of x is denoted by $\mathbb{W}(x)$. If nothing else is specified, the symbols of a word w are indexed by $[0, |w|[$, i.e. $w = w_{[0,|w|[}$.

Definition 2.1.8 (Language)**.** Let $X \subseteq \Sigma^{\mathbb{Z}}$. The set of finite words occurring in elements of X is denoted by $\mathbb{W}(X)$, and called the language of X. The set of all words occurring in elements of X is denoted by $\mathbb{W}^*(X)$. We use the following notation:

$$\begin{aligned} \mathbb{W}(X)^c &= \mathbb{W}(\Sigma^{\mathbb{Z}}) - \mathbb{W}(X), \\ \mathbb{W}_n(X) &= \{w \in \mathbb{W}(X) \mid |w| = n\}, \\ \mathbb{W}_{\geq n}(X) &= \{w \in \mathbb{W}^*(X) \mid |w| \geq n\} \text{ and} \\ \mathbb{W}_{\leq n}(X) &= \{w \in \mathbb{W}(X) \mid |w| \leq n\}. \end{aligned}$$

If X is a shift space, then the set $\mathbb{W}_1(X)$ is called the alphabet of X and is denoted by Σ_X, or simply Σ, when the shift space is understood.

If u and v are words over Σ, uv denotes the concatenation of them. When $n \in \mathbb{N}$, u^n denotes u concatenated with itself $n-1$ times and u^∞ denotes u concatenated with itself infinitely many times to the left, right or both depending on the context. When dealing with subwords of words in $\mathbb{W}^*(X)$, things like $ax_{[i,j]}b$ mean what they say when i and j are finite and $x_{]-\infty,j]}b$, $ax_{[i,\infty[}$ and $x_{]-\infty,\infty[}$, when $i=-\infty$, $j=\infty$ and $[i,j] =]-\infty,\infty[$ respectively. This is done in order to avoid cluttering the text with special cases.

Definition 2.1.9 (X_F)**.** Let X be a shift space and $F \subseteq \mathbb{W}(\Sigma_X^{\mathbb{Z}})$. Then X_F denotes the maximal subset of X, which satisfies that no word from F occurs in any element. i.e.

$$X_F = \{x \in X \mid \mathbb{W}(x) \cap F = \emptyset\}.$$

The elements in F are called forbidden words.

Definition 2.1.9 enables us to formulate the promised equivalent definition of shift spaces:

Theorem 2.1.10. *A subset $X \subseteq \Sigma^{\mathbb{Z}}$ is a shift space if and only if there exists a set $F \subseteq \mathbb{W}(\Sigma^{\mathbb{Z}})$, such that $X = \Sigma_F^{\mathbb{Z}}$.*

Proof. $\Rightarrow$: Let $F \subseteq \mathbb{W}(\Sigma^{\mathbb{Z}})$. Then $\Sigma_F^{\mathbb{Z}}$ is clearly shift invariant. And the argument used to prove that $\Sigma^{\mathbb{Z}}$ is compact in Lemma 2.1.2 also proves that $\Sigma_F^{\mathbb{Z}}$ is compact, and therefore closed. $\Leftarrow$: Let $X \subseteq \Sigma^{\mathbb{Z}}$ be a shift space. Then it is closed, which means that $\Sigma^{\mathbb{Z}} - X$ is open. We can therefore for each $y \in \Sigma^{\mathbb{Z}} - X$ find a $k_y \in \mathbb{N}$ such that $\forall z \in \Sigma^{\mathbb{Z}}\colon z_{[-k_y,k_y]} = y_{[-k_y,k_y]} \Rightarrow z \in \Sigma^{\mathbb{Z}} - X$. Define $F = \{y_{[-k_y,k_y]} \mid y \in \Sigma^{\mathbb{Z}} - X\}$. I claim that $X = \Sigma_F^{\mathbb{Z}}$. Let $x \in \Sigma^{\mathbb{Z}} - X$. Then $x_{[-k_x,k_x]} \in F$ and thus $x \notin \Sigma_F^{\mathbb{Z}}$. Let $x \in \Sigma^{\mathbb{Z}} - \Sigma_F^{\mathbb{Z}}$. Then we can find $i,j \in \mathbb{N}$ such that $x_{[i,j]} \in F$. But then $\sigma^{-\frac{i+j}{2}}(x) \notin X$ by definition of F. So by shift invariance $x \notin X$. □

Theorem 2.1.10 makes it easier to construct concrete shift spaces as illustrated in the following example:

Example 2.1.11. Let X_{gm} be the set of elements in $\{0,1\}^{\mathbb{Z}}$, for which two consecutive 1's are separated by at least one 0. Then it is a shift space, as $X_{\text{gm}} = \{0,1\}^{\mathbb{Z}}_{\{11\}}$. X_{gm} is called the golden mean shift.
Let X_{2n} be the set of elements in $\{0,1\}^{\mathbb{Z}}$, for which two consecutive 1's are separated by an even number of 0's. Then it is a shift space, since $X_{\text{2n}} = \{0,1\}^{\mathbb{Z}}_{\{10^{2k+1}1 \mid k \in \mathbb{N}\}}$. X_{2n} is called the even shift.

Lemma 2.1.12. *Let X be a shift space and Σ be an alphabet containing Σ_X. Then*

$$X = \Sigma^{\mathbb{Z}}_{\mathbb{W}(X)^c}$$

Proof. $\subseteq$ is obviously true for all subsets of $\Sigma^{\mathbb{Z}}$. $\supseteq$: Since X is a shift space, $X = \Sigma_F^{\mathbb{Z}}$ for some $F \in \mathbb{W}(\Sigma^{\mathbb{Z}})$ by Theorem 2.1.10. Let $x \in \Sigma^{\mathbb{Z}}_{\mathbb{W}(X)^c}$. Then $x \in \Sigma_F^{\mathbb{Z}}$ since clearly $F \subseteq \mathbb{W}(X)^c$. $\square$

Lemma 2.1.12 implies that an element $x \in \Sigma^{\mathbb{Z}}$ is in a shift space X if and only if each subword of x is in $\mathbb{W}(X)$. Thus a shift space is uniquely given by its language.

Definition 2.1.13 (Irreducible)**.** A shift space X is called irreducible if

$$\forall a, b \in \mathbb{W}(X) \exists w \in \mathbb{W}(X) \colon awb \in \mathbb{W}(X).$$

Definition 2.1.14 (Mixing)**.** A shift space X is called mixing if

$$\forall a, b \in \mathbb{W}(X) \exists N \in \mathbb{N} \forall n \geq N \exists w \in \mathbb{W}_n(X) \colon awb \in \mathbb{W}(X).$$

If N can be chosen independently of a and b, it is called a transition length of X. The minimal transition length is denoted by $\mathrm{TL}(X)$.

Definition 2.1.15 (Synchronizing)**.** Let X be a shift space and $s \in \mathbb{W}(X)$. Then s is called synchronizing if

$$\forall a, b \in \mathbb{W}(X) \colon as, sb \in \mathbb{W}(X) \Rightarrow asb \in \mathbb{W}(X).$$

The set of synchronizing words in X is denoted by $S(X)$.

If a word is synchronizing, then each word in which it occurs is synchronizing:

Lemma 2.1.16. *Let X be a shift space and $u, v, w \in \mathbb{W}(X)$, with the property $uwv \in \mathbb{W}(X)$. Then*

$$w \in S(X) \Rightarrow uwv \in S(X)$$

Proof. Assume that $w \in S(X)$ and let $s, t \in \mathbb{W}(X)$ be arbitrary. If $suwv$, $uwvt \in \mathbb{W}(X)$, then $suw, wvt \in \mathbb{W}(X)$, which implies that $suwvt \in \mathbb{W}(X)$, since $w \in S(X)$. $\square$

As we will see later on, the synchronizing words in a shift space are very useful when constructing maps between shift spaces.

2.2 Morphisms

Definition 2.2.1 (Morphism)**.** Let X be a shift space. A map $\varphi \colon X \to \Sigma^{\mathbb{Z}}$ is called a morphism if it commutes with the shift map and it is continuous.

Note that the composition of two morphisms is a morphism.

Definition 2.2.2 (Factor, Embedding, Conjugacy)**.** Let X and Y be shift spaces. If there is a morphism $\varphi\colon X \to Y$, we say that X maps to Y, which is denoted by $X \to Y$. If φ is surjective, it is called a factor map, and Y is called a factor of X, which is denoted by $X \twoheadrightarrow Y$. If φ is injective it is called an embedding, and X is said to embed into Y, which is denoted by $X \hookrightarrow Y$. If φ is invertible, it is called a conjugacy, and X and Y are called conjugate, which is denoted by $X \simeq Y$.

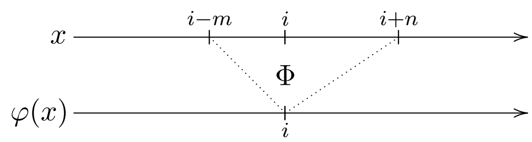
As with shift spaces the topological viewpoint of Definition 2.2.2 is not the way we will work with morphisms. We are going to construct morphisms by constructing sliding block codes as defined below.

Let $n \in \mathbb{N}$. A map $\Phi\colon \mathbb{W}_n(X) \to \Sigma$, called a block map, induces a map $\varphi\colon X \to \Sigma^{\mathbb{Z}}$, called a sliding block code, in the following way:

Definition 2.2.3 (Sliding Block Code)**.** Let X be a shift space and let $\Phi\colon \mathbb{W}_{m+n+1}(X) \to \Sigma$ be a map for some $m, n \in \mathbb{N}$. Then the map $\varphi\colon X \to \Sigma^{\mathbb{Z}}$ defined by

$$\varphi(x) = \{\Phi(x_{[i-m,i+n]})\}_{i\in\mathbb{Z}}$$

is called an (m, n)-sliding block code. m is called the memory and n the anticipation of φ.



Theorem 2.2.4. *Let X be a shift space. A map $\varphi\colon X \to \Sigma^{\mathbb{Z}}$ is a morphism if and only if it is a sliding block code.*

Proof. $\Rightarrow$: Let $\varphi\colon X \to \Sigma^{\mathbb{Z}}$ be a morphism. Then φ is continuous and therefore uniformly continuous, since X is compact. Since φ is shift-commuting, we can therefore find a $k \in\ N$, such that for all $x \in X$ and $i \in \mathbb{Z}$, $\varphi(x)_i$ is function of $x_{[i-k,i+k]}$. This block map induces φ.
$\Leftarrow$: Left to the reader. $\square$

Example 2.2.5. The shift map, σ, is a morphism, as it is the $(0, 1)$-sliding block code induced by the block map $x_1x_2 \mapsto x_2$. It is a conjugacy, since the (1,0)-sliding block code induced by the block map $x_1x_2 \mapsto x_1$ is an inverse.

Definition 2.2.6 (n-block map)**.** Let X be a shift space and $n \in \mathbb{N}$. A morphism $\varphi\colon\ X \to \Sigma^{\mathbb{Z}}$ is called an n-block map if it is an $(0, n-1)$-sliding block code.

Lemma 2.2.7. *Let X be a shift space and $\varphi\colon X \to \Sigma^{\mathbb{Z}}$ be an (m, n)-sliding block code for some $m, n \in \mathbb{N}$. Then $\varphi \circ \sigma^m\colon X \to \Sigma^{\mathbb{Z}}$ is an $m + n + 1$-block map with the same image.*

Proof. Let Φ be the block map, which induces φ. Then clearly

$$\varphi \circ \sigma^m(x) = \{\Phi(x_{[i,i+m+n]})\}_{i \in \mathbb{Z}}.$$

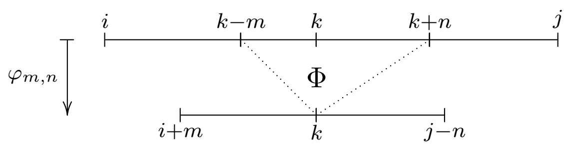
For all $x \in X$. Thus $\varphi \circ \sigma^m$ is an $m+n+1$ block map. And since X is shift invariant, $(\varphi \circ \sigma^m)(X) = \varphi(X)$. □

This implies that we can always recode a morphism to an n-block map for some $n \in \mathbb{N}$. Note that an n-block map, φ, can be considered an m-block map for all $m \geq n$ by defining a block map which ignores the last $m - n$ symbols and otherwise works as the block map inducing φ.

Definition 2.2.8 (Word Map)**.** Let X and Y be shift spaces. If $\varphi\colon X \to Y$ is an (m,n)-sliding block code, then $\varphi_{m,n}$ denotes the map $\mathbb{W}_{\geq m+n+1}(X) \to \mathbb{W}^*(Y)$, defined by

$$\varphi_{m,n}(x_{[i,j]}) = \{\Phi(x_{[k-m,k+n]})\}_{k \in [i+m, j-n]}.$$

$\varphi_{m,n}$ is called the (m,n)th word map of φ. To simplify notation we define $\varphi_{0,n-1} = \varphi_n$.



The following lemma shows that the set of shift spaces as well as the subsets of irreducible and mixing shift spaces are invariant under morphisms.

Lemma 2.2.9. *Let X be a shift space and $\varphi\colon X \to \Sigma^{\mathbb{Z}}$ be a morphism. Then $\varphi(X)$ is a shift space. And if X is irreducible or mixing, then so is $\varphi(X)$.*

Proof. let X be a shift space and $\varphi\colon X \to \Sigma^{\mathbb{Z}}$ a morphism. As φ commutes with the shift, $\sigma(\varphi(X)) = \varphi(\sigma(X)) = \varphi(X)$. Thus $\varphi(X)$ is shift invariant. And since it is the image of a compact set under a continuos map, it is also closed and therefore a shift space.

By Lemma 2.2.7 we can without loss of generality assume that φ is an n-block morphism for some $n \in \mathbb{N}$. Assume that X is irreducible. Let $a, b \in \mathbb{W}(Y)$ and find $a', b' \in \mathbb{W}(X)$ such that $\varphi_n(a') = a$ and $\varphi_n(b') = b$. Since X is irreducible, we can find an $x' \in \mathbb{W}(X)$, such that $a'x'b' \in \mathbb{W}(X)$. Define $x \in \mathbb{W}(Y)$ by $axb = \varphi_n(a'x'b')$. Then $axb \in \mathbb{W}(Y)$. Thus Y is irreducible.

Assume that X is mixing and let $a, b \in \mathbb{W}(Y)$. Find $a', b' \in \mathbb{W}(X)$, such that $\varphi_n(a') = a$ and $\varphi_n(b') = b$. Since X is mixing, we can find an N', such that we for all $k \geq N'$ can find a word $x' \in \mathbb{W}_k(X)$, such that $a'x'b' \in \mathbb{W}(X)$. Define $N = N' + n - 1$, and let $k \geq N$. Choose an $x' \in \mathbb{W}_{k-n+1}(X)$, such that $a'x'b' \in \mathbb{W}(X)$. Define $x \in \mathbb{W}_k(Y)$ by $axb = \varphi_n(a'x'b')$. Then $axb \in \mathbb{W}(Y)$. Thus Y is mixing. □

Definition 2.2.10 (Higher block shift)**.** Let X be a shift space and $n \in \mathbb{N}$. The nth higher block shift of X, $X^{[n]}$, is defined by:

$$X^{[n]} = \beta_n(X),$$

where $\beta_n \colon X \to \mathbb{W}_n(X)^{\mathbb{Z}}$ is the n-block map, induced by the block map $x_{[0,n[} \mapsto x_{[0,n[}$.

When $x \in X$, we use the notation $x^{[n]}$ for the word $\beta_n(x)$.

It is easier to imagine the points in $X^{[n]}$, if we write the symbols in the alphabet $\mathbb{W}_n(X)$ vertically;

$$x^{[n]} = \beta_n(x) = \cdots \begin{bmatrix} x_{n-2} \\ x_{n-3} \\ \cdots \\ x_{-1} \end{bmatrix} . \begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \cdots \\ x_0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \\ \cdots \\ x_1 \end{bmatrix} \cdots \in \mathbb{W}_n(X)^{\mathbb{Z}},$$

where the symbol to the right of the dot corresponds to index 0.

Note that the higher block shifts are shift spaces by Lemma 2.2.9 and that if $n, m \in \mathbb{N}$, then $(X^{[n]})^{[m]} \simeq X^{[n+m-1]}$.

β_n is a conjugacy, since the 1-block map, β_n^{-1}, induced by $x_{[0,n[} \mapsto x_0$ is an inverse of β_n. The symbol β_n is also used to denote the nth word map of β_n, $(\beta_n)_n \colon \mathbb{W}^*_{\geq n}(X) \to \mathbb{W}^*(X^{[n]})$ and $\beta_{n,1}^{-1}$ denotes the inverse of $(\beta_n)_n$. i.e. $\beta_{n,1}^{-1}(x^{[n]}_{[1,m]}) = x_{[1,m+n-1]}$, for all $m \geq 1$. Note that $\beta_{n,1}^{-1} \colon \mathbb{W}^*(X^{[n]}) \to \mathbb{W}^*_{\geq n}(X)$ is not the the 1st word map of β_n^{-1}, $(\beta_n^{-1})_1$. This is illustrated by the following example.

Example 2.2.11. Consider $0100 \in \mathbb{W}_4(X_{\mathrm{gm}})$. Then $\beta_2(0100) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ is a word in $\mathbb{W}_3(X_{\mathrm{gm}}^{[2]})$ and $\beta_{2,1}^{-1}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right) = 0100 \neq 010 = (\beta_2^{-1})_1\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right)$.

Remark 2.2.12. If $\varphi \colon X \to Y$ is an n-block map, then $\varphi \circ \beta_d^{-1} \colon X^{[d]} \to Y$ is an $m = \max\{1, n-d+1\}$-block map, since it is induced by the map $x^{[d]}_{[1,m]} \mapsto \varphi_n(x_{[1,n]}) = \varphi_n(\beta_{d,1}^{-1}(x^{[d]}_{[1,m]})_{[1,n]})$. So when we are in a situation where only the conjugacy class of X matters, we can without loss of generality assume that φ is 1-block, or k-block for any $k \in \mathbb{N}$ for that matter.

Lemma 2.2.13. *Let X be a shift space and $n \in \mathbb{N}$. Then*

$$\beta_n(S_{\geq n}(X)) = S(X^{[n]})$$

Proof. Left to the reader. □

Definition 2.2.14 (Higher Power Shift)**.** Let X be a shift space and $n \in \mathbb{N}$. The nth higher power shift of X, X^n, is defined by:

$$X^n = \lambda_n(X),$$

where $\lambda_n \colon X \to \mathbb{W}_n(X)^{\mathbb{Z}}$ is defined by $\lambda_n(x)_i = x_{[in,in+n-1]}$.

Again it is easier to imagine λ_n if we write the symbols in the alphabet $\mathbb{W}_n(X)$ vertically;

$$\lambda_n(x) = \cdots \begin{bmatrix} x_{-1} \\ x_{-2} \\ \cdots \\ x_{-n} \end{bmatrix} . \begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \cdots \\ x_0 \end{bmatrix} \begin{bmatrix} x_{2n-1} \\ x_{2n-2} \\ \cdots \\ x_n \end{bmatrix} \cdots \in (\mathbb{W}_n(X))^{\mathbb{Z}}.$$

As for β_n, λ_n also denotes the map $\mathbb{W}^*_{\geq n}(X) \to \mathbb{W}(X^n)$ induced by λ_n.

Note that λ_n is *not* a morphism $X \to X^n$. It is however a conjugacy between (X, σ^n) and X^n, where (X, σ^n) is the set X seen as a shift space, where the shift map shifts n steps to the left instead of one.

2.3 Periodic Points

The structure of periodic points in shift spaces turns out to play a crucial role in determining whether a shift space X is a factor of or embeds into another shift space Y. In fact when X and Y are irreducible shifts of finite type of different entropy[1], then it is possible to decide whether $X \twoheadrightarrow Y$ or $X \hookrightarrow Y$, when knowing only the sets of periodic points in X and Y, as shown by Boyle and Krieger in [B] and [K].

Definition 2.3.1 (Periodic Point)**.** Let X be a shift space and $n \in \mathbb{N}$. A point $x \in X$ is called n-periodic if $\sigma^n(x) = x$. The set of n-periodic points in X is denoted by $\mathrm{Per}_n(X)$. A point is called periodic, if it is n-periodic for some $n \in \mathbb{N}$, and the set of all periodic points in X is denoted by $\mathrm{Per}\, X$. The least number n for which an $x \in X$ is n-periodic, is called the minimal period x and is denoted by $\mathrm{period}(x)$. $Q_n(X)$ denotes the set of points in X, which have minimal period n. If $x \in Q_n(X)$, then p_x denotes a subword of x of length n. p_x is called a minimal period of x.

The minimal period of a point $x \in \mathrm{Per}\, X$ divides any other period of x:

Lemma 2.3.2. *Let X be a shift space and $n \in \mathbb{N}$. Then*

$$\mathrm{Per}_n(X) \subseteq \bigcup_{m|n} Q_m(X).$$

Proof. Let $n \in \mathbb{N}$ and $x \in \mathrm{Per}_n(X)$. Let m be the minimal period of x. Assume that m does not divide n. Then x is (n mod m)-periodic, which contradicts the minimality of m. □

Lemma 2.3.3. *Let X and Y be shift spaces, $\varphi \colon X \to Y$ be a morphism and $n \in \mathbb{N}$. Then*

$$\varphi(Q_n(X)) \subseteq \bigcup_{m|n} Q_m(Y).$$

[1]Shifts of finite type is the topic of the next section and entropy is defined in section 2.8

Proof. Let $x \in Q_n(X)$. Then $\sigma^n(\varphi(x)) = \varphi(\sigma^n(x)) = \varphi(x)$, which means that $\varphi(x) \in \mathrm{Per}_n(Y)$. Thus $\varphi(x) \in \bigcup_{m|n} Q_m(Y)$ by Lemma 2.3.2. $\square$

Lemma 2.3.4. *Let X and Y be shift spaces and $\varphi\colon X \to Y$ be an embedding. Then*

$$\forall n \in \mathbb{N}\colon \varphi(Q_n(X)) \subseteq Q_n(Y).$$

In particular $|Q_n(X)| \leq |Q_n(Y)|$.

Proof. Assume that $\varphi\colon X \to Y$ is injective and let $x \in Q_n(X)$. Then $\varphi(x) \in Q_m(Y)$, for some $m|n$, by Lemma 2.3.3. As φ is injective we must have $m = n$, since otherwise $\varphi(x) = \varphi(\sigma^m(x))$ and $x \neq \sigma^m(x)$. $\square$

Corollary 2.3.5. *Let X and Y be shift spaces and $\varphi\colon X \to Y$ be a conjugacy. Then*

$$\forall n \in \mathbb{N}\colon \varphi(Q_n(X)) = Q_n(Y).$$

In particular $|Q_n(X)| = |Q_n(Y)|$.

Definition 2.3.6. Let X be a shift-invariant subset of a full shift. We call the greatest common divisor of $\{|p_x| \mid x \in \mathrm{Per}\, X\}$ the period of X and denote it by $\mathrm{period}(X)$.

2.4 Shifts of Finite Type

Definition 2.4.1 (SFT)**.** A shift space X is called a shift of finite type, if there exists a finite set $F \subseteq \mathbb{W}(\Sigma^{\mathbb{Z}})$ such that $X = \Sigma^{\mathbb{Z}}_F$.

I will use SFT both as the set of all shifts of finite type and as an abreviation of 'shift of finite type'. Thus $X \in \mathrm{SFT}$ and 'X is an SFT' means the same thing.

Note that the definition only requires that there exists a finite set F - not that any set of forbidden words should be finite. If $F \neq \emptyset$ we can always extend F to be infinite. (If $b \in F$, then $\Sigma^{\mathbb{Z}}_F = \Sigma^{\mathbb{Z}}_{F'}$, where $F' = F \cup \{b^n | n \in \mathbb{N}\}$.)

Example 2.4.2. Since $X_{\mathrm{gm}} = \{0,1\}^{\mathbb{Z}}_{\{11\}}$, $X_{\mathrm{gm}} \in \mathrm{SFT}$. And since $\emptyset$ is finite, any full shift is an SFT.

Definition 2.4.3 (M-step)**.** Let X be a shift space and $M \in \mathbb{N}$. X is said to be M-step, if there exists an $F \subseteq \mathbb{W}_{M+1}(\Sigma^{\mathbb{Z}}_X)$, such that $X = \Sigma^{\mathbb{Z}}_F$. M is called a step length of X. $\mathrm{SL}(X)$ denotes the minimal step length of X. SFT_M denotes the set of all M-step shift spaces.

The great thing about M-step shifts is that they are locally recognizable in the sense that it is possible to tell if an element in $\Sigma^{\mathbb{Z}}$ is an element of the shift, by looking only at subwords of length $M + 1$. This is due to the fact that if X is M-step, then $\mathbb{W}_M(X) \subseteq S(X)$, which in fact characterizes the set SFT_M by the following lemma.

Lemma 2.4.4. *Let X be shift space and $M \in \mathbb{N}$. Then $X \in \mathrm{SFT}_M$ if and only if*

$$\mathbb{W}_M(X) \subseteq S(X)$$

Proof. $\Rightarrow$ follows by definition. $\Leftarrow$: Assume that $\mathbb{W}_M(X) \subseteq S(X)$. Define $F = \mathbb{W}_{M+1}(\Sigma_X^{\mathbb{Z}}) - \mathbb{W}_{M+1}(X)$. Then obviously $X \subseteq \Sigma_F^{\mathbb{Z}}$. Let $x \in \Sigma_F^{\mathbb{Z}}$. Then each subword $x_{[i,i+M]}$ is a word in $\mathbb{W}(X)$. So as each $x_{[i,i+M[}$ is synchronizing, each subword of x must be in $\mathbb{W}(X)$. This implies that $x \in X$ by Lemma 2.1.12. Thus $X = \Sigma_F^{\mathbb{Z}}$, which establishes the claim, since $F \subseteq \mathbb{W}_{M+1}(\Sigma_X^{\mathbb{Z}})$. $\square$

This kind of local recognizability turns out to be very useful when constructing morphisms.

Any SFT is an M-step shift for some $M \in N$ and vice versa:

Lemma 2.4.5. *Let X be a shift space. Then*

$$X \in \mathrm{SFT} \Leftrightarrow \exists M \in \mathbb{N}\colon X \in \mathrm{SFT}_M\,.$$

Proof. $\Rightarrow$: Assume that X is an SFT and find a finite set $F \subseteq \mathbb{W}(\Sigma^{\mathbb{Z}})$ such that $X = \Sigma_F^{\mathbb{Z}}$. If $|F| = 0$, we define $M = 0$. If $|F| > 0$, we set $M = \max\{|f| \mid f \in F\} - 1$ and define $F' = \{w \in \mathbb{W}_{M+1}(\Sigma^{\mathbb{Z}}) \mid \exists u \in F\colon u \subseteq w\}$. Then clearly $X = \Sigma_F^{\mathbb{Z}} = \Sigma_{F'}^{\mathbb{Z}}$.
$\Leftarrow$: If X is M-step, then $X \in \mathrm{SFT}$, as any $F \subseteq \mathbb{W}_{M+1}(\Sigma_X^{\mathbb{Z}})$ is finite, since $\mathbb{W}_{M+1}(\Sigma_X^{\mathbb{Z}})$ is. $\square$

Example 2.4.6. Lemma 2.4.5 can be used to show that the even shift $X_{2\mathrm{n}}$, is not an SFT. Because if it was an SFT, Lemma 2.4.5 would provide an $M \in \mathbb{N}$, such that $X_{2\mathrm{n}}$ was M-step. But that would imply that $x = 0^\infty 10^{2M+1}10^\infty$ is in $X_{2\mathrm{n}}$, since all subwords of length $M+1$ are in $\mathbb{W}(X_{2\mathrm{n}})$. But since $2M+1$ is odd, x is not in $X_{2\mathrm{n}}$.

The two preceding lemmas, together with Lemma 2.1.16, give a useful characterization of the set of shifts of finite type:

Theorem 2.4.7. *Let X be a shift space. Then*

$$X \in \mathrm{SFT} \Leftrightarrow \exists M \in \mathbb{N}\colon \mathbb{W}_{\geq M}(X) \subseteq S(X).$$

This implies that X is an SFT if and only if each $x \in X$ contains a synchronizing word:

Corollary 2.4.8. *Let X be a shift space. Then*

$$X \in \mathrm{SFT} \Leftrightarrow X_{S(X)} = \emptyset.$$

Proof. $\Rightarrow$ follows directly from Theorem 2.4.7. $\Leftarrow$: If $X \notin \mathrm{SFT}$ is a shift space, then Theorem 2.4.7 lets us choose a sequence $w^{\mathbb{N}}$ of points in X, such that $w^i_{[-i,i]} \notin S(X)$ for all $i \in \mathbb{N}$. Since X is compact, we can choose a convergent subsequence w^{i_j}. Let w be the limit of w^{i_j}. Then $w \in X$ and $\mathbb{W}(w) \cap S(X) = \emptyset$, by Lemma 2.1.16. $\square$

Using Lemma 2.2.13, the reader easily verifies:

Corollary 2.4.9. *A shift space X is an SFT if and only if there is an $M \in \mathbb{N}$ such that $X^{[N]} \in \mathrm{SFT}_1$, for all $N \geq M$.*

Corollary 2.4.10. *SFT is closed under conjugacies.*

Proof. Let $X \in \mathrm{SFT}$, Y be a shift space and $\varphi\colon X \to Y$ be a conjugacy. Let M be a step length for X. We wish to find an $M' \in \mathbb{N}$ which is a step length for Y. Let n be the largest number among the memory and anticipation of φ and φ^{-1} and consider them both as (n,n)-block maps. Define $M' = M + 2n$ and let $y \in \mathbb{W}_{\geq M}(Y)$ be arbitrary. I claim that $y \in S(Y)$. In order to establish the claim, let $a, b \in \mathbb{W}(Y)$ such that $ay, yb \in \mathbb{W}(Y)$. Choose words $s, t \in \mathbb{W}_{2\mathrm{n}}(Y)$ such that say and ybt are words in $\mathbb{W}(Y)$. Then $\varphi_{n,n}(say) = a'\varphi_{n,n}(y)$ and $\varphi_{n,n}(ybt) = \varphi_{n,n}(y)b'$ are words in $\mathbb{W}(X)$. So as $|\varphi_{n,n}(y)| = M$, $\varphi_{n,n}(y) \in S(X)$, which implies that $a'\varphi_{n,n}(y)b' \in \mathbb{W}(X)$. Thus $\varphi_{n,n}^{-1}(a'\varphi_{n,n}(y)b') = ayb \in \mathbb{W}(Y)$, which establishes the claim. $\square$

It turns out that there is a strong connection between SFTs and (finite directed) graphs.

Definition 2.4.11 (Graph)**.** Let Σ be an alphabet. A graph is a pair (V,E) of finite sets V and $E \subseteq V^2 \times \Sigma$, such that $(v_1, v_2, n) \neq (w_1, w_2, m) \in E \Rightarrow n \neq m$. The elements in V are called vertices and the elements in E are called edges. Let $e = (v_1, v_2, n)$ be an edge. v_1 and v_2 are called the initial and terminal vertex of the edge and n is called the name of the edge. The initial vertex of an edge is denoted by $i(e)$ and the terminal vertex by $t(e)$.

The reason the edges have names is that there can be several edges between two vertices.

Example 2.4.12. Graphs can of course be illustrated in the standard way:

$$(\{0,1\}, \{(0,0,a), (0,1,b), (0,1,c), (1,1,d)\}) \rightsquigarrow \begin{tikzcd} \cdot \arrow[loop left, "a"] \arrow[r, bend left, "b"] \arrow[r, bend right, "c"'] & \cdot \arrow[loop right, "d"] \end{tikzcd}$$

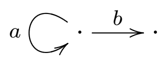
The names of the vertices have been omitted, because we will only be interested in the names of the edges.

Definition 2.4.13 (Path, Edge Shift)**.** A path in a graph $G = (E, V)$ is a sequence of edges $e_{[1,n]} \in \mathbb{W}^*(E^{\mathbb{Z}})$, for which $t(e_i) = i(e_{i+1})$ for all $i \in \{1, \cdots, n-1\}$. A path $p = e_{[1,n]}$ is said to go through a vertex I, if I is the initial or terminal vertex of one of the edges in p. This is denoted by $I \in p$. $i(e_1)$ is called the initial vertex of p and is denoted by $i(p)$. Similarly $t(p)$ denotes $t(e_n)$ and is called the terminal vertex of p. We will not distinguish between an edge and its name. A path can therefore be thought of as a word in $\mathbb{W}^*(\Sigma^{\mathbb{Z}})$. The set of all finite paths in G is denoted by $\mathbb{W}(G)$. The set of all doubly infinite paths in G is denoted by Σ_G and is called the edge shift of G.

Note that Σ_G is a 1-step SFT, since we can choose $F = \{ef \in E^2 | t(e) \neq i(f)\} \subseteq \mathbb{W}_2(\Sigma^{\mathbb{Z}})$ as the forbidden words.

The following example shows that it is not true in general that $\mathbb{W}(G) = \mathbb{W}(\Sigma_G)$.

Example 2.4.14. The G be the following graph.



Then $b \in \mathbb{W}(G)$, but $b \notin \mathbb{W}(\Sigma_G) = \mathbb{W}(\{a^\infty\})$.

Definition 2.4.15 (Essential Graph)**.** A graph G, is called essential if $\mathbb{W}(\Sigma_G) = \mathbb{W}(G)$.

Remark 2.4.16. If we in a graph G delete all vertices, which are either not the initial vertex of any edge or not the terminal vertex of any edge, and then delete all edges, which started or ended in a deleted vertex, then the resulting graph presents the same shift as G. So as a graph is essential if and only if $\forall v \in V \exists e, f \in E \colon i(e) = v = t(f)$, we can repeat this process a finite number of times and end up with an essential graph, which presents the same shift as G.

Example 2.4.17. The process described in Remark 2.4.16 used on the graph from example 2.4.14 deletes the terminal vertex of b and then b, which leaves us with the graph presenting $\{a^\infty\}$.

The promised connection between SFTs and graphs is:

Proposition 2.4.18. *Let X be a shift space. Then*

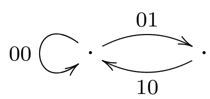
$$X \in \mathrm{SFT} \Leftrightarrow \exists G \colon X \simeq \Sigma_G.$$

Proof. $\Leftarrow$: Follows from corollary 2.4.10 and the fact that edge shifts are 1-step. $\Rightarrow$: Let $X \in \mathrm{SFT}$. Then there exists an $M \in \mathbb{N}$ such that $X^{[M]} \in \mathrm{SFT}_1$ by Lemma 2.4.9. Define $V = \mathbb{W}_M(X)$ and

$$E = \{(v, w, n) | v, w \in V \wedge vw \in \mathbb{W}_2(X^{[M]}) \wedge n = vw\}$$

Then $G = (V, E)$ is a graph and $X \simeq X^{[M+1]} \simeq (X^{[M]})^{[2]} = \Sigma_G$. $\square$

Example 2.4.19 ($X_{\mathrm{gm}}^{[2]}$)**.** The edge shift of the following graph is $X_{\mathrm{gm}}^{[2]}$, and therefore conjugate to X_{gm}.



The following gives us a canonical way of approximating a shift space from above by SFTs.

Lemma 2.4.20. *Let S be a shift space. Then the sequence $S_{\mathbb{N}}$ of* SFT*s defined by $S_n = \Sigma^{\mathbb{Z}}_{\mathbb{W}_n(S)^c}$ is decreasing, i.e. $S_{n+1} \subseteq S_n$, for all $n \in \mathbb{N}$, and*

$$\bigcap_{n\in\mathbb{N}} S_n = S.$$

Proof. Clearly each S_n is an SFT and $S_{n+1} \subseteq S_n$. Lemma 2.1.12 implies, that $\cap_{n\in\mathbb{N}} S_n = \Sigma^{\mathbb{Z}}_{\cup_n \mathbb{W}_n(S)^c} = S$. □

Note that the points in S_n, for a given $n \in \mathbb{N}$, are the points in $\Sigma^{\mathbb{Z}}$ for which it is impossible to rule out that they are in S by looking at subwords of length n.

Lemma 2.4.21. *Let $X \subseteq \Sigma^{\mathbb{Z}}$ be a finite shift invariant set. Then X is an SFT with step length $2\max\{|p_x| \mid x \in X\} - 1$.*

Proof. Clearly all points in X are periodic. Thus $k = 2\max\{|p_x| \mid x \in X\} - 1$ is well-defined. Since X is a finite set, it is closed. So as it is shift invariant, it is a shift space. By lemma 2.4.4 we therefore only need to show that all words in $\mathbb{W}_k(X)$ are synchronizing.

Let $v \in \mathbb{W}_k(X)$ and $u, w \in \mathbb{W}(X)$, such that $uv, vw \in \mathbb{W}(X)$. Then we can find $x, y \in X$, such that $x_{[-|u|,0[} = u$, $x_{[0,k[} = v = y_{[0,k[}$ and $y_{[k,k+|w|[} = w$. We want to show that $x = y$, since that would imply that $uvw \in \mathbb{W}(X)$, because uvw would be equal to $x_{[-|u|,k+|w|[}$.

Let p and q denote the minimal period of x and y, respectively. Then $x = x^{\infty}_{[0,p[}$ and $y = y^{\infty}_{[0,q[} = x^{\infty}_{[0,q[}$. So if we could prove that $p = q$, we would be done.

Let $i \in \mathbb{Z}$. Then

$$x_i = x_{[i]_p} = y_{[i]_p} = y_{[i]_p+q} = x_{[i]_p+q} = x_{i+q},$$

since $0 \leq [i]_p, [i]_p + q \leq p - 1 + q \leq k$. Thus $x \in \operatorname{Per}_q X$, which implies that $p|q$ by Lemma 2.3.2. By replacing x by y and p by q, the same argument shows that $q|p$. Thus $p = q$. □

2.5 Sofic Shifts

We will mainly be interested in a particular kind of shift spaces called sofic shifts. They are defined in terms of labeled graphs.

Definition 2.5.1 (Labeled graph)**.** Let Σ be an alphabet. A labeled graph is a triple $G = (V, E, L)$, where (V, E) is a graph and L is a map $E \to \Sigma$. The elements of Σ are called labels and the morphism $\Sigma_{(V,E)} \to \Sigma^{\mathbb{Z}}$ induced by L

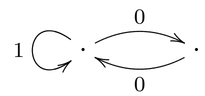
is called the label map and is also denoted by L. A labeled graph is said to present the set $L(\Sigma_{(V,E)})$, which is denoted by Σ_G. The set $L_1(\mathbb{W}(V,E))$ is denoted by $\mathbb{W}(G)$.

In order to simplify notation, we also denote the first word map of L, L_1, by L. Thus if u_I is a (possibly infinite) path in a labeled graph (V,E,L), then $L(u) = \{L(u_i)\}_{i\in I}$.

The main difference between labeled graphs and non-labeled graphs is that several edges can have the same label. Note that a graph G can be turned into a labeled graph (G,L), which presents Σ_G, by defining L to be the identity. Note also that if G is a labeled graph, then Σ_G is a shift space since it is the image of a shift space under a morphism.

Definition 2.5.2 (Sofic shift)**.** A shift space X is called sofic if it can be presented by a labeled graph.

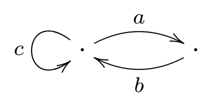
Example 2.5.3. As the even shift, $X_{2\mathrm{n}}$, is presented by the labeled graph below, it is a sofic shift space.



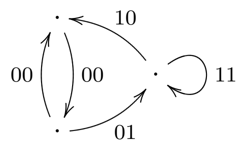
The original motivation for introducing sofic shift spaces was to find the smallest set of shift spaces containing SFT which was closed under factor maps. They were therefore defined as the shift spaces, which are factors of shifts of finite type. That is however equivalent to our definition. To prove that, we need the following definition:

Definition 2.5.4 (Higher Block Graph)**.** Let $G = (V,E,L)$ be a labeled graph and $n \geq 2$. The nth higher block graph of G, $G^{[n]} = (V',E',L')$ is defined by $V' = \mathbb{W}_{n-1}(V,E)$, $E' = \{(u_{[1,n[}, v_{[1,n[}, m) \in V'^2 \times \mathbb{W}_n(V,E) | u_{[2,n-1]} = v_{[1,n-2]}, m = uv_{n-1}\}$ and $L' = L$.

Example 2.5.5. Let $G = (V,E,L)$ be the labeled graph from Example 2.5.3 presenting the even shift. We wish to construct the second higher block graph of G. To do that we need the names of the edges of the underlying graph. Let's assume that (V,E) is the following graph:



Then the vertex set of $G^{[2]}$ is $\mathbb{W}_1(V,E) = \{a,b,c\}$. And since the condition $u_{[2,2-1]} = v_{[1,2-2]}$ is trivial, for all $u, v \in V'$, we only need to check whether $uv \in \mathbb{W}_2(V,E)$, to decide whether (u,v,uv) is an edge in $G^{[2]}$. Thus the edge set is $\{(a, b, ab), (b, a, ba), (b, c, bc), (c, a, ca), (c, c, cc)\}$. So as the label map is the first word map of L. i.e. $L'(uv) = L(u)L(v)$, $G^{[2]}$ is the following:



The reader easily verifies the following:

Lemma 2.5.6. *Let G be a labeled graph and $n \in \mathbb{N}$. Then $\Sigma_{G^{[n]}} = \Sigma_G^{[n]}$.*

Theorem 2.5.7. *A shift space is sofic if and only if it is a factor of an* SFT.

Proof. $\Rightarrow$: Let X be a sofic shift space. Then the label map induces a factor map from the edge shift of the underlying graph onto X. $\Leftarrow$: Let X be a shift space, $S \in$ SFT and $\varphi \colon S \to X$ a factor map. Let G be the graph from Theorem 2.4.18 such that $S \simeq \Sigma_G$. Define $\pi \colon \Sigma_G \to X$ by $\pi = \varphi \circ \alpha$, where α is the conjugacy $\Sigma_G \to S$. Recode π to an n-block code for some $n \in \mathbb{N}$. Then π induces a 1-block map $\Sigma_{G^{[n]}} \to X$ by remark 2.2.12 and Lemma 2.5.6. By identifying edges in $G^{[n]}$ with their name, this gives us a map $L \colon E_{G^{[n]}} \to \Sigma_X$. Thus $(V_{G^{[n]}}, E_{G^{[n]}}, L)$ is a labeled graph, which presents X. $\square$

Corollary 2.5.8. *All SFTs are sofic.*

Corollary 2.5.9. *The set of sofic shifts is closed under conjugacies.*

2.6 Fischer Cover

Let X be an irreducible sofic shift. Among all the labeled graphs presenting X there is one with particularly pleasant properties. The definition of this graph, called the Fischer cover of X, is the goal of this section.

Definition 2.6.1 (Essential labeled graph)**.** A labeled graph is called essential if the underlying graph is essential.

Definition 2.6.2 (Right-, and left-resolving)**.** Let $G = (V, E, L)$ be a labeled graph. G is called right-resolving in a vertex $v \in V$ if L is injective on the set of edges with v as their initial vertex, $\{e \in E \mid i(e) = v\}$. i.e. if all edges starting at v has different labels. If G is right-resolving in all vertices, it is called right-resolving.

Left-resolving is defined analogously. If a labeled graph is both right- and left-resolving, it is called bi-resolving.

Definition 2.6.3 (Irreducible)**.** A labeled graph (V, E, L) is called irreducible if for each pair of vertices $v_1, v_2 \in V$ there exists a path from v_1 to v_2.

Note that if G is irreducible, then it is essential and Σ_G is an irreducible sofic shift.

Definition 2.6.4 (Follower Set of States)**.** Let $G = (V, E, L)$ be a labeled graph and $v \in V$. The follower set of v, $F(v)$, is defined by

$$F(v) = \{L(p) \mid p \in \mathbb{W}(G) \wedge i(p) = v\}.$$

Definition 2.6.5 (Follower Separated)**.** A labeled graph $G = (V, E, L)$ is said to be follower separated if different states have different follower sets. i.e.

$$\forall v, v' \in V\colon\ v \neq v' \Rightarrow F(v) \neq F(v')$$

Definition 2.6.6 (Graph Isomorphism)**.** Let G and H be labeled graphs. A pair of maps $\partial\Phi\colon V_G \to V_H$ and $\Phi\colon E_G \to E_H$ is called a graph homomorphism if for all edges $e \in E_G$ the maps satisfy $i(\Phi(e)) = \partial\Phi(i(e))$, $t(\Phi(e)) = \partial\Phi(t(e))$ and $L_H(\Phi(e)) = L_G(e)$. If both Φ and $\partial\Phi$ are invertible, then G and H are called isomorphic, which is denoted by $G \simeq H$.

It turns out that there is (up to graph isomorphism) at most one irreducible, right-resolving and follower separated labeled graph, which presents an irreducible sofic shift space.

Proposition 2.6.7 (Fischer)**.** *Let G and G' be irreducible, right-resolving and follower-separated labeled graphs, such that $\Sigma_G = \Sigma_{G'}$. Then $G \simeq G'$.*

Proof. Proposition 3.3.17 in [LM]). □

In order to prove that any irreducible sofic shift has such a presentation, we need some graph constructions.

Definition 2.6.8 (Subgraph)**.** Let $G = (V, E, L)$. Then a triple (V', E', L') is called a subgraph of G if it is a labeled graph and $V' \subseteq V$, $E' \subseteq E$ and $L' = L_{|E'}$.

Lemma 2.6.9. *Let X be an irreducible sofic shift space. If G is a labeled graph, which presents X, then there exists a subgraph of G, which is irreducible and presents X.*

Proof. Assume that $G = (V, E, L)$ presents X. If a vertex $I \in V$ satisfies $\forall u \in \mathbb{W}(X) \exists p \in L^{-1}(u)\colon I \notin p$, then the subgraph we get by deleting I and any edge starting or ending in I will still present X. By repeating this process at most $|V| - 1$ times, we get a subgraph $(V', E', L') \subseteq G$, which presents X and satisfies $\forall I \in V' \exists u_I \in \mathbb{W}(X) \forall p \in L'^{-1}(u_I)\colon I \in p$. I claim that G' is irreducible. Let $I, J \in V'$ be arbitrary and find $u_I, u_J \in \mathbb{W}(X)$. Since X is irreducible we can find a $w \in \mathbb{W}(X)$ such that $u_I w u_J \in \mathbb{W}(X)$. Let p be a path which presents $u_I w u_J$. Then a subpath of p will be a path from I to J, by our choice of u_I and u_J. □

Definition 2.6.10 (Sensible labeled graph)**.** Let $G = (V, E, L)$ be a labeled graph. We call G sensible, if given $i, j \in V$ and $a \in \Sigma$, there is at most one edge labeled a from i to j.

Lemma 2.6.11. *1. Let G be a labeled graph. Then there exists a subgraph of G, which is sensible, and which presents the same shift as G.*

2. Let $G = (V, E, L)$ be a sensible labeled graph. Then G can be represented by the pair (V, E'), where $E' = \{(i, j, a) \in V^2 \times L(E) \mid \exists e \in E \colon i(e) = i, t(e) = j, L(e) = a\}$.

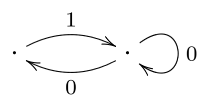
3. Let V and Σ be finite sets and $E \subseteq V^2 \times \Sigma$. Then (V, E) corresponds to a sensible labeled graph (V, E', L), defined by $E' = \{(i, j, (i, j, a)) \mid (i, j, a) \in E\}$ and $L((i, j, (i, j, a))) = a$.

Proof. 1.: Simply remove all but one of the edges with the same label between two vertices. 2.: Given (V, E') we can restore E and L (except that the name of the edges may be different), by $E = \{(i, j, (i, j, a)) \mid (i, j, a) \in E'\}$ and $L((i, j, (i, j, a))) = a$. 3.: Left to the reader. □

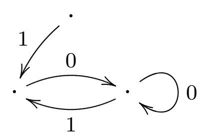
Thus sensible labeled graphs are simply graphs for which we have weakened the requirement on the names of the edges. Instead of them all being different, only edges with the same initial and terminal vertex have to have different names. 3. simplifies the notation quite a bit, when constructing labeled graphs, since it makes it possible to define both the edges and the label map in one step without finding unique names for the edges.

Definition 2.6.12 (Subset graph)**.** Let G be a labeled graph. We define a new labeled graph 2^G, called the subset graph of G, in the following way: Define V to be the non-empty subsets of V_G. Let for each $I \subseteq V_G$ and $a \in \Sigma$, I_a denote the set of vertices in V_G, which are reachable from I by edges labeled a. Define $E = \{(I, I_a, a) \mid I \in V, a \in \Sigma, I_a \neq \emptyset\}$. Then (V, E) corresponds to a labeled graph by Lemma 2.6.11.

Example 2.6.13. Let G be the following non-right-resolving graph:



Then 2^G looks like:



Which is right-resolving.

The reader easily verifies the following lemma.

Lemma 2.6.14. *Let G be a labeled graph. Then $\Sigma_{2^G} = \Sigma_G$ and 2^G is right-resolving.*

Definition 2.6.15 (Merged graph)**.** Let G be a labeled graph. We define a new labeled graph M_G, called the merged graph of G, in the following way: Define a relation $\sim$ on vertices of G by $i \sim j \Leftrightarrow F(i) = F(j)$. Then $\sim$ is clearly an equivalence relation. Define $V = V_G/\sim$ and $E = \{(I, J, a) \mid \exists i \in I, j \in J, e \in E_G \colon i(e) = i, t(e) = j, L(e) = a\}$. Then (V, E) corresponds to a labeled graph by Lemma 2.6.11.

Lemma 2.6.16. *Let G be a labeled graph. Then $\Sigma_{M_G} = \Sigma_G$ and M_G is follower-separated. Furthermore, if G is right-resolving, then so is M_G. And if G is irreducible, then so is M_G.*

Proof. Left to the reader. □

Proposition 2.6.17. *Let X be an irreducible sofic shift space. Then there exists an irreducible, right-resolving and follower-separated labeled graph, which presents X.*

Proof. Let X be an irreducible sofic shift space and G a labeled graph, which presents X. Let G' be the irreducible sub-graph of 2^G from Lemma 2.6.9. Then the merged graph $M_{G'}$ is irreducible, right-resolving and follower-separated by the preceding lemmas. □

By Proposition 2.6.17 and 2.6.7 the following is well-defined.

Definition 2.6.18 (Fischer Cover)**.** Let X be an irreducible sofic shift space. The irreducible, right-resolving and follower-separated labeled graph, which presents X is called the Fischer cover of X and is denoted by F_X. The label map of the Fischer cover is denoted by π.

One very useful property of the Fischer cover G of an irreducible sofic shift X is, that a word w is synchronizing for X if and only if all paths with label w terminate in the same vertex. Words with that property are called magic words.

Definition 2.6.19 (Magic)**.** Let $G = (V, E, L)$ be a right-resolving labeled graph. A word $w \in \mathbb{W}(\Sigma_G)$ is called magic for L if all paths presenting w have the same terminal vertex.

Magic words are obviously synchronizing. It is the converse that takes some work. In order to prove that synchronizing words in irreducible sofic shift spaces are magic for π, we need the following lemma:

Lemma 2.6.20. *Let $G = (V, E, L)$ be a right-resolving and follower-separated labeled graph and $w \in \mathbb{W}(\Sigma_G)$. Then w extends to the right to a word wu, which is magic for L.*

Proof. Let for all $v \in \mathbb{W}(\Sigma_G)$, $T(v)$ be the set of terminal vertices of paths in $L^{-1}(v) \subseteq \mathbb{W}(G)$.

If $T(w)$ contains only one vertex, w is already magic. If not, we can find $v_1, v_2 \in T(w)$ such that $F(t(v_1)) \nsubseteq F(t(v_2))$, since G is follower separated. Choose $u' \in F(t(v_1)) - F(t(v_2))$. Then the set $T(wu')$ contains less elements than $T(w)$, since G is right-resolving. So since $T(w)$ is finite, because it is a subset of V, we can repeat this process a finite number of times and obtain a word u such that $wu \in \mathbb{W}(\Sigma_G)$ and $T(L^{-1}(wu))$ contains exactly one element. □

Lemma 2.6.20 together with the subset graph and merged graph give us:

Lemma 2.6.21. *Let X be a sofic shift space. Then $S(X) \neq \emptyset$.*

Proposition 2.6.22. *Let X be an irreducible sofic shift and let $\pi\colon G \to X$ be the Fischer cover of X. Then a word is synchronizing if and only if it is magic for π.*

Proof. That magic words for π are synchronizing for X is trivial. In order to prove the converse, we take an arbitrary $s \in S(X)$ and prove that it is magic for π. Let $\pi^{-1}(s) = \{p_i\}$. If all p_i's terminate in the same vertex, we are done. If not, we can choose p_i and p_j such that $F(t(p_i)) \nsubseteq F(t(p_j))$, since G is follower separated. Choose $u \in F(t(p_i))$ such that $u \notin F(t(p_j))$. By Lemma 2.6.20 we can find a word w, such that suw is magic for π. Since G is irreducible we can find a $v \in \mathbb{W}(G)$ such that $\pi^{-1}(suw)vp_j \in \mathbb{W}(G)$. Then $\pi(\pi^{-1}(suw)vp_i) = suw\pi(v)s \in \mathbb{W}(X)$, since G is essential. So as $s \in S(X)$, $suw\pi(v)su \in \mathbb{W}(X)$, which implies that $u \in F(t(p_j))$, since G is right-resolving. But this contradicts our choice of u. Thus $t(p_i)$ is the same for all i, which means that s is magic for π. □

Proposition 2.6.22 implies that the following is well-defined.

Definition 2.6.23 (Terminal Vertex)**.** Let s be a synchronizing word in an irreducible sofic shift space X. We call the common terminal vertex of the paths in the Fischer cover, which present s, the terminal vertex of s.

The following is a convenient property of mixing sofic shifts, that we will use extensively:

Lemma 2.6.24. *A mixing sofic shift space has a transition length.*

Proof. Follows from the fact that the Fischer Cover contains only finitely many vertices. □

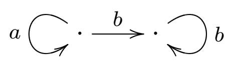
2.7 The Derived Shift, Irreducible Components and Affiliation

Definition 2.7.1 (Non-wandering)**.** Let X be a shift space. The non-wandering part of X is defined by $R(X) = \overline{\operatorname{Per} X}$.

$R(X)$ is a shift space, since it is both shift-invariant and closed. Note that $R(X) = X$, when X is irreducible.

When X is a sofic shift, there is a simple way of finding a labeled graph, which presents $R(X)$. Simply take a labeled graph, which presents X, and delete all edges for which there is no path from the terminal vertex to the initial vertex.

Example 2.7.2. Let X be the sofic shift presented by the following labeled graph:



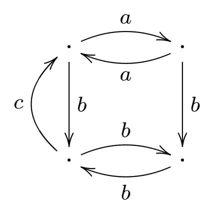
Then $R(X)$ is $\{a^\infty, b^\infty\}$.

Definition 2.7.3 (Derived shift)**.** Let X be a shift space. The derived shift is defined by:

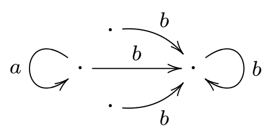
$$\partial X = R(X)_{S(R(X))}.$$

∂X is a shift space, since $\partial X = R(X) \cap \Sigma^{\mathbb{Z}}_{S(R(X))}$. It is in fact a sofic shift space, as shown by Klaus Thomsen in [T]. He also constructs an algorithm for finding a labeled graph, which presents ∂X: When X is irreducible sofic one simply takes the subset-graph of the Fischer cover of X and deletes all vertices, which correspond to subsets with anything but two elements, and all edges starting or ending in a deleted vertex. That works by Proposition 2.6.22.

Example 2.7.4. Let X be the irreducible sofic shift space presented by the following graph:



Then the algorithm described above produces the following graph:



Thus ∂X is the SFT $\{a, b\}^{\mathbb{Z}}_{\{ba\}}$, which is illustrated in Example 2.7.2.

Definition 2.7.5 ($\mathbf{S}(X)$)**.** Let X be a shift space and $\sim$ be the equivalence relation on $S(R(X))$ defined by $s \sim t \Leftrightarrow \exists x \in R(X) \colon s, t \subseteq x$. Define

$$\mathbf{S}(X) = S(R(X))/\sim .$$

Definition 2.7.6 (Irreducible Components)**.** Let X be a shift space and $\alpha \in \mathbf{S}(\partial^n X)$. Then $X_{(\alpha,n)}$ is the set of elements in $R(\partial^n X)$ for which

$$\sup_{i\in\mathbb{Z}}(\inf\{j \geq i \mid \exists \omega \in \alpha, \omega \subseteq x_{[i,j[}\} - i) < \infty.$$

With the convention that $\inf \emptyset = \infty$. The subsets $X_{(\alpha,n)}$, $\alpha \in \mathbf{S}(\partial^n X)$ are called the irreducible components at level n in X.

Definition 2.7.7 (Top Component)**.** When X is an irreducible sofic shift space, $\mathbf{S}(X) = S(X)/\sim$ contains exactly one element. We call the corresponding irreducible component the top component of X and denote it by X_0.

Example 2.7.8. Let X be the shift from Example 2.7.4. Then X has three irreducible components:

At level 0 there is only one irreducible component, namely the top component $X_0 = X_{([c],0)}$, which consists of the points in X for which there is an upper bound on the distance between occurrences of c.

At level 1, there are two irreducible components: $X_{([a],1)} = \{a^\infty\}$ and $X_{([b],1)} = \{b^\infty\}$, since $R(\partial X) = \{a^\infty, b^\infty\}$ and $a, b \in S(R(\partial X))$.

And since $\partial^2 X = \emptyset$, there are no more.

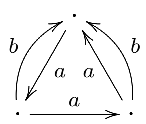
Definition 2.7.9 (Affiliation)**.** Let $d \in \mathbb{N}$, $F \subseteq \{0, 1, \ldots d-1\}$ and $X_{(\alpha,n)}$ be an irreducible component in X. A point $x \in \operatorname{Per} X$ is said to be (d, F)-affiliated to $X_{(\alpha,n)}$ if there exists words $u, v \in \alpha$ such that

$$u p_x^{kd+f} v \in \mathbb{W}(X),$$

for all $k \in \mathbb{N}, f \in F$. The set of points which are (d, F)-affiliated to $X_{(\alpha,n)}$ is denoted by $X_{(\alpha,n)}^{(d,F)}$. To simplify notation we define $X_{(\alpha,k)}^{(d,\emptyset)} = X_{(\alpha,k)}^{(d)}$.

I leave it to the reader to verify that this is well defined in the sense that the choice of p_x is irrelevant.

Example 2.7.10. Let X be the irreducible sofic shift space presented by the following graph:



Then $a \in X_0^{(3,\{1,2\})}$, since $b \in S(X)$ and $ba^{3k+1}b, ba^{3k+2}b \in \mathbb{W}(X)$ for all $k \in \mathbb{N}$.

This affiliation concept is a generalization of the one defined in §5 in [T]. Thomsen's definition corresponds to $F = \emptyset$. As we will see it behaves much like the old one under morphisms. The following is a generalization of Lemma 7.2 in [T].

Lemma 2.7.11. *Let X be an irreducible shift space and Y a shift space. Let $\varphi\colon X \to Y$ be a morphism, such that $\mathbb{W}(\varphi(X)) \cap S(Y) \neq \emptyset$. Then*

$$\varphi(Q_n(X_0^{(d,F)})) \subseteq \bigcup_{m|n} Q_m(Y_0^{\frac{n}{m}(d,F)}),$$

holds for all $n, d \in \mathbb{N}$ and $F \subseteq \{0, 1, \dots d-1\}$.

Proof. Let $\varphi\colon X \to Y$ be an N-block morphism, which satisfies $\mathbb{W}(\varphi(X)) \cap S(Y) \neq \emptyset$. Let $x \in Q_n(X_0^{(d,F)}))$ and let $y = \varphi(x)$. Let m denote the least period of y and fix an $i_x \in \mathbb{Z}$, such that $p_x = x_{[i_x, i_x+|p_x|[}$. Choose p_y such that $\varphi_N(x_{[i_x,i_x+|p_x|+N-1[}) = p_y^{\frac{n}{m}}$. Find $u, v \in S(X)$, such that $up_x^{kd+f}v \in \mathbb{W}(X)$, for all $k \in \mathbb{N}$ and $f \in F$. Since $\mathbb{W}(\varphi(X)) \cap S(Y) \neq \emptyset$, we can find an $s \in S(Y)$ and an $s_0 \in \mathbb{W}(X)$, such that $\varphi_N(s_0) = s$. Since X is irreducible we can find $u', v' \in W(X)$, such that $s_0u'u, vv's_0 \in \mathbb{W}(X)$. Then $s_0u'up_x^{kd+f}vv's_0 \in \mathbb{W}(X)$, for all $k \in \mathbb{N}$ and $f \in F$, since $u, v \in S(X)$. Thus $\varphi_N(s_0u'up_x^{kd+f}vv's_0) \in \mathbb{W}(Y)$, for all $k \in \mathbb{N}$ and $f \in F$. Define $a = \varphi_N(s_0u'ux_{[i_x,i_x+d|p_x|+N-1[})$ and $b = \varphi_N(p_x^{cd}vv's_0)$, where $c \in \mathbb{N}$ is chosen such that $cd|p_x| \geq N$. Then $a, b \in S(Y)$, since s occurs in both of them, and $\varphi_N(s_0u'up_x^{kd+f}vv's_0) = ap_y^{\frac{n}{m}((k-c-1)d+f)}b$, for all $k \geq c+1$ and $f \in F$. Thus $\varphi(x) = y \in \bigcup_{m|n} Q_m(Y_0^{\frac{n}{m}(d,F)})$. □

Pairs of shift spaces X, Y for which there is a morphism $X \to Y$, which hits a synchronizing word will come up again later. I therefore introduce the notation $X \rightharpoonup Y$ for them.

Definition 2.7.12. Let X and Y be shift spaces. We say that $X \rightharpoonup Y$ if there exists a morphism $\varphi\colon X \to Y$, such that $\mathbb{W}(\varphi(X)) \cap S(Y) \neq \emptyset$.

Similar to the definition of $\mathrm{Per\,X} \overset{(\mathrm{d})}{\twoheadrightarrow} \mathrm{Per\,Y}$ in definition 9.12 in [T] we define:

Definition 2.7.13. Let X and Y be irreducible sofic shift spaces and $A \subseteq X$. We say that $\operatorname{Per} A \overset{(d,F)}{\rightarrow} \operatorname{Per} Y$, when the following holds:

$$\bigcap_{(d,F)\in G} Q_n(X_0^{(d,F)} \cap A) \neq \emptyset \Rightarrow \bigcup_{m|n} \bigcap_{(d,F)\in G} Q_m(Y_0^{\frac{n}{m}(d,F)}) \neq \emptyset,$$

for all $G \subseteq \{(d,F) \in \mathbb{N} \times 2^{\mathbb{N}} \mid F \subseteq \{0, \dots, d-1\}\}$.

By Lemma 2.7.11 we obtain the following:

Corollary 2.7.14. *Let X be an irreducible shift space and Y a shift space. Then*

$$X \rightharpoonup Y \Rightarrow \mathrm{Per\,X} \overset{\mathrm{(d,F)}}{\to} \mathrm{Per\,Y}\,.$$

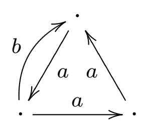
In particular

$$X \twoheadrightarrow Y \Rightarrow \mathrm{Per\,X} \overset{\mathrm{(d,F)}}{\twoheadrightarrow} \mathrm{Per\,Y},$$

when $S(Y) \neq \emptyset$.

Corollary 2.7.14 gives a stronger necessary condition than $\mathrm{Per\,X} \overset{\mathrm{(d)}}{\twoheadrightarrow} \mathrm{Per\,Y}$ from [T], as illustrated by the following example.

Example 2.7.15. Let X be the shift from Example 2.7.10 and Y be the shift presented by the following graph:



Then $\mathrm{Per\,X} \overset{\mathrm{(d)}}{\twoheadrightarrow} \mathrm{Per\,Y}$ holds, but $\mathrm{Per\,X} \overset{\mathrm{(d,F)}}{\twoheadrightarrow} \mathrm{Per\,Y}$ does not. Thus Corollary 2.7.14 implies that X does not factor onto Y, and Lemma 7.2 in [T] does not tell us anything.

The following technical lemma is a counterpart to Lemma 9.8 in [T]:

Lemma 2.7.16. *Let X be an irreducible sofic shift. There exists a number $R \in \mathbb{N}$ such that for any $x \in \mathrm{Per}\,X$ we have: $x \in Q_n(X_0^{(d,F)}) \Leftrightarrow$ there exists words $u', v' \in S(X)$ of length at most R, such that $u' p_x^{kd+f} v' \in \mathbb{W}(X)$, for all $k \in \mathbb{N}$ and $f \in F$.*

Lemma 2.7.16 follows easily from the following general result, which shows that there is a constant $R \in \mathbb{N}$ such that any synchronizing word s can be replaced by a synchronizing word of length at most R which glues together all words that s does.

Lemma 2.7.17. *Let X be an irreducible sofic shift space. Then there exists a number R, such that*

$$\forall w \in S(X) \exists w' \in S_{\leq R}(X) \forall a, b \in \mathbb{W}^*(X)\colon awb \in \mathbb{W}^*(X) \Rightarrow aw'b \in \mathbb{W}^*(X).$$

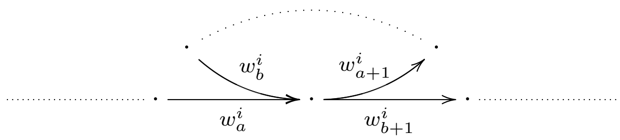
Proof. Let F_X be the Fischer cover of X, and let π be its label map. Fix some synchronizing word $s \in S(X)$, and a path $s_0 \in \pi^{-1}(s)$. Define $R = V^V + 2V + |s_0|$, where V is the number of vertices in F_X.

Let $w \in S(X)$ and set

$$\pi^{-1}(w) = \{w^1, \dots, w^N\}.$$

Then all the w^i's end at the same vertex v_0 by Proposition 2.6.22.

If $|w| > V^N$, then $\exists a < b \in \{0, \dots, V^N\} \forall i \in \{1, \dots, N\} \colon t(w^i_a) = t(w^i_b)$, by the pigeon hole principle.



So by defining a new set of paths $\{v^1, \dots, v^N\}$ by skipping $w^i_{a+1} \cdots w^i_b$ i.e.

$$v^i_j = \begin{cases} w^i_j & , j \leq a \\ w^i_{j+(b-a)} & , j > a \end{cases}$$

we achieve that $t(v^i) = v_0$ and $i(v^i) = i(w^i)$ for all i, and that all the $\pi(v^i)$'s are equal to some word $v \in \mathbb{W}(X)$ with the property $|v| \leq |w| - 1$. By repeating this process a finite number of times we obtain paths $v^1, \dots, v^N$, such that $\pi(v^i) = \pi(v^1)$, $|v^i| \leq V^N \leq V^V$, $t(v^i) = v_0$ and $i(v^i) = i(w^i)$, for all i. The word $\pi(v^1)$ might not be synchronizing, so to ensure that we get a word in $S(X)$, we use the irreducibility of F_X to find paths p_1 and p_2 of length less than V, such that $v^i p_1 s_0 p_2$ is a path in F_X with terminal vertex v_0, for all i. Define $w' = \pi(v^1)\pi(p_1 s_0 p_2)$. Then $|w'| \leq R$ and $w' \in S(X)$, since s occurs in w'.

In order to verify that w' satisfies the condition, we let $a, b \in \mathbb{W}^*(X)$ be arbitrary and assume that $awb \in \mathbb{W}^*(X)$.

Since $awb \in \mathbb{W}^*(X)$, there is an element p in $\pi^{-1}(awb)$. We can therefore find paths $a' \in \pi^{-1}(a)$, $b' \in \pi^{-1}(b)$, and an $i \in \{1, \cdots, N\}$, such that $p = a' w^i b'$. Thus, by definition of v^i, $a' v^i p_1 s_0 p_2 b'$ is a path in F_X, which implies that $aw'b = \pi(a' v^i p_1 s_0 p_2 b') \in \mathbb{W}(X)$. □

2.8 Entropy

Entropy is a general concept used in many different forms in different scientific areas. In symbolic dynamics it is a kind of measure of the size or complexity of a shift space. It measures the exponential growth rate of the number of blocks of a given length.

Definition 2.8.1 (Entropy)**.** Let X be a shift space. The entropy of X, $\mathrm{h}(X)$, is defined by

$$\mathrm{h}(X) = \lim_{n \to \infty} \tfrac{1}{n} \log |\mathbb{W}_n(X)|.$$

This limit will always exist. I will however not prove that here. See Proposition 4.1.8 in [LM]. When X is mixing sofic the entropy equals the growth rate of the number of periodic points:

Lemma 2.8.2. *Let X be a mixing sofic shift space. Then*

$$\mathrm{h}(X) = \lim_{n\to\infty} \tfrac{1}{n} \log |Q_n(X)|.$$

Proof. Corollary 4.5.13 in [LM]. □

The entropy of shift spaces behaves nicely under morphisms. It is in fact a conjugacy invariant by the following lemma:

Lemma 2.8.3. *Let X and Y be shift spaces and $\varphi\colon X \to Y$ be a morphism. If φ is a factor map, then $\mathrm{h}(X) \geq \mathrm{h}(Y)$.*
If φ is an embedding, then $\mathrm{h}(X) \leq \mathrm{h}(Y)$.
If φ is a conjugacy, then $\mathrm{h}(X) = \mathrm{h}(Y)$.

Proof. If φ is a factor map induced by an m-block map, then $|\mathbb{W}_n(Y)| \leq |\mathbb{W}_{n+m-1}(X)|$ for all $n \in \mathbb{N}$, since each word in $\mathbb{W}_n(Y)$ is the image of a word in $\mathbb{W}_{n+m-1}(X)$ under φ_m. Thus

$$\begin{aligned} h(Y) &= \lim_{n\to\infty} \tfrac{1}{n} \log |\mathbb{W}_n(Y)| \leq \lim_{n\to\infty} \tfrac{1}{n} \log |\mathbb{W}_{n+m-1}(X)| \\ &= \lim_{n\to\infty} \frac{n+m-1}{n} \frac{1}{n+m-1} \log |\mathbb{W}_{n+m-1}(S)| = h(S) \end{aligned}$$

This implies, that if φ is a conjugacy, then $\mathrm{h}(X) = \mathrm{h}(Y)$. So if φ is an embedding, then $h(X) = \mathrm{h}(\varphi(X)) \leq \mathrm{h}(Y)$, as it is a conjugacy on its image and $|\mathbb{W}_n(\varphi(X))| \leq |\mathbb{W}_n(Y)|$ for all $n \in \mathbb{N}$, since $\varphi(X) \subseteq Y$.[2] □

Thus $\mathrm{h}(X) \geq \mathrm{h}(Y)$ is necessary for a shift space X to factor another shift space Y. I focus on the problem of finding necessary and sufficient conditions when $\mathrm{h}(X) > \mathrm{h}(Y)$. Because then there is a subshift W of X which factors onto Y, by the following lemma. So if I can find a way of mapping the rest of X into Y in a continuous and shift-commuting way, then I have a factor map $X \twoheadrightarrow Y$. That turns out to be a sound strategy.

Lemma 2.8.4. *Let X and Y be sofic shift spaces. Y mixing and X irreducible such that $\mathrm{h}(X) > \mathrm{h}(Y)$. Then there exists an SFT, W, with the following properties:*

1. $W \subseteq X$
2. $W \twoheadrightarrow Y$.
3. $\exists D \in \mathbb{N} \colon \mathbb{W}_D(W) \subseteq S(X)$
4. $\exists TL \in \mathbb{N} \forall a, b \in \mathbb{W}(W) \exists x \in \mathbb{W}_{TL}(W) \colon axb \in \mathbb{W}(W)$

[2] Note that if $\varphi\colon X \to Y$ is an embedding, then it is not necessarily true that φ_n is injective, which means that $|\mathbb{W}_m(Y)| \geq |\mathbb{W}_{m+n-1}(X)|$ is not true in general.

Note that the third property is stronger than saying $W \in \mathrm{SFT}$ and that the fourth is weaker than saying that W is mixing, but stronger than saying that W is irreducible.

Proof. Since Y is mixing sofic, we can find a mixing SFT, Z, which factors onto Y and has the same entropy as Y. Let $p = \mathrm{period}(X_0)$. By Theorem 3.2 in [T] there is a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of irreducible SFTs inside X_0 such that $\cup_{i\in\mathbb{N}} A_i = X_0$, $\lim_{i\to\infty} \mathrm{h}(A_i) = \mathrm{h}(X)$ and for each $i \in \mathbb{N}$ there is an n_i such that $\mathbb{W}_{n_i}(A_i) \subseteq S(X)$. Choose $I \in \mathbb{N}$ such that $\mathrm{h}(A_I) > \mathrm{h}(Y) = \mathrm{h}(Z)$ and such that $\mathrm{period}(A_I) = p$.

Since A_I is an SFT with period p, we can find p closed and disjoint sets $\{C_i\}_{i=0}^{p-1}$ such that $A_I^p = \bigsqcup C_i$, $C_i = \sigma^i(C_0)$ for all $0 \le i < p$ and $C_0 = \sigma^p(C_0)$. Consider C_0 as shift space under the action of σ^p. Then C_0 is a mixing SFT by Lemma 3.6 in [T] and $\mathrm{h}(C_i) = \mathrm{h}(A_I^p) = p\,\mathrm{h}(A_I) > p\,\mathrm{h}(Z) = \mathrm{h}(Z^p)$.

By Lemma 2.8.2 there is an $N \in \mathbb{N}$ so large that $|Q_n(C_0)| > |Q_n(Z^p)|$ for all $n \ge N$. But as $n \mid |Q_n(X)|$ for any shift space X, this implies that $|Q_n(C_0)| \ge |Q_n(Z^p)| + n$ for all $n \ge N$. By using Boyle's Covering Lemma (Lemma 2.1 in [B]) repeatedly, we construct a mixing SFT W_0, with the properties: $W_0 \twoheadrightarrow Z^p$, $\mathrm{h}(W_0) = \mathrm{h}(Z^p)$ and $|Q_n(C_0)| \ge |Q_n(W_0)|$ for all $n \in \mathbb{N}$. By Krieger's embedding theorem, W_0 is a subshift of C_0, i.e. $W_0 \subseteq C_0 \subseteq A_I^p$.

The factor map $W_0 \twoheadrightarrow Y^p$ induces a factor map from the shift $W = \bigcup_{i=0}^{p-1} \sigma^i(W_0)$ onto Y. Note that W is a subshift of A_I, since $\sigma^i(W_0) \subseteq C_i$ for each i. Thus $D = n_I$ works in the third property. When TL_0 is a transition length for W_0, $TL = pTL_0$ has the desired property. $\square$

2.9 The Marker Lemma

Definition 2.9.1 (Periodic word)**.** Let $j, T \in \mathbb{N}$. A word $w \in \mathbb{W}^*(X)$ is called j-periodic, if $j \le |w|$ and w is a subword of $w_{[0,j[}^\infty$. We say that w is $<T$-periodic if it is j-periodic for some $j < T$.

The following result, due to Krieger, is traditionally called the marker lemma.

Lemma 2.9.2 (Marker Lemma)**.** *Let X be a shift space and $k, T \in \mathbb{N}$, such that $k > T > 1$. Then there exists a clopen set $F \subseteq X$, which satisfies:*

1. *The sets $\sigma^i(F)$, $0 \le i < T$, are disjoint.*
2. *If a point $x \in X$ satisfies*
$$\sigma^i(x) \notin \bigcup_{-T<j<T} \sigma^j(F),$$
for some $i \in \mathbb{Z}$, then $x_{[i-k,i+k]}$ is $<T$-periodic.

Proof. Let $k > T > 1$ and let $\{w_1, \ldots, w_L\}$ be the set of words in $\mathbb{W}_{2k+1}(X)$ which are not n-periodic for any $n < T$. Define $C_i = \{x \in X \mid x_{[-k,k]} = w_i\}$ for $1 \le i \le L$. Then each C_i is clopen and the sets $\sigma^j(C_i)$, $0 \le j < T$, are disjoint since each w_i is not $<T$-periodic. Define $F_1 = C_1$ and inductively

$$F_{i+1} = F_i \cup \Big(C_{i+1} - \bigcup_{-T<j<T} \sigma^j(F_i)\Big).$$

Then each F_i is clopen, since σ is a homeomorphism and each of the C_i's are clopen. A simple induction shows that each F_i satisfies 1. Let $F = F_L$. I claim that F also satisfies 2. Let $x \in X$ and $i \in \mathbb{Z}$, such that $x_{[i-k,i+k]}$ is not n-periodic for any $n < T$. Then we can find an $1 \le m \le L$ such that $\sigma^i(x) \in C_m$. If $\sigma^i(x) \notin \bigcup_{-T<j<T} \sigma^j(F_{m-1})$, then $\sigma^i(x) \in F_m \subseteq F$ by definition of the F_i's. And if $\sigma^i(x) \in \bigcup_{-T<j<T} \sigma^j(F_{m-1})$, then $\sigma^i(x) \in \bigcup_{-T<j<T} \sigma^j(F)$ because the F_i's are increasing. So either way $\sigma^i(x) \in \bigcup_{-T<j<T} \sigma^j(F)$, which establishes the claim. □

Definition 2.9.3 (Marker, Marker interval)**.** Let X be a shift space, $x \in X$ and $k > T > 1$. Let F be the set from the marker lemma. Then the elements in the set $M_{k,T}(x) = \{i \in \mathbb{Z} \mid \sigma^i(x) \in F\}$ are called the (k,T)-markers in x. An interval $[i, j[$ is called a (k,T)-marker interval in x if $[i, j] \cap M_{k,T}(x) = \{i, j\}$. The set of all (k,T)-marker intervals in x is denoted by $I_{k,T}(x)$.

Remark 2.9.4. The marker intervals in a point $x \in X$ are disjoint and the union of all the marker intervals is $\mathbb{Z}$. Thus each $i \in \mathbb{Z}$ is contained in exactly one marker interval in x. Hence if $I_{\mathbb{Z}}$ denotes the set of intervals in $\mathbb{Z}$, then $I_{k,T}$ can be seen as map $X \to I_{\mathbb{Z}}$, which splits points in X into intervals.

Lemma 2.9.5. *Let X be a shift space, $k \ge T \in \mathbb{N}$ and $w \in \mathbb{W}^*(X)$. If each subword of w of length $2k+1$ is $<T$-periodic, then w is $<T$-periodic.*

Proof. Lemma 2.3 in [B]. □

Lemma 2.9.6. *Let X be an irreducible sofic shift space and $T \in \mathbb{N}$. Then there exists a $k > T$, such that if a word $x \in \mathbb{W}_{\ge k}(X)$ is j-periodic, for some $j < T$, then $x^{\infty}_{[0,j[} \in X$.*

Proof. Let $G = (V, E, \pi)$ be the Fischer cover of X. Define $k = (|V|+1)T$. Let $x \in \mathbb{W}_{\ge k}(X)$ be a j-periodic word for some $j < T$. Choose a $p \in \pi^{-1}(x)$. Then $p_n := p_{[0,jn[}$ presents $x^n_{[0,j[}$ for each $n \le |V|+1$. The pigeon hole principle gives us two numbers $n_1 < n_2 \le |V| + 1$, such that $t(p_{n_1}) = t(p_{n_2})$, which means that the path $p_{[jn_1,jn_2[}$ has the same initial and terminal vertex. Thus $p^{\infty}_{[jn_1,jn_2[}$ is a path in G, which implies that $\pi(p^{\infty}_{[jn_1,jn_2[}) = x^{\infty}_{[0,j[} \in X$. □

The marker lemma and the two preceding lemmas imply:

Lemma 2.9.7. *Let X be an irreducible sofic shift space, $x \in X$ and $T > 1$. Then there exists a $K > T$, such that for any $k \geq K$, all (k,T)-marker intervals, $[i, j[$, in x satisfy:*

1. $j - i \geq T$.
2. $j - i > 2T \Rightarrow \exists z \in \mathrm{Per}_{<T}(X) \colon x_{[i+T-k, j-T+k[} \subseteq z$.

Lemma 2.9.8. *Let X be an irreducible sofic shift space and $k > T > 1$. Then there exists a number $|F|$, such that we for each $i \in \mathbb{N}$ can determine whether $i \in M(x)$ by looking at $x_{[i-|F|, i+|F|]}$.*

Proof. Let F be the clopen set from the marker lemma. Since F is open we can for each $f \in F$ find a $k_f \in \mathbb{N}$, such that the set $C_{k_f}(f) = \{w \in \Sigma^{\mathbb{Z}} \mid w_{[-k_f, k_f]} = f_{[-k_f, k_f]}\}$ is a subset of F. And since $F \subseteq \Sigma^{\mathbb{Z}}$ is closed, it is compact. We can therefore find a finite set $\{f_1, \ldots, f_n\} \subseteq F$ such that $F = \bigcup C_{k_{f_i}}(f_i)$. Define $|F| = \max\{k_{f_i}\}$. □

2.10 The Marker Strategy

The marker lemma is very useful when constructing morphisms $\varphi \colon X \to Y$. In this section I will first describe one way of doing this, and illustrate it by proving a simple result. Afterwards I generalize the strategy to make it useful in more cases.

When constructing a morphism $X \to Y$, it is not very practical to start by constructing a block map $\mathbb{W}_n(X) \to \Sigma_Y$. Because it is difficult to check whether the resulting sliding block code actually maps all points in X to points in Y. Therefore we don't construct sliding block codes one symbol at a time, but in stead on words. As we will see this makes the verification a lot easier, and the resulting map is still a sliding block code. It also makes the definition of the morphism a lot simpler, as there are fewer cases to consider.

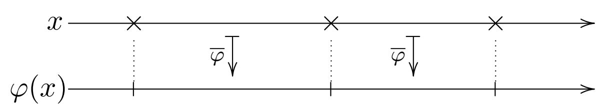
In the simplest applications, one defines φ on each word in $\mathbb{W}^*(X)$, which corresponds to a marker interval.

Let $k > T > 1$ and let $\mathbb{W}_{k,T}$ denote the set of words in X corresponding to the (k, T)-marker intervals. i.e. $\mathbb{W}_{k,T} = \bigcup_{x \in X}\{x_{[i,j[} | [i, j[\in I_{k,T}(x)\}$. The elements in $\mathbb{W}_{k,T}$ are called marker words.

The basic idea is to construct a length preserving map, $\overline{\varphi} \colon \mathbb{W}_{k,T} \to \mathbb{W}^*(Y)$, such that the map

$$\varphi \colon x \mapsto \ldots \overline{\varphi}(x_{[i_{-1}, i_0[})\overline{\varphi}(x_{[i_0, i_0[})\overline{\varphi}(x_{[i_0, i_1[}) \ldots,$$

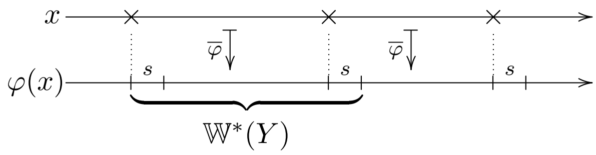
where $\{[i_k, i_{k+1}[\}_{k \in \mathbb{N}} = I_{k,T}(x)$, is a morphism from X to Y. The idea is illustrated by the following, where an $\times$ corresponds to a marker:



Since $\overline{\varphi}$ preserves the length of words, it makes sense to use the same indices for the image of a word as for the word itself. i.e. $\overline{\varphi}(x_{[i,j[})$ is thought of as a word $y_{[i,j[}$. With that convention the definition of φ becomes notationally simpler:

$$\varphi(x)_i = \overline{\varphi}(x_{[i_k,i_{k+1}[})_i, \text{ where } i \in [i_k, i_{k+1}[.$$

To ensure that φ is a morphism, we need to make sure that the image words glue together to form a point in Y and that $\overline{\varphi}$ acts as a block map. The gluing is done by inserting a synchronizing word s at the beginning of each image word w and making sure that ws is a word in Y, as illustrated by the following figure:



Formally $\overline{\varphi}$ is constructed such that it satisfies the following for some $s \in S(Y)$ and all $w \in \mathbb{W}_{k,T}$:

1. $\overline{\varphi}(w)_{[0,|s|[} = s$. i.e. the image of a word begins with s.

2. $\overline{\varphi}(w)s \in \mathbb{W}^*(Y)$.

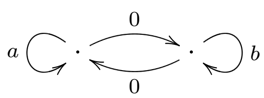
3. $|\overline{\varphi}(w)| = |w|$. i.e. it preserves the length of words.

4. $\exists K \in \mathbb{N}$ such that $\overline{\varphi}(w)_k$ depends only on $w_{[k-K,k+K]\cap[0,|w|[}$.

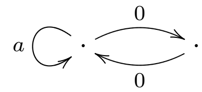
Because then φ maps points in X to points in Y, thanks to 1. and 2. And by 3., 4. and Lemma 2.9.8 it is a (K+|F|,K+|F|)-sliding block code, and thus a morphism.

In most cases the 'short' marker words, i.e. the marker words of length less than or equal to kT, for some fixed $k \geq 2$, are easy to handle. The difficulty is to make the map satisfy 4. on the long intervals. We know by the marker lemma, that the long marker intervals correspond to periodic words. The problem is that there is generally no way of saying anything interesting about the amount of periods in a periodic block, by looking at it locally. This makes 1. and 2. difficult when we want to satisfy 4. as illustrated by the following example:

Example 2.10.1. Let X be the sofic shift:



And Y be the sofic shift:



Then it is easy to see that for any choice of k and T, 0^n is a marker word in X, for all large enough n's. I claim that if we want to use the strategy above to define a non-trivial morphism from X to Y, which maps 0^∞ to 0^∞, then we need to find synchronizing words $s_1, s_2 \in S(Y)$, such that $s_1 0^n s_2$ is a word in Y for all large enough n's, which is clearly impossible, since Y allows only even powers of 0.

The requirement that 0^∞ has to map to 0^∞ is used to simplify the graphs involved in order to make it easier for the reader to see the point of the example. It is easy to construct a similar example, where 0^∞ would be forced to map to itself by lemma 2.3.3; one simply replaces the a and b loops by loops of length 3. But such an example involves larger graphs and morphisms with longer block length.

To establich the claim, note that the fourth condition implies that the middle part of $\overline{\varphi}(0^n)$ has to be of the form 0^m, i.e. $\overline{\varphi}(0^n)_{[K,n-K]} = 0^{n-2K}$. Thus condition 1. and 2. imply that we need to find words u and v, such that $su0^{n-2K}vs$ is a word in Y for all large n's. Note that u and v cannot depend on n, by property 4. By defining $s_1 = su$ and $s_2 = vs$, we have established the claim.

Thus in this example it is impossible to define a map $\overline{\varphi}$, which satisfies the four conditions above, even though there is a morphism from X to Y, which maps 0^∞ to itself. Y is in fact a factor of X via the morphism induced by the following block map:

$$\begin{array}{ll} a? \mapsto a & 0b \mapsto a \\ 00 \mapsto 0 & bb \mapsto a \\ 0a \mapsto 0 & b0 \mapsto 0 \end{array}$$

where ? can be any symbol.

One way of avoiding complications like the one in the preceding example is to make some assumption on the affiliation of the periodic words in X or Y. If we for instance know that for each periodic point $z \in Q_n(X)$ there is a periodic point $z' \in \bigcup_{m|n} Q_m(Y)$, which is 1-affiliated (or $\frac{n}{m}$-affiliated) to the top component of Y, then there is a morphism from X to Y by the following result:

Proposition 2.10.2. *Let X and Y be sofic shift spaces with Y mixing. If $\forall n \in \mathbb{N}\colon Q_n(X) \neq \emptyset \Rightarrow \bigcup_{m|n} Q_m(Y_0^{(\frac{n}{m})}) \neq \emptyset$, then there exists a morphism $\varphi\colon X \to Y$.*

Proof. As with most proofs of similar results the relatively simple idea drowns in a bunch of preliminary choices of constants and functions. The idea is to use the assumption on the periodic points to define a map, λ, from the periodic points in X to the periodic points in Y which pairs a point $x \in Q_n(X)$ with a point $y \in \bigcup_{m|n} Q_m(Y_0^{(\frac{n}{m})})$. This map is used to define $\overline{\varphi}$ on the middle part of words corresponding to long marker intervals. And to connect these words with s at both ends, we use the definition of affiliation and that Y is mixing. In symbols:

$$\overline{\varphi}(x_{[i,j[}) = su'u\lambda(x_{[i_0,j_0[})vv',$$

where u and v are the synchronizing words from the definition of affiliation and u' and v' are found using that Y is mixing. The precise definition of the involved variables can be found below. To define $\overline{\varphi}$ on the remaining marker words, we simply use that Y is mixing to find words a_n of length $n \geq \mathrm{TL}(Y)$, such that $sa_ns \in \mathbb{W}(Y)$ and map $x_{[i,j[}$ to sa_n for a suitable value of n.

Now for the details:

For the reader's convenience I have marked the definition of each variable in the margin. Choose $s \in S(Y)$ arbitrarily. Let R be the number from Lemma 2.7.16 corresponding to Y. Define $T = 2\,\mathrm{TL}(Y) + 2R + |s|$ and choose $k > \max\{T, K\}$, where K is the number from Lemma 2.9.7 corresponding to Y. Choose for each $n \geq \mathrm{TL}(Y)$ a word $a_n \in \mathbb{W}_n(Y)$, such that $sa_ns \in \mathbb{W}(Y)$. Using the assumption on the periodic points, we can define a shift commuting map λ: $\operatorname{Per} X \to \operatorname{Per} Y$, such that $z \in Q_n(X) \Rightarrow \lambda(z) \in \bigcup_{m|n} Q_m(Y_0^{(\frac{n}{m})})$. Choose for each $z \in \operatorname{Per} X$ a p_z, such that $p_z = p_{z'}$ when $z = \sigma(z')$, and an $i \in \mathbb{Z}$ such that $z_{[i,i+|p_z|[} = p_z$. Find using Lemma 2.9.7 a pair of words $u, v \in S_R(Y)$ such that $u\lambda(z)^k_{[i,i+|p_z|[}v \in \mathbb{W}(Y)$ for all $k \in \mathbb{N}$. Find for each $k \geq \mathrm{TL}(Y)$ and each u and v, $u', v' \in \mathbb{W}_k(Y)$, such that $su'u, vv's \in \mathbb{W}(Y)$.

Margin: s, R — T — K, k — a_n — λ — p_z — u, v — u', v'

Let for each $x \in X$ and each $[i, j[\in I_{k,T}(x)$, which is longer than $2T$, $z \in \operatorname{Per}_{<T}(X)$ be a point, such that $x_{[i,j[} = z_{[i,j[}$. z

Define for each $x_{[i,j[} \in \mathbb{W}_{k,T}$

$$\overline{\varphi}(x_{[i,j[}) = \begin{cases} sa_{j-i-|s|} & \text{, if } j - i \leq 3T \\ su'u\lambda(z)_{[i_0,j_0[}vv' & \text{, if } j - i > 3T \end{cases}$$

Where $i_0 = \min\{k \geq i + |s| + \mathrm{TL}(Y) + R | x_{[k,k+p_z[} = p_z\}$ and $j_0 = \max\{k \leq j - \mathrm{TL}(Y) - R \mid x_{[k-|p_z|,k[} = p_z\}$. This is well-defined because $i_0 \leq j_0$, since $|[i + |s| + \mathrm{TL}(Y) + R, j - \mathrm{TL}(Y) - R]| \geq 2T > 2|p_z|$, which implies that $p_z \subseteq [i + |s| + \mathrm{TL}(Y) + R, j - \mathrm{TL}(Y) - R]$.

Now $\overline{\varphi}$ clearly satisfies 1. and 2. and by choosing u' and v' to be of the correct length, (such that $|su'u| = |[i, i_0[|$ and $|vv'| = |[j_0, j[|$) 3. is also satisfied.

I claim that $K = 3T + 1$ works in 4.

To establish the claim, I let $x \in X$ and $k \in \mathbb{Z}$ be arbitrary and $[i, j]$ be the marker interval in x such that $k \in [i, j]$. Define $[a, b] = [k - K, k + K] \cap [i, j]$. I need to show that $\overline{\varphi}(x_{[i,j[})_k$ can be determined using only $x_{[a,b]}$.

If $b-a \leq 3T$, then $a=i$ and $b=j$ and $\overline{\varphi}(x_{[i,j[})_k$ is the symbol with index k in $sa_{b-a-|s|}$, seen as a word $w_{[a,b[}$.

If $b-a > 3T$ Lemma 2.9.5 implies that we can find the $z \in \mathrm{Per}_{<T}(X)$ such that $x_{[a,b[} = z_{[a,b[}$, by looking only at $x_{[a,b[}$. The argument splits into three cases:

1. $k \in [a, a+T[$. Then $a=i$, since $K > T$. By looking at $x_{[a,b[}$ we can find i_0 as in the definition of $\overline{\varphi}$ and hence determine u' and u. So as we also know $\lambda(z)$, $\overline{\varphi}(x_{[i,j[})_k$ is simply the symbol with index k in the word $su'u\lambda(z)_{[i_0,\infty[}$ seen as a word $w_{[a,\infty[}$.

2. $k \in [b-T, b[$. Then $b=j$, since $K > T$. By looking at $x_{[a,b[}$ we can find j_0 as in the definition of $\overline{\varphi}$ and hence determine v and v'. So as we also know $\lambda(z)$, $\overline{\varphi}(x_{[i,j[})_k$ is simply the symbol with index k in the word $\lambda(z)_{]-\infty,j_0,[}vv'$ seen as a word $w_{]-\infty,b]}$.

3. $k \in [a+T, b-T[$. $\overline{\varphi}(x_{[i,j[})_k = \lambda(z)_k$. □

By Proposition 2.10.2 we see that what hinders us from using the marker lemma directly to define a morphism $X \to Y$ is the set of periodic points

$$E_{X,Y} = \{x \in \mathrm{Per}\, X \mid \bigcup_{m||p_x|} Q_m(Y_0^{(\frac{|p_x|}{m})}) = \emptyset\}.$$

So we need a way of handling those. In the first chapter we saw how Boyle and Thomsen did that[3], and in the next chapters we will see a new way of handling them.

When dealing with more complicated problems, as for example the one in Example 2.10.1, it is not possible in general to construct a map $\overline{\varphi}\colon \mathbb{W}_{k,T} \to \mathbb{W}^*(Y)$, which satisfies the four properties on page 35. One way of getting around that is to come up with a different way of splitting points into intervals, with properties similar to those of the marker intervals. It turns out that it is convenient to define different types of intervals, similar to the short and long marker intervals, and then define $\overline{\varphi}$ on each type of interval like in the proof of Proposition 2.10.2. Thus $\overline{\varphi}$ depends on the type of the interval, which means that we have to be able to tell the type of the interval by local inspection, since $\overline{\varphi}$ has to work like a sliding block code. i.e. the intervals have to be locally recognizable in the following sense, where $I_{\mathbb{Z}}$ denotes the intervals in $\mathbb{Z}$, i.e. the set $\{[i,j] \mid i,j \in \mathbb{Z}\}$:

Definition 2.10.3 (Locally recognizable)**.** Let X be a shift space. A map $I\colon X \to I_{\mathbb{Z}}$ is called locally recognizable if for each $x \in X$ the intervals in $I(x)$ are disjoint and there is an $N \in \mathbb{N}$ such that it by inspection of $x_{[k-N,k+N]}$ for any $x \in X$ and $k \in \mathbb{Z}$ is possible to decide whether k occurs in an interval in $I(x)$ and if so to determine whether k is an endpoint of the interval.

[3]Boyle by assuming that $E_{X,Y} = \emptyset$ and Thomsen by assuming that all points in $E_{X,Y}$ have marked entries and exits

Remark 2.10.4. The last requirement may seem superfluous because if it is possible to tell whether a $k \in \mathbb{N}$ is in one of the intervals in $I(x)$ by looking at $x_{[k-N,k+N]}$, then it should be possible to tell whether k is an endpoint of the interval by looking at $x_{[k-N-1,k+N+1]}$; one would simply check whether $k-1$ or $k+1$ did not occur in an interval from $I(x)$.

That strategy does not work, however. Because it may be the case that two intervals in $I(x)$ lie right next to each other without space between them. Think of the marker intervals. And we need to be able to find the endpoints in order for the strategy to work.

Definition 2.10.5 (Splitting Map)**.** Let X be a shift space, τ be a finite set and let $\{I_t\}_{t\in\tau}$ be a family of maps $X \to I_{\mathbb{Z}}$. Then the map $I_\tau\colon X \to I_{\mathbb{Z}} \times \tau$ defined by

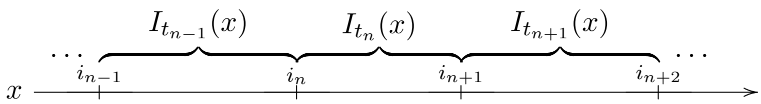
$$I_\tau(x) = \bigsqcup_{t\in\tau} I_t(x) = \bigcup_{t\in\tau} I_t(x) \times t$$

is called a splitting map if each I_t is locally recognizable and for any $x \in X$ the intervals in $\bigcup_{t\in\tau} I_t(x)$ are disjoint and their union is $\mathbb{Z}$.

Note that when I_τ is a splitting map and $x \in X$ then $I_\tau(x)$ can be arranged into a sequence

$$\{([i_n, i_{n+1}[, t_n)\}_{n\in\mathbb{Z}},$$

such that $i_n < i_{n+1}$ for all $n \in \mathbb{Z}$ and $\cup_{n\in\mathbb{Z}}[i_n, i_{n+1}[= \mathbb{Z}$.



And there is an $N \in \mathbb{N}$ such that it is possible for each $x \in X$ and $k \in \mathbb{Z}$ to tell the type of the interval in which k occurs and to decide whether $k = i_n$ for some $n \in \mathbb{Z}$, by looking at $x_{[k-N,k+N]}$.

The i_n's are called the endpoints in x.

The endpoints correspond to markers and N to the $|F|$ of lemma 2.9.8. Note that $I_{k,T}$ can be seen as a splitting map I_τ in two different ways:

1. There is only one type; $\tau = \{marker\}$ and $I_{k,T}\colon X \to I_{\mathbb{Z}} \times \{marker\}$.
2. $\tau = \{short, long\}$ and $I_{k,T}\colon X \to I_{\mathbb{Z}} \times \{short, long\}$.

since both the short and long marker intervals are locally recognizable and the union of the marker intervals in a point is all of $\mathbb{Z}$.

Definition 2.10.6 (t and τ intervals and words)**.** Let X be a shift space, τ be a finite set, $x \in X$, and let for each $t \in \tau$, I_t be a map $X \to I_{\mathbb{Z}}$. Then $I_t(x)$ is called the t intervals in x and the words corresponding to t intervals

$$\mathbb{W}_t = \bigcup_{x\in X} \{x_{[i,j]} \mid [i,j] \in I_t(x)\}$$

are called the t words.

Let I_τ be a map $X \to I_{\mathbb{Z}} \times \tau$. Then $I_\tau(x)$ is called the τ intervals in x and

$$\mathbb{W}_\tau = \bigsqcup_{t\in\tau} \mathbb{W}_t \subseteq \mathbb{W}^*(X) \times \tau$$

are called the τ words.

Example 2.10.7. This example is a continuation of Example 2.10.1. So X is the shift:

$$\begin{tikzcd} \cdot \arrow[loop left, "a"] \arrow[r, bend left, "0"] & \cdot \arrow[l, bend left, "0"] \arrow[loop right, "b"] \end{tikzcd}$$

I define two types of intervals:

Type '0': The intervals corresponding to words of the form 0^n, which are maximal in the sense that both the preceding and following symbol are not 0.

Type 'ab': The intervals corresponding to words in which 0 never occur and for which both the previous and next symbol are 0.

In other words for each $x \in X$:

$$\begin{aligned} I_0(x) &= \{[i,j] \mid x_{i-1}, x_{j+1} \neq 0, \forall n \in [i,j]\colon x_n = 0\} \text{ and} \\ I_{ab}(x) &= \{[i,j] \mid x_{i-1}, x_{j+1} = 0, \forall n \in [i,j]\colon x_n \neq 0\}. \end{aligned}$$

Then both I_0 and I_{ab} are locally recognizable: Clearly any two intervals in $I_0(x)$ or $I_{ab}(x)$ are disjoint. And given $x \in X$ and $n \in \mathbb{N}$ it is possible to tell the type of the interval in which n occurs by simply checking whether the symbol is 0 or not, and one can tell whether it is an endpoint by looking at the preceding and following symbol. Thus $N = 1$ works for both.

$I_{\{0,ab\}}\colon X \to I_{\mathbb{Z}} \times \{0, ab\}$ is therefore a splitting map, since the intervals in $I_0(x) \cup I_{ab}(x)$ are disjoint and their union is $\mathbb{Z}$.

The 0, ab and $\{0, ab\}$ words are given by:

$$\begin{aligned} \mathbb{W}_0 &= \{0^n \mid n \in \mathbb{N}_\infty\}, \\ \mathbb{W}_{ab} &= \{c^n \mid c \in \{a,b\}, n \in \mathbb{N}_\infty\} \text{ and} \\ \mathbb{W}_{\{0,ab\}} &= \bigcup_{n\in\mathbb{N}_\infty} \{(0^n, 0), (a^n, ab), (b^n, ab)\}. \end{aligned}$$

Since a splitting map I_τ has enough of the properties of the marker intervals I could show that if a map $\overline{\varphi}\colon \mathbb{W}_\tau \to \mathbb{W}^*(Y)$ satisfies the four properties on page 35, then the induced map φ is a morphism $X \to Y$. But it turns out that it is not only the restriction to marker intervals that keeps us from constructing morphisms; the requirement in property 4. is also too restrictive. Sometimes more information than just $(x_{[i,j]}, t)$ is needed to be able to define $\overline{\varphi}(x_{[i,j]}, t)$. I therefore introduce the notation:

$$\mathbb{W}_{M,\tau} = \bigcup_{x\in X} \{(x_{[i-M,j+M]}, t) \mid (x_{[i,j]}, t) \in \mathbb{W}_\tau\}.$$

Example 2.10.8. In the setup of Example 2.10.7 τ is the set $\{0, ab\}$ and $\mathbb{W}_{1,\tau}$ is the set

$$\bigcup_{n\in\mathbb{N}_\infty} \{(a0^{2n}a, 0), (b0^{2n}b, 0), (a0^{2n+1}b, 0), (b0^{2n+1}a, 0), (0a^n0, ab), (0b^n0, ab)\}.$$

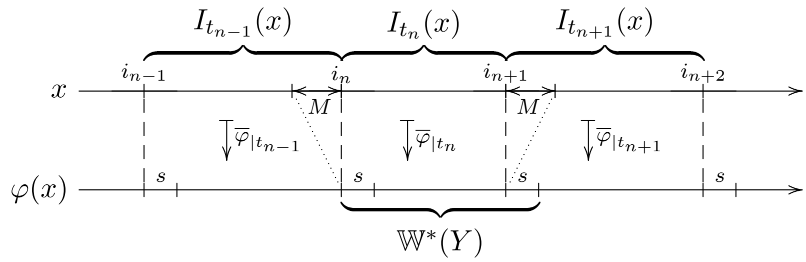
$\overline{\varphi}\colon \mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$ is then defined in such a way that it satisfies the following four properties, for some $M \in \mathbb{N}$, $s \in S(Y)$ and all $(x_{[i-M,j+M]}, t) \in \mathbb{W}_{M,\tau}$:

1. $\overline{\varphi}(x_{[i-M,j+M]}, t)_{[i,i+|s|[} = s$.
2. $\overline{\varphi}(x_{[i-M,j+M]}, t)s \in \mathbb{W}^*(Y)$.
3. $|\overline{\varphi}(x_{[i-M,j+M]}, t)| = j - i$.
4. There is a $K \in \mathbb{N}$ such that $\overline{\varphi}(x_{[i-M,j+M]}, t)_k$ depends only on t and $x_{[k-K,k+K]\cap[i-M,j+M]}$.

Where I use the convention that the symbols in $\overline{\varphi}(x_{[i-M,j+M]})$ have indices in $[i, j]$.

When I_τ is a splitting map and $\overline{\varphi}$ satisfies those four properties, I define a map $\varphi\colon X \to \Sigma_Y^{\mathbb{Z}}$ by $\varphi(x)_{[i_n,i_{n+1}[} = \overline{\varphi}(x_{[i_n-M,i_{n+1}+M[}, t_n)$, for all $n \in \mathbb{Z}$, where $\{([i_n, i_{n+1}[, t_n)\}_{n\in\mathbb{Z}} = I(x)$.

The definition of φ is illustrated in the following figure:



φ is a morphism $X \to Y$ by the following lemma:

Lemma 2.10.9. *Let X and Y be shift spaces, $I_\tau\colon X \to I_{\mathbb{Z}} \times \tau$ be a splitting map for some finite set τ and $\overline{\varphi}\colon \mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$ satisfy the following four properties for some $M \in \mathbb{N}$, $s \in S(Y)$ and all $(w, t) \in \mathbb{W}_{M,\tau}$:*

1. $\overline{\varphi}(w, t)_{[0,|s|[} = s$.
2. $\overline{\varphi}(w, t)s \in \mathbb{W}^*(Y)$.
3. $|\overline{\varphi}(w, t)| = |w| - 2M$.
4. *There is a $K \in \mathbb{N}$ such that $\overline{\varphi}(w, t)_k$ depends only on t and $w_{[k-K,k+K]\cap[0,|w|[}$.*

Then φ is a morphism from X to Y.

Proof. Note first that φ maps points in X to points in Y because of property 1.-3. I will prove that φ is a morphism by showing that it is induced by a block map. To do that I will show, that there is a $B \in \mathbb{N}$, such that the value of $\varphi(x)_i$ depends only on $x_{[i-B,i+B]}$, for each $x \in X$ and $i \in \mathbb{Z}$.

Let for each $t \in \tau$, N_t be a number that works in the definition of locally recognizability of I_t. Define $N = \max\{N_t\}_{t\in\tau}$.

I claim that $B = K + N$ works, where K is the one from property 4.

To see that I let $x \in X$, $i \in \mathbb{Z}$ and $k \in \mathbb{N}$ be the number, such that $i \in [i_k, i_{k+1}[$. By inspection of $x_{[i-K-N,i+K+N]}$ I can determine if $[i-K, i+K]$ contains any endpoints, and thereby determine $[i-K, i+K] \cap [i_k-M, i_{k+1}+M[$ including its type, perhaps without knowing neither i_k nor i_{k+1}. Thus B works by the fourth property. □

Example 2.10.10. In the setup of Example 2.10.8 I define $\overline{\varphi}\colon \mathbb{W}_{1,\tau} \to \mathbb{W}^*(Y)$ by:

$$\overline{\varphi}(w,t) = \begin{cases} a0^{n-2}a, & w = a0^n a,\ t = 0, \\ a0^{n-3}aa, & w = a0^n b,\ t = 0, \\ aa0^{n-3}a, & w = b0^n a,\ t = 0, \\ aa0^{n-4}aa, & w = b0^n b,\ t = 0, \\ a^{|w|-2}, & t = ab. \end{cases}$$

Then $\overline{\varphi}$ satisfies the four properties above with $s = a$ and $K = 2$. Thus the map φ induced by $\overline{\varphi}$ is a morphism from X to y by Lemma 2.10.9.

Hence this more general strategy allows us to handle the problem in Example 2.10.1, which could not be solved with the basic marker strategy. The point is that the more general approach lets us see the beginning and end of a periodic word (though not at the same time). I am going to use this idea of using information about these 'edges' of certain words extensively in my results.

Lemma 2.10.9 shows that the strategy, of splitting points in X into intervals of different types and inserting synchronizing words at each transition between intervals, is sound in the sense that it is sufficient to guarantee that the resulting map is a morphism. An interesting consequence of my results in the following chapters is that the strategy is also necessary in the sense that if there is a morphism from X to Y, then there is a non-trivial locally recognizable way of splitting points in X and a way of inserting synchronizing words as described above.

2.10.1 Extension Strategy

In the following chapters I will often need to construct a morphism $\varphi\colon X \to Y$ which extends some other morphism $\varphi_S\colon S \to Y$ for some subshift $S \subseteq X$.

That is easy to do using Lemma 2.10.9:

Define a locally recognizable map $I_S \colon X \to I_{\mathbb{Z}}$ which maps points in S to $\mathbb{Z}$ and then define $\overline{\varphi}$ such that it behaves like φ_S on the words in $\mathbb{W}_S$. Because then if x is a point in S, then x is in $\mathbb{W}_S$, which implies that $\varphi(x) = \overline{\varphi}(x) = \varphi_S(x)$. Thus φ extends φ_S.

Chapter 3

The Marked Property

The goal of this chapter is to investigate the problem of extending a morphism $\varphi\colon \partial X \to Y$ to a morphism $X \to Y$, when X is irreducible sofic and Y is mixing sofic. But because the techniques used to do that applies to the problem of extending morphisms from general subshifts $S \subseteq X$, I develop the theory in a wider set up.

3.1 Local Words

I want to use the marker strategy as described in section 2.10. In order to do that, I need to find a smart way of splitting words from X into locally recognizable intervals, such that long words from $\mathbb{W}^*(S)$ are not split. It is however generally not possible to recognize words in $\mathbb{W}^*(S)$ by local inspection, when $S \notin \mathrm{SFT}$. I therefore define a larger set of words, containing $\mathbb{W}^*(S)$, for which it is possible. I call them the local words of S.

Definition 3.1.1. Let X be a shift space. Then for each subshift $S \subseteq X$ and $x \in X$, I define

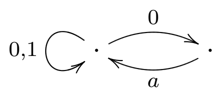
$$\begin{aligned}
\mathrm{L}(S,n) &= \{x \in X \mid \mathbb{W}_n(x) \subseteq \mathbb{W}(S)\} \\
\mathrm{LW}(S,n) &= \{x \in \mathbb{W}_{\geq n}(X) \mid \mathbb{W}_n(x) \subseteq \mathbb{W}(S)\} \\
\mathrm{MW}(S,x) &= \{[i,j] \subseteq \mathbb{Z} \mid x_{[i,j]} \in \mathbb{W}^*(S), x_{[i-1,j]}, x_{[i,j+1]} \notin \mathbb{W}^*(S)\} \\
\mathrm{MLW}(S,n,x) &= \{[i,j] \subseteq \mathbb{Z} \mid x_{[i,j]} \in \mathrm{LW}(S,n), x_{[i-1,j]}, x_{[i,j+1]} \notin \mathrm{LW}(S,n)\}
\end{aligned}$$

Elements in $\mathrm{L}(S,n)$ and $\mathrm{LW}(S,n)$ are called local points and local words of S, and elements in $\mathrm{MW}(S,x)$ and $\mathrm{MLW}(S,n,x)$ are called maximal words and maximal local words of S in x. n is called the order of the local word or point.

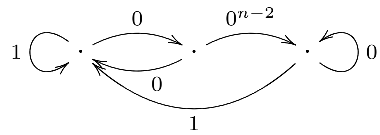
The reader should think of the elements in $\mathrm{LW}(S,n)$ as the words in $\mathbb{W}^*(X)$ which locally look like words from S, in the sense that it is impossible to rule out, that a word w is in S, by looking at subwords of length n.

The sets in Definition 3.1.1 are illustrated in the following example:

Example 3.1.2. Let X be the the sofic shift:



and let S be the even shift. Then $S \subseteq X$ and the set of local points of S of order n is presented by the following graph:



i.e

$$\mathrm{L}(S,n) = S \cup \{0,1\}^{\mathbb{Z}}_{\{10^i 1\}_{i<n-1}}.$$

So for example $0^\infty 10^3 10^\infty$ is in $\mathrm{L}(S,4)$ and clearly not in S.

The local words of order n is the set

$$\mathrm{LW}(S,n) = \mathbb{W}_{\geq n}(S) \cup \mathbb{W}_{\geq n}(\{0,1\}^{\mathbb{Z}}_{\{10^i 1\}_{i<n-1}}) = \mathbb{W}_{\geq n}(\mathrm{L}(S,n)).$$

Let x be the point

$$0^\infty a 0^4 10^3 10 a 0^\infty = 0^\infty_{]-\infty,0[} a_0 0_1 0_2 0_3 0_4 1_5 0_6 0_7 0_8 1_9 0_{10} a_{11} 0^\infty_{[12,\infty[},$$

where I have included the index of each symbol for easy reference. Then the set of maximal words of S in x is

$$\mathrm{MW}(S,x) = \{]-\infty,0[,[1,8],[6,10],[12,\infty[\},$$

corresponding to the words $0^\infty, 0^4 10^3, 0^3 10$ and 0^∞ respectively.

The maximal local words of S of order 4 and 5 in x are

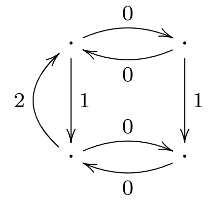
$$\begin{aligned}\mathrm{MLW}(S,4,x) &= \{]-\infty,0[,[1,10],[12,\infty[\}\\ \mathrm{MLW}(S,5,x) &= \mathrm{MW}(S,x).\end{aligned}$$

Remark 3.1.3. The set of doubly infinite sequences in $\mathrm{LW}(S,n)$, $\mathrm{L}(S,n)$, is a sort of SFT within X, in the sense that only finitely many more words are forbidden. Specifically $\mathrm{L}(S,n) = X_{\mathbb{W}_n(S)^c}$, which means that $\mathrm{L}(S,n) = X \cap S_n$, where the S_n's are the SFTs from Lemma 2.4.20.

Note that $\mathrm{L}(S,n) \subseteq \mathrm{L}(S,m)$ for all $n \geq m$ and $\bigcap_{n\in\mathbb{N}} \mathrm{L}(S,n) = S$, by Lemma 2.1.12. So since $\mathrm{L}(S,n)$ is a shift space for all $n \in \mathbb{N}$, $\{\mathrm{L}(S,n)\}_{n\in\mathbb{N}}$ is a decreasing sequence of shift spaces within X, which converges to S.

When $S \in \mathrm{SFT}$, the local word concept adds nothing new, since $\mathrm{L}(S,n) = S$ and $\mathrm{MLW}(S,n,x) = \mathrm{MW}_{\geq n}(S,x)$ for all $n > \mathrm{SL}(S)$, because $\mathrm{LW}(S,n) = \mathbb{W}_{\geq n}(S)$, for all $n > \mathrm{SL}(S)$ by the definition of step length. i.e. words that look like words in S are words in S. S does not have to be an SFT for this to happen, as illustrated by the following example:

Example 3.1.4. Let X be the sofic shift space presented by the following graph:



And let $S = \partial X$. i.e. the sequences in $\{0,1\}^{\mathbb{Z}}$, which contain at most one '1'. Then $S \notin \mathrm{SFT}$, since 0^n is not synchronizing for any $n \in \mathbb{N}$. But clearly $\mathrm{L}(S,n) = S$, $\mathrm{LW}(S,n) = \mathbb{W}_{\geq n}(S)$ and $\mathrm{MLW}(S,n,x) = \mathrm{MW}_{\geq n}(S,x)$, for all $n \geq 1$.

Subshifts with the property $\mathrm{LW}(S,n) = \mathbb{W}_{\geq n}(S)$, when n is large, turn out to be useful. So let's name them:

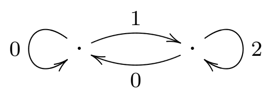
Definition 3.1.5 (SFT-like)**.** Let X be a shift space and S a subshift of X. S is called SFT relative to X, or simply SFT-like, when X is understood, if there is a number $N \in \mathbb{N}$, such that

$$\mathrm{LW}(S, N+1) = \mathbb{W}_{\geq N+1}(S).$$

The minimal such N is called the step length of S in X and is denoted by $\mathrm{SL}_X(S)$

The name 'SFT-like' is motivated by the fact that a shift space S is an SFT if and only if it is SFT relative to the full shift over its alphabet, by Theorem 2.4.7. In fact when S is an SFT, it is clearly SFT relative to any shift space in which it is a subshift, since $\mathrm{SL}(S)$ works for N in the definition of step length. This implies that $\mathrm{SL}_X(S) \leq \mathrm{SL}(S)$. By the following example, the inequality can be strict:

Example 3.1.6. Let X be the shift presented by the following graph:

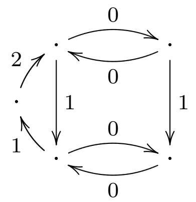


Then the golden mean shift $X_{\mathrm{gm}} = \{0,1\}^{\mathbb{Z}}_{\{11\}}$ is a subshift of X and $X_{\mathrm{gm}} = X_{\{a\}}$. Thus

$$\mathrm{SL}_X(X_{\mathrm{gm}}) = 0 < 1 = \mathrm{SL}(X_{\mathrm{gm}}).$$

In both Example 3.1.4 and Example 3.1.2 it is true that $\mathrm{LW}(S,n) = \mathbb{W}_{\geq n}(\mathrm{L}(S,n))$. That is not true in general by the following example, which shows that a local word doesn't necessarily occur in a local point. Or in other words: The local words are not extendable.

Example 3.1.7. Let X be the sofic shift presented by the following graph:



Define $S = \partial X$ and let $n \in \mathbb{N}$. Then $\mathrm{L}(S, n) = S$ and $10^{n-1}1 \in \mathrm{LW}(S, n)$, but $10^{n-1}1 \notin \mathbb{W}(S)$. Thus $\mathrm{LW}(S, n) \neq \mathbb{W}_{\geq n}(S) = \mathbb{W}_{\geq n}(\mathrm{L}(S, n))$ for all $n \in \mathbb{N}$.

Note that even though $\mathrm{LW}(S, n) \neq \mathbb{W}_{\geq n}(\mathrm{L}(S, n))$ in Example 3.1.7, they are 'almost equal' in the sense that if one removes the last symbol from a word in $\mathrm{LW}(S, n)$ then it is a word in $\mathbb{W}_{\geq n-1}(\mathrm{L}(S, n)) = \mathbb{W}_{\geq n-1}(S)$. This kind of 'almost equality' is no coincidence. Lemma 3.1.8 below shows that it will always be the case that there is a number $m \in \mathbb{N}$, such that if one trims the first and last m symbols from the words in $\mathrm{LW}(S, n)$, then they become words in $\mathrm{L}(S, n)$.

Denote the map $\mathbb{W}^*(X) \to \mathbb{W}^*(X)$, which trims off the first and last m symbols, by c_m. i.e. $c_m(x_{[i,j]}) = x_{[i+m,j-m]}$.

Lemma 3.1.8. *Let X be a sofic shift space. Then for each $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that*

$$c_m(\mathrm{LW}(S, n)) = \mathbb{W}^*(\mathrm{L}(S, n)).$$

Proof. Let $n \in \mathbb{N}$. Since $\supseteq$ clearly holds when $m \geq n$ I only need to find an $m \geq n$ such that $\subseteq$ holds. Let G be a presentation of $X^{[n]}$ and let E be the number of edges in G. Define $m = E + n$. Let $w \in \mathrm{LW}(S, n)$. If $|w| \leq 2m$, then $c_m(w)$ is the empty word, which is in $\mathbb{W}^*(\mathrm{L}(S, n))$ by definition. Assume therefore that $|w| \geq 2m + 1$ Then $v = \beta_n(w) \in \mathrm{LW}_{\geq 2(E+1)+1}(S^{[n]}, 1)$.

Let p be a path in G which presents v. By the pigeon hole principle I can find $a, b, c, d \in \mathbb{N}$ such that

$$0 \leq a \leq b < E + 1,$$
$$|v| - (E + 1) \leq c \leq d < |v|$$

and such that $p_a = p_b$ and $p_c = p_d$. This implies that $v^{\infty}_{[a,b[} v_{[b,c]} v^{\infty}_{]c,d]}$ is in $\mathrm{L}(S^{[n]}, 1)$. Thus $c_{E+1}(v) \in \mathbb{W}^*(\mathrm{L}(S^{[n]}, 1))$ by my choice of a, b, c and d, which implies that $c_{E+1}(w) \in \mathbb{W}^*(\mathrm{L}(S, n))$. So as $m \geq E + 1$, $c_m(w) \in \mathbb{W}^*(\mathrm{L}(S, n))$, which is what I needed to prove. $\square$

The derived shift of the sofic shift in Example 3.1.7 is not SFT relative to X, since $\mathrm{LW}(S, n) \neq \mathbb{W}_{\geq n}(S)$. But it is 'almost' SFT-like in two ways:

1. Because local points of S are points in S, since $\mathrm{L}(S, 1) = S$. i.e.

$$\exists N \in \mathbb{N} \colon \mathrm{L}(S, N) = S.$$

2. Because $c_1(\mathrm{LW}(S,1)) = \mathbb{W}^*(S)$, i.e.

$$\exists m, N \in \mathbb{N}\colon c_m(\mathrm{LW}(S,N)) = \mathbb{W}^*(S).$$

Both of these properties could serve as a meaningful definition of 'almost SFT-like'. But by Lemma 3.1.8 the two conditions are equivalent.

I therefore define:

Definition 3.1.9 (Almost SFT-like)**.** Let X be a shift space and S be a subshift of X. S is said to be almost SFT relative to X, or almost SFT-like, if there is a number $N \in \mathbb{N}$ such that

$$\mathrm{L}(S,N) = S.$$

Remark 3.1.3 implies that a subshift $S \subseteq X$ is almost SFT relative to X if and only if there is an SFT W, such that $S = X \cap W$.

Example 3.1.2 shows that not all subshifts are almost SFT-like and Example 3.1.7 shows that a subshift can be almost SFT-like without being SFT-like.

The reader easily verifies the following upper limit on the overlap of two intervals in $\mathrm{MLW}(S,n,x)$:

Lemma 3.1.10. *Let $S \subseteq X$ be shift spaces, $x \in X$ and $n \in \mathbb{N}$. Then*

$$u, v \in \mathrm{MLW}(S,n,x), u \neq v \Rightarrow |u \cap v| < \max\{1, n-1\}.$$

Thus by trimming $\max\{0, n-2\}$ symbols from the beginning of each interval in $\mathrm{MLW}(S,n,x)$, I achieve that any two such 'MLW-intervals' are disjoint. In section 4.1 I show that the resulting intervals can be used to construct a locally recognizable set of intervals, with the properties from the marker strategy section.

3.2 Visualizing Local Words

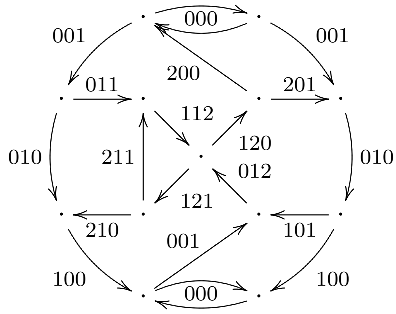
One way of visualizing $\mathrm{LW}(S,n)$, when X is an irreducible sofic shift, is by constructing the n-th higher block graph of the Fischer cover of X and then deleting all edges whose labels are not in $\mathbb{W}_n(S)$. The label of any path in the resulting graph will be a word in $\mathrm{LW}(S^{[n]},1) = \beta_n(\mathrm{LW}(S,n))$ and vice versa. In order to illustrate the set

$$\mathrm{MLW}(S,n) = \bigcup_{x \in X} \{x_{[i,j]} \mid [i,j] \in \mathrm{MLW}(S,n,x)\},$$

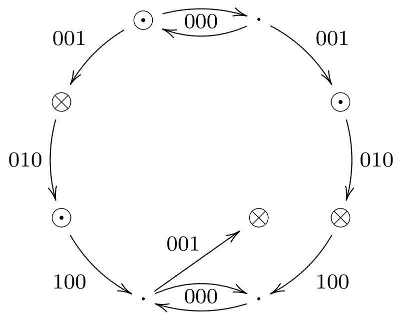
i.e the words, which correspond to maximal local words in points in X, mark each vertex in which one of the deleted edges ended by a $\odot$ and each vertex where a deleted edge started by a $\otimes$. Then the label of any path starting at a $\odot$ and ending at a $\otimes$ will correspond to a word in $\mathrm{MLW}(S,n)$, and any word in $\mathrm{MLW}(S,n)$ is presented by a path from a $\odot$ to a $\otimes$.

I call the resulting graph the nth local word graph of S, or the $\mathrm{MLW}(S,n)$-graph.

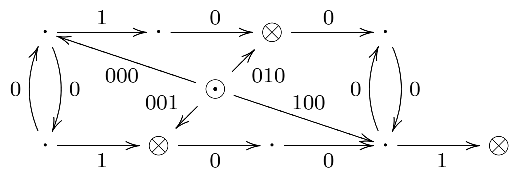
Example 3.2.1. Let X be the sofic shift from example 3.1.7 and $S = \partial X = \{0,1\}_{\{10^n1\}_{n\in\mathbb{N}}}$. Then the third higher block graph of the Fischer cover of X looks like:



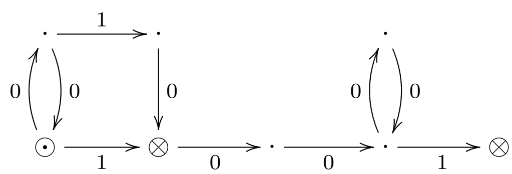
By deleting all edges not in $\mathbb{W}_3(S)$ and inserting $\odot$'s and $\otimes$'s as instructed I get the third local word graph of S:



To make it easier to check whether a given word is in $\mathrm{MLW}(S,3)$ I gather the $\odot$'s in one starting vertex and replace each label, except the ones on edges from the starting vertex, by its last symbol:



Which can be simplified if I add the restriction that I am only interested in paths of length at least 3:



Thus a word $x \in \mathbb{W}_{\geq 3}(X)$ is in $\mathrm{MLW}(S,3)$ if and only if it can be presented by a path from the $\odot$ to a $\otimes$.

3.3 A Necessary Condition

Let X and Y be shift spaces and S be a subshift of X. If $\varphi\colon S \to Y$ is an (m,n)-sliding block code, then the word map $\varphi_{m,n}$ extends in the obvious way to a map $\widetilde{\varphi}_{m,n}\colon \mathrm{LW}(S, m+n+1) \to \mathbb{W}^*(\Sigma_Y^{\mathbb{Z}})$.

When φ extends to a morphism $X \to Y$, then $\widetilde{\varphi}_{m,n}$ has to map into $\mathbb{W}^*(Y)$, when m and n are large enough by the following lemma:

Lemma 3.3.1. *Let X and Y be shift spaces and S be a subshift of X. If a morphism $\varphi\colon S \to Y$ extends to a morphism $X \to Y$, then there exists $M, N \in \mathbb{N}$ such that*

$$\widetilde{\varphi}_{m,n}(\mathrm{LW}(S, m+n+1)) \subseteq \mathbb{W}^*(Y),$$

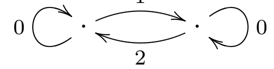
for all $m \geq M$ and $n \geq N$.

Proof. Let $S \subseteq X$ and Y be shift spaces. And let $\varphi\colon S \to Y$ be a morphism, which extends to a morphism $\phi\colon X \to Y$, with memory M and anticipation N.

I claim that $\widetilde{\varphi}_{m,n}$ must be equal to $\phi_{m,n}$ on words in $\mathrm{LW}(S, m+n+1)$, when $m \geq M$ and $n \geq N$. Because if it is not, then I can find a word $s \in \mathbb{W}_{m+n+1}(S)$ for which $\varphi_{m,n}(s) \neq \phi_{m,n}(s)$, which would imply that $\varphi(x) \neq \phi(x)$, for x's in S in which s occurs. But that would contradict the assumption that φ is the restriction of ϕ to S.

Thus $\widetilde{\varphi}_{m,n}$ has to map $\mathrm{LW}(S, m+n+1)$ to $\mathbb{W}^*(Y)$, when m is larger than the memory of ϕ and n is larger than the anticipation of ϕ, since $\phi_{m,n}$ maps $\mathbb{W}^*(X)$ into $\mathbb{W}^*(Y)$. $\square$

Example 3.3.2. Let X be a shift space, such that $\{0,1\}^{\mathbb{Z}} \subseteq X$, S be the points in $\{0,1\}^{\mathbb{Z}}$, which contain at most one '1', Y be the shift presented by the following graph



and $\varphi\colon S \to Y$ be the identity. Then $\widetilde{\varphi}_{m,n}$ is the identity map on the set $\mathrm{L}(S, m+n+1) = \{0,1\}^{\mathbb{Z}}_{\{10^i1\}_{i \leq m+n}} \nsubseteq Y$, for all $n, m \in \mathbb{N}$. Thus φ does not extend to a morphism $X \to Y$.

The condition from Lemma 3.3.1 can be quite difficult to check for a given map $\varphi\colon S \to Y$. But since $\mathrm{LW}(S, m) \subseteq \mathrm{LW}(S, n)$, for all $m \geq n$, it is equivalent to

$$\exists m, n \in \mathbb{N}\colon \widetilde{\varphi}_{m,n}(\mathrm{LW}(S, m+n+1)) \subseteq \mathbb{W}^*(Y). \tag{3.1}$$

So if $\mathrm{LW}(S, n) = \mathbb{W}_{\geq n}(\mathrm{L}(S, n))$, for n larger than some number, as in Example 3.1.2, then one only needs to check, whether φ maps $\mathrm{L}(S, n)$ to Y, for one $n \in \mathbb{N}$.

Note that if S is SFT relative to X, then condition (3.1) is trivial since any morphism $S \to Y$ maps words in $\mathrm{LW}(S,n)$ to words in $\mathbb{W}^*(Y)$ because $\mathrm{LW}(S,n) = \mathbb{W}^*(S)$ for large n's.

In the following I will not distinguish between $\varphi_{m,n}$ and $\widetilde{\varphi}_{m,n}$, because they are equal on the intersection of their domains.

3.4 Definition of the Marked Property

I want to use the marker strategy from section 2.10 to extend morphisms from S to Y, for some $S \subseteq X$, to morphisms from X to Y. Therefore I need to define a splitting map $I_\tau \colon X \to I_{\mathbb{Z}} \times \tau$, which doesn't split large words of S that occur in points in X. And then I need to define a map $\overline{\varphi} \colon \mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$, for some $M \in \mathbb{N}$, with the four properties from Lemma 2.10.9:

1. $\overline{\varphi}(w,t)$ begins with s.
2. s can be put at the end of $\overline{\varphi}(w,t)$.
3. $\overline{\varphi}$ preserves length.
4. The kth symbol of $\overline{\varphi}(w,t)$ can be determined by local inspection of w.

To make sure that I does not split the long words of S, I could define

$$I_{\mathrm{LW}}(x) = \mathrm{MLW}(S, m+n+1, x)$$

for some large $m, n \in \mathbb{N}$. Then I_{LW} would be locally recognizable. And when m and n are large enough, $\varphi_{m,n}$ has to map $\mathrm{MLW}(S, m+n+1)$ into $\mathbb{W}^*(Y)$ by Lemma 3.3.1. So if I use $\varphi_{m,n}$ on the words in $\mathbb{W}_{\mathrm{LW}}$, then 4. is satisfied on that type of intervals.

But I have to make the map satisfy the other three conditions as well. To make sure it satisfies 1., 2. and 3. I will find a way of putting synchronizing words at the beginning and end of words in $\varphi_{m,n}(\mathrm{MLW}(S, m+n+1))$, in a way such that 4. is still satisfied. The way I do that is by defining maps from the beginnings and ends of maximal local words to $S(Y)$. Morphisms $S \to Y$ for which that is possible will be called 'marked'.

Similar to the entries and exits of periodic points defined by Thomsen, I call the beginnings and ends of maximal local words of S in X, the S-entries and S-exits.

Definition 3.4.1 (Entry, Exit)**.** Let S be a subshift of a shift space X and $n, k, L \in \mathbb{N}$. The words in the set

$$A_S(n,k,L) = \{x_{[-L,k]} \in \mathbb{W}(X) \mid x_{[0,k]} \in \mathrm{LW}(S,n), x_{[-1,k]} \notin \mathrm{LW}(S,n)\}.$$

are called S-entries. And the words in

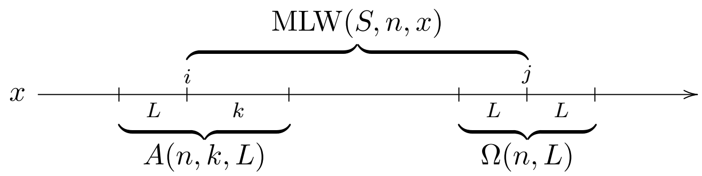
$$\Omega_S(n,L) = \{x_{[-L,L]} \in \mathbb{W}(X) \mid x_{[-L,0]} \in \mathrm{LW}(S,n), x_{[-L,1]} \notin \mathrm{LW}(S,n)\}$$

are called S-exits.

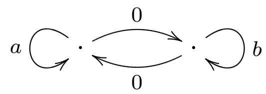
Define $A_S(n, L, L) = A_S(n, L)$.

To simplify notation I omit the S and write simply $A(n, L)$ and $\Omega(n, L)$, when S is understood.

The definition of entries and exits is illustrated in the following figure:



Example 3.4.2. Let X be the sofic shift from Example 2.10.1:



and let $S = \partial X = \{0^\infty\}$. Then

$$A_S(1, 1) = \{a00, b00\} = A_S(2, 1) \text{ and}$$
$$\Omega_S(1, 1) = \{00a, 00b\} = \Omega_S(2, 1).$$

Definition 3.4.3 (Marked)**.** Let X and Y be shift spaces and S a subshift of X. A morphism $\varphi\colon S \to Y$ is called (m, n)-marked, if there for each $n' \geq n$ exists an $L \in \mathbb{N}$ and maps $\alpha_{n'}\colon A_S(n' + m + 1, L) \to S(Y)$ and $\omega_{n'}\colon \Omega_S(n' + m + 1, L) \to S(Y)$ such that

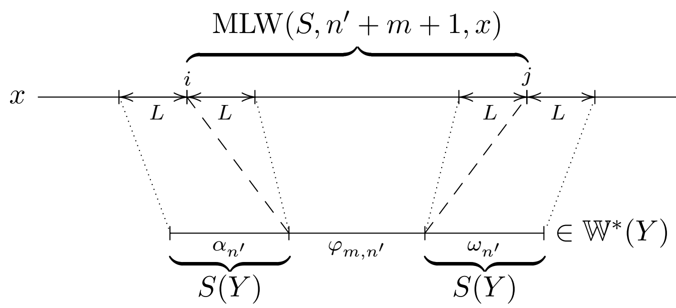
$$\alpha_{n'}(x_{[i-L,i+L]})\varphi_{m,n'}(x_{[i,j]})\omega_{n'}(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $x \in X$ and $[i, j] \in \mathrm{MLW}(S, n' + m + 1, x)$.

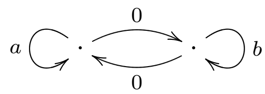
φ is called marked if it is (m, n)-marked for some $n, m \in \mathbb{N}$ and m-marked if it is $(0, m - 1)$-marked.

The definition of the marked property is quite a mouthful, but I can console the reader that under mild assumption on X and S, the definition simplifies significantly. That is the point of section 3.8. But in order to make my main result as strong as possible I am forced to work with Definition 3.4.3.

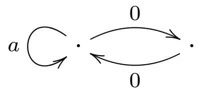
The following figure illustrates the functions in the definition of marked.



Example 3.4.4. Let's look at the problem from Example 2.10.1 again. i.e. X is the sofic shift:



And Y is the sofic shift:



And we wish to extend the map $0^\infty \mapsto 0^\infty$ to a non-trivial morphism from X to Y. So in this example $S = \{0^\infty\} = \partial X$, and we have a morphism $\varphi\colon S \to Y$, which turns out to be 1-marked:

For $n' = 1$, define $L = 1$, $\alpha_1\colon A_S(1,1) = \{a00, b00\} \to S(Y)$ by

$$a00 \mapsto a, b00 \mapsto a0$$

and $\omega_1\colon \Omega_S(1,1) = \{00a, 00b\} \to S(Y)$ by

$$00a \mapsto a, 00b \mapsto 0a.$$

To verify that these definitions work let $x \in X$ and $[i,j] \in \mathrm{MLW}(S,1,x)$. Then $x_{[i,j]}$ has the form 0^n for some $n \in \mathbb{N}$ and $x_{[i-L,i+L]}$ is either $a00$ or $b00$ and $x_{[j-L,j+L]}$ is either $00a$ or $00b$. If both the entry and exit contain the same non-zero symbol, then n is even and if they contain different non-zero symbols, then n is odd. So either way

$$\alpha_1(x_{[i-L,i+L]})\varphi_1(x_{[i,j]})\omega_1(x_{[i-L,i+L]}) = \alpha_1(x_{[i-L,i+L]})0^n\omega_1(x_{[i-L,i+L]})$$

has the form $a0^m a$ for some even m, which means that it is in $\mathbb{W}^*(Y)$.

I leave it to reader to generalise to arbitrary $n' > 1$.

Remark 3.4.5. If a morphism $\varphi\colon S \to Y$ is (m,n)-marked for some $m, n \in \mathbb{N}$, then it is (m,n')-marked for all $n' \geq n$.

Note that if $\varphi\colon S \to Y$ is (m,n)-marked, then $\varphi \circ \sigma^m$ is $n+m+1$-marked. Thus when trying to extend marked maps, I can always assume that they are m-marked for some $m \in \mathbb{N}$. Because if I extend $\varphi \circ \sigma^m$ to a morphism $\phi\colon X \to Y$, then $\phi \circ \sigma^{-m}$ extends φ. This makes the definition somewhat easier to work with:

Lemma 3.4.6. *Let X and Y be shift spaces and S a subshift of X. A morphism $\varphi\colon S \to Y$ is m-marked, if and only if there for each $n \geq m$ exists an $L \in \mathbb{N}$ and maps $\alpha_n\colon A_S(n,L) \to S(Y)$ and $\omega_n\colon \Omega_S(n,L) \to S(Y)$, such that*

$$\alpha_n(x_{[i-L,i+L]})\varphi_n(x_{[i,j]})\omega_n(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y)$$

for all $x \in X$ and $[i,j] \in \mathrm{MLW}(S,n,x)$.

The attentive reader will have noticed that the definition of marked makes it dificult to ensure that $\overline{\varphi}$ becomes length preserving, since I have no control over the length of $\alpha_n(x_{[i-L,i+L]})\varphi_n(x_{[i,j]})\omega_n(x_{[j-L,j+L]})$. The following lemma takes care of that in the case when Y is irreducible sofic by providing a common length for all the words in the image of α_n and ω_n, for any given $n \in \mathbb{N}$, and by letting me cut away as much as I like from the left side of $\varphi_n(x_{[i,j]})$.

Lemma 3.4.7. *Let S be a subshift of a shift space X and Y be an irreducible sofic shift space. A morphism $\varphi\colon S \to Y$ is m-marked if and only if there for each $n \geq m$ exist $L, N \in \mathbb{N}$ and maps $\alpha_{n,k}\colon A_S(n, L', L) \to S_N(Y)$ for each $k \in \mathbb{N}$ and $\omega_n\colon \Omega_S(n, L) \to S_N(Y)$ such that*

$$\alpha_{n,k}(x_{[i-L,i+L']})\varphi_n(x_{[i+k,j]})\omega_n(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $k \in \mathbb{N}$, $x \in X$ and $[i, j] \in \mathrm{MLW}(S, n, x)$, where $L' = \max\{L, k+n-2\}$.

Proof. $\Leftarrow$: The definition corresponds to letting $k = 0$.
$\Rightarrow$: Assume that φ is m-marked. Let $n \geq m$ and find $L, \alpha_n^{\mathrm{def}}$ and ω_n^{def} as in the definition of the marked property. Since $A(n, L)$ and $\Omega(n, L)$ are finite sets, so are $\alpha_n^{\mathrm{def}}(A(n, L))$ and $\omega_n^{\mathrm{def}}(\Omega(n, L))$. Thus it makes sense to talk about the longest word in each set. Let N' be the length of the longest word in $\alpha_n^{\mathrm{def}}(A(n, L)) \cup \omega_n^{\mathrm{def}}(\Omega(n, L))$ and define $N = N' + E$, where E is the number of edges in the Fischer cover of Y. Let $k \in \mathbb{N}$ and define $L' = \max\{L, k+n-2\}$.

Let $x_{[-L,L']} \in A(n, L', L)$. Then $x_{[-L,L]} \in A(n, L)$ and $\alpha_n^{\mathrm{def}}(x_{[-L,L]}) \in S_a(Y)$ for some $a \leq N'$. Find a path, p_1, of length at most E from the terminal vertex of $\alpha_n^{\mathrm{def}}(x_{[-L,L]})$ to the terminal vertex of $\alpha_n^{\mathrm{def}}(x_{[-L,L]})\varphi_n(x_{[0,k+n-2]})$. And then find a path, p_2, of length $N - |p_1| - a$, which terminates in a vertex, which is the initial vertex of a path presenting $\alpha_n'(x_{[-L,L]})$.

Define

$$\alpha_{n,k}(x_{[-L,L']}) = \pi_Y(p_2)\alpha_n^{\mathrm{def}}(x_{[-L,L]})\pi_Y(p_1).$$

Then $\alpha_{n,k}(x_{[-L,L']}) \in S_N(Y)$.

Define for each $x_{[-L,L]} \in \Omega(n, L)$,

$$\omega_n(x_{[-L,L]}) = \omega_n^{\mathrm{def}}(x_{[-L,L]})\pi_Y(q),$$

where q is an arbitrary path of length $N - |\omega_n^{\mathrm{def}}(x_{[-L,L]})|$ starting in the terminal vertex of $\omega_n^{\mathrm{def}}(x_{[-L,L]})$. Then $\omega_n\colon \Omega(n, L) \to S_N(Y)$.

Let $x \in X$ and $[i, j] \in \mathrm{MLW}(S, n, x)$. Then

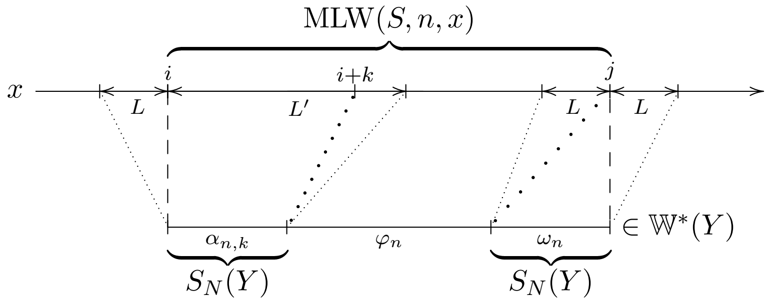
$$\alpha_{n,k}(x_{[i-L,i+L']})\varphi_n(x_{[i+k,j]})\omega_n(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

by our choice of p's and q's. $\square$

Thus by choosing k appropriately ($k = 2N - (n - 1)$), I can achieve that

$$|\alpha_{n,k}(x_{[i-L,i+L']})\varphi_n(x_{[i+k,j]})\omega_n(x_{[j-L,j+L]})| = |x_{[i,j]}|,$$

if $x_{[i,j]}$ is long enough.



3.5 Necessity of the Marked Property

It turns out that when $S \subseteq X$ has the property that all sufficiently long S-entries and S-exits are synchronizing for X, then a morphism $S \to Y$ has to be marked in order to have a chance of extending to a morphism $X \rightharpoonup Y$.

Definition 3.5.1 (Synchronizing Edges)**.** Let S be a subshift of a shift space X. S is said to have synchronizing edges if there is an $N \in \mathbb{N}$ and a sequence $\{L_n\}_{n \geq N} \subseteq \mathbb{N}$ such that:

$$A_S(n, L_n), \Omega_S(n, L_n) \subseteq S(X),$$

for all $n \geq N$.

Example 3.5.2. Let X and S be as in Example 3.4.2. Then S has synchronizing edges, since both a and b are synchronizing for X.

Lemma 3.5.3. *Let X be an irreducible shift space, Y be a shift space and $\varphi\colon X \to Y$ be a morphism, which hits a sychronizing word in Y. If $S \subseteq X$ has synchronizing edges, then $\varphi_{|S}$ is marked.*

Proof. Since S has synchronizing edges, I can find $N \in \mathbb{N}$ and $\{L_n\}_{n \geq N} \subseteq \mathbb{N}$ such that $A_S(n, L_n), \Omega_S(n, L_n) \subseteq S(X)$ for all $n \geq N$. And since φ is a morphism, I can find $n, m \in \mathbb{N}$, such that it is an (m, n)-sliding block map and $m + n + 1 \geq N$.

I claim that $\varphi_{|S}$ is (m, n)-marked.

Let $n' \geq n$ and define $L = \max\{L_{n'+m+1}, n' + m\}$. Since φ hits a synchronizing word, I can find an $s \in S(Y)$ and an $s_0 \in \mathbb{W}(X)$, such that $\varphi_{m,n'}(s_0) = s$. By using that X is irreducible, I can for each entry $x_{[-L,L]} \in A_S(n' + m + 1, L)$ find a $u \in \mathbb{W}(X)$, such that $s_0 u x_{[-L,L]} \in \mathbb{W}(X)$. And for each exit $x_{[-L,L]} \in \Omega_S(n' + m + 1, L)$ I can find a $v \in \mathbb{W}(X)$, such that $x_{[-L,L]} v s_0 \in \mathbb{W}(X)$. Define $\alpha_{n'}\colon A_S(n' + m + 1, L) \to S(Y)$ and $\omega_{n'}\colon \Omega_S(n' +$

$m+1, L) \to S(Y)$ by

$$\alpha_{n'}(x_{[-L,L]}) = \varphi_{m,n'}(s_0 u x_{[-L,m+n'[}) \text{ and}$$
$$\omega_{n'}(x_{[-L,L]}) = \varphi_{m,n'}(x_{]-(m+n'),L]} v s_0).$$

Then they both map into $S(Y)$, since $\alpha_{n'}(x_{[-L,L]})$ and $\omega_{n'}(x_{[-L,L]})$ always contain s.

Let $[i,j] \in \mathrm{MLW}(S, n'+m+1, x)$. Then

$$\alpha_{n'}(x_{[i-L,i+L]})\varphi_{m,n'}(x_{[i,j]})\omega_{n'}(x_{[j-L,j+L]}) = \varphi_{m,n'}(s_0 u x_{[i-L,j+L]} v s_0),$$

which is in $\mathbb{W}^*(Y)$ since $s_0 u x_{[i-L,j+L]} v s_0 \in \mathbb{W}^*(X)$ because both $x_{[i-L,i+L]}$ and $x_{[j-L,j+L]}$ are synchronizing as $L \geq L_{n'+m+1}$. This establishes the claim since $\varphi_{|S}$ is also an (m,n)-sliding block map, and $(\varphi_{|S})_{m,n'} = \varphi_{m,n'}$ on words in $\mathrm{LW}(S, n'+m+1)$. □

Corollary 3.5.4. *Let X be an irreducible shift space, Y be a sofic shift space and $S \subseteq X$ be a shift space with synchronizing edges. If $\varphi\colon X \to Y$ is a factor map, then $\varphi_{|S}$ is marked.*

Proof. Follows from Corollary 2.6.21 and Lemma 3.5.3. □

It turns out that when X is irreducible and sofic, ∂X has synchronizing edges. In order to prove that, I need the following lemma:

Lemma 3.5.5. *Let X be an irreducible sofic shift, $x \in X$ and V the number of vertices in the Fischer cover of X. Then*

$$x_{[i,j]} \notin \mathbb{W}(\partial X) \Rightarrow x_{[i-V^2, j+V^2]} \in S(X).$$

Proof. Assume that $x_{[i-V^2,j+V^2]} \notin S(X)$. Then it isn't magic for π according to Proposition 2.6.22. This enables me to find paths $u, v \in \pi^{-1}(x_{[i-V^2,j+V^2]})$ such that $u_k \neq v_k, \forall k$. Since there are less than V^2 possibilities for the pair (t_{u_k}, t_{v_k}) of terminal vertices for u_k and v_k, I can find numbers a, b, c and d such that

$$i - V^2 \leq a < b \leq i,$$
$$j \leq c < d \leq j + V^2,$$

with the property that $u^\infty_{[a,b[} u_{[b,c]} u^\infty_{]c,d]}$ and $v^\infty_{[a,b[} v_{[b,c]} v^\infty_{]c,d]}$ are paths in the Fischer cover of X. So as $u_k \neq v_k, \forall k$, I have found two disjoint paths in the Fischer cover of X, which present the same point y. Thus $y \in \partial X$ by Proposition 6.5 in [T], which implies that $x_{[i,j]} \in \mathbb{W}(\partial X)$, since $x_{[i,j]} \subseteq y$, by my choice of b and c. □

Lemma 3.5.6. *Let X be an irreducible sofic shift. Then ∂X has synchronizing edges.*

Proof. Let V denote the number of vertices in the Fischer cover of X. Define $L_n = n-2+V^2$ for each $n \in \mathbb{N}$. I claim that $A_{\partial X}(n, L_n), \Omega_{\partial X}(n, L_n) \subseteq S(X)$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $x_{[-L_n,L_n]} \in A_{\partial X}(n, L_n)$. Then $x_{[0,L_n]} \in LW(\partial X, n)$ and $x_{[-1,L_n]} \notin \mathrm{LW}(\partial X, n)$ by definition of entries. By definition of $\mathrm{LW}(\partial X, n)$ this implies that $x_{[-1,n-1[} \notin \mathbb{W}_n(\partial X)$. Thus $x_{[-L_n,L_n]} \supseteq x_{[-1-V^2,n-1+V^2[} \in S(X)$ by Lemma 3.5.5, which means that $x_{[-L_n,L_n]} \in S(X)$ by lemma 2.1.16.

The verification of $\Omega_{\partial X}(n, L_n) \subseteq S(X)$ is analogous. □

Now we know that in order for a morphism $\partial X \to Y$ to be able to extend to a morphism $X \rightharpoonup Y$, it has to be marked. Thus a necessary condition for X to factor onto Y is that there exists a marked morphism $\partial X \to Y$.

3.6 Comparison Between the Marked Property and Previous Necessary Conditions

The idea behind the marked property is similar to the idea of affiliation. The following lemma shows that it is in fact a strengthening of $\operatorname{Per} S \overset{(d,F)}{\to} \operatorname{Per} Y$ as defined in Definition 2.7.13 and hence stronger than $\mathrm{Per\,X} \overset{\mathrm{(d)}}{\twoheadrightarrow} \mathrm{Per\,Y}$ as defined in [T] (p. 4 in this text) for all periodic points in S, when $S \subseteq \partial X$.

Lemma 3.6.1. *Let X and Y be shift spaces and S be a subshift of ∂X. If a morphism $\varphi\colon S \to Y$ is marked, then*

$$\varphi(Q_n(X_0^{(d,F)} \cap S)) \subseteq \bigcup_{m|n} Q_m(Y_0^{\frac{n}{m}(d,F)}),$$

for all $d \in \mathbb{N}$ and $F \subseteq \{0, \dots, d-1\}$. In particular $\operatorname{Per} S \overset{(d,F)}{\to} \operatorname{Per} Y$.

Proof. Let $d \in \mathbb{N}$, $F \subseteq \{0, \dots, d-1\}$ and $x \in Q_n(X_0^{(d,F)} \cap S)$. Define $y = \varphi(x)$ and $m = |p_y|$. Then m divides n by Lemma 2.3.3.

Since $x \in Q_n(X_0^{(d,F)})$, we can find $u, v \in S(X)$, such that $up_x^{kd+i}v \in \mathbb{W}(X)$, for all $k \in \mathbb{N}$ and $i \in F$. Assume that φ is (a,b)-marked and $a+b+1$ is larger than the length of both u and v. That is possible by Remark 3.4.5. As $S \subseteq \partial X$, neither u nor v occur in words in $\mathrm{LW}(S, a+b+1)$. There is therefore some suffix u' of u and some prefix v' of v, such that $u'p_x^{kd+i}v' \in MLW(\partial X, a+b+1, x')$, for some $x' \in X$ and all $k \in \mathbb{N}$ and $i \in F$. So since φ is marked, we can find $s_1, s_2 \in S(Y)$, such that $s_1 p_y^{\frac{n}{m}(kd+i)} s_2 \in \mathbb{W}(Y)$, for all $k \in \mathbb{N}$. Thus $y \in Q_m(Y_0^{\frac{n}{m}(d,F)})$. □

Corollary 3.6.2. *Let X and Y be shift spaces. If $\varphi\colon S \to Y$ is marked for some subshift $S \subseteq X$ for which $E_{X,Y} \subseteq S \subseteq \partial X$, then* $\operatorname{Per} X \overset{(d,F)}{\to} \operatorname{Per} Y$.

Proof. Assume that x is in $\bigcap_{(d,F)\in G} Q_n(X_0^{(d,F)})$ for some subset G of the set $\{(d,F) \in \mathbb{N} \times 2^{\mathbb{N}} \mid F \subseteq \{0,\dots,d-1\}\}$.

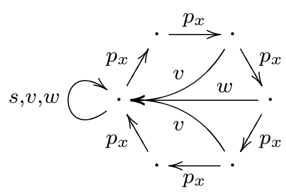
If $x \notin E_{X,Y}$, then

$$\emptyset \neq \bigcup_{m|n} Q_m(Y_0^{(\frac{n}{m})}) \subseteq \bigcup_{m|n} \bigcap_{(d,F)\in G} Q_m(Y_0^{\frac{n}{m}(d,F)}),$$

by definition of $E_{X,Y}$. And if $x \in E_{X,Y}$, then $\bigcup_{m|n} \bigcap_{(d,F)\in G} Q_m(Y_0^{\frac{n}{m}(d,F)}) \neq \emptyset$ by Lemma 3.6.1. □

Remark 3.6.3. If X is irreducible, the assumption $S \subseteq \partial X$ in Lemma 3.6.1 and Corollary 3.6.2 can be replaced by $S \subseteq X$.

Remark 3.6.4. The argument used to prove Lemma 3.6.1 implies that the existence of a marked map $S \to Y$, for some $S \subseteq \partial X$, actually requires a lot more of the periodic points than $\operatorname{Per} X \overset{(d,F)}{\to} \operatorname{Per} Y$ does. If for example some synchronizing word $s \in S(X)$ satisfies $sp_x^{id+j}v, sp_x^{id'+j'}w \in \mathbb{W}(X)$, for some $x \in \operatorname{Per} S$ and $v, w \in S(X)$ and all $i \in \mathbb{N}$ and j, j' in some subsets of $\mathbb{N}$, as in the following sofic shift

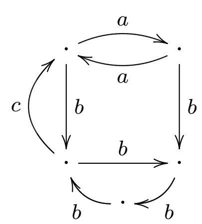


then there must be a $y \in \operatorname{Per} Y$ and words $s', v', w' \in S(Y)$, such that both $s'p_y^{\frac{|p_x|}{|p_y|}(id+j)}v'$ and $s'p_y^{\frac{|p_x|}{|p_y|}(id'+j')}w'$ are in $\mathbb{W}(Y)$, for all $i \in \mathbb{N}$ and j,j'.

Similarly if some synchronizing word $s \in S(X)$ satisfies $up_x^{id+j}s, vp_x^{id'+j'}s \in \mathbb{W}(X)$, for some $x \in \operatorname{Per} S$ and $u, v \in S(X)$ and all $i \in \mathbb{N}$ and j, j' in some subsets of $\mathbb{N}$, then there must be a $y \in \operatorname{Per} Y$ and words $s', u', v' \in S(Y)$, such that both $u'p_y^{\frac{|p_x|}{|p_y|}(id+j)}s'$ and $v'p_y^{\frac{|p_x|}{|p_y|}(id'+j')}s'$ are in $\mathbb{W}(Y)$.

This stronger requirement on individual periodic points is however not the primary strength of the marked condition. As we will see next, the marked condition is fundamentally different from the previous necessary conditions in that it looks at several different periodic points at the same time.

Take a look at the shift X defined by the following graph:



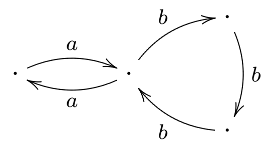
The strongest previously known necessary condition for X to factor onto a sofic shift space Y comes from Theorem 7.4 in [T] :

Theorem 3.6.5 (Thomsen)**.** *Let X and Y be sofic shift spaces, and $\varphi\colon X \to Y$ a morphism. It follows that there is a map $X_c \to Y_{\varphi(c)}$ from the irreducible components of X to the irreducible components of Y, such that*

$$\varphi(Q_n(X_c^{(d)})) \subseteq \bigcup_{m|n} Q_m(Y_{\varphi(c)}^{(\frac{n}{m}d)}). \tag{3.2}$$

In particular $\varphi(\overline{X_c}) \subseteq \overline{Y_{\varphi(c)}}$.

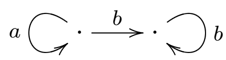
Thus before the marked condition, all that we knew was required from an irreducible sofic shift Y in order for Y to be a factor of X, was that there for each periodic point in X had to be a periodic point in Y with similar affiliation properties. If for example Y' is the shift:



Then X and Y' satisfy Thomsen's condition: X and Y' both have three irreducible components, namely their top component, to which all periodic points except a^∞ and b^∞ are 1-affiliated, since they contain synchronizing words, and two components at level 1: $\{a^\infty\}$ and $\{b^\infty\}$. So by defining φ to be the identity map on the components we see that (3.2) is satisfied, since $\bigcup_{m|n} Q_m(Y_0'^{(\frac{n}{m})}) \neq \emptyset$ for all $n \geq 2$.

Thus X and Y' satisfy the strongest previously known necessary condition for X to factor onto Y. But what does the marked condition tell us?

∂X is the following shift:



And the marked condition implies that if X is to factor onto a sofic shift Y, then there must be a marked morphism from ∂X to Y. I claim that if there exists such a marked map, then there is a word $g \in \mathbb{W}(Y)$ and words $u, v \in S(Y)$, such that

$$ua^i g b^j v \in \mathbb{W}(Y),$$

for all $i, j \in \mathbb{N}\colon ([i]_2, [j]_3) \in \{(0,1), (1,0)\}$.

Note that we cannot expect to be able to choose g to be synchronizing (take for example $Y = X$).

Thus the marked condition links the periodic points a^∞ and b^∞ by requiring something about allowed multiples of their periodis at the same time.

Since Y' clearly does not contain such a g, the marked condition tells us that X does not factor onto Y', even though X and Y' satisfy Thomsens condition.

In [T] it is suggested that Theorem 3.6.5 might describe all obstructions to the existence of factors and embeddings between sofic shift spaces of different entropy. This example shows that it is not the case.

To establish the claim, assume that $\varphi\colon \partial X \to Y$ is m-marked, and let $n \geq m$ such that $6|n-1$. Define $a^\infty := \varphi(a^\infty)$ and $b^\infty := \varphi(b^\infty)$. Then $\varphi_n(a^i) = a^{i-n+1}$ and $\varphi_n(b^i) = b^{i-n+1}$, for each $i \geq n-1$.

Define $g = \varphi_n(a^{n-1}b^{n-1})$. Then $\varphi_n(a^i b^j) = a^{i-n+1} g b^{j-n+1}$, for all $i, j \geq n-1$. Note that $\mathrm{MLW}(\partial X, m) = \{a^i b^j \mid ([i]_2, [j]_3) \in \{(0,1),(1,0)\}\}$. Let L and α_n, ω_n be ones from the definition of φ being marked, corresponding to n. Define $u = \alpha_n(b^{L-1}ca^{L+1})$ and $v = \omega_n(b^{L+1}ca^{L-1})$.

Let $i, j \in \mathbb{N}$, with the property $([i]_2, [j]_3) \in \{(0,1),(1,0)\}$. Then $([i+n-1]_2, [j+n-1]_3) \in \{(0,1),(1,0)\}$ because $6|n-1$. Thus $a^{i+n-1}b^{j+n-1} \in \mathrm{MLW}(\partial X_2, n)$. And by the definition of the marked property

$$ua^i g b^j v = \alpha_n(b^{L-1}ca^{L+1})\varphi_n(a^{i+n-1}b^{j+n-1})\omega_n(b^{L+1}ca^{L-1}) \in \mathbb{W}(Y).$$

This establishes the claim.

So the marked condition is a kind of generalization of affilliation, which works with more than one periodic point at a time in that it works with words of the form

$$up_1^{k_1}w_1p_2^{k_2}w_2\cdots p_n^{k_n}v, \tag{3.3}$$

where u and v are synchronizing, the w_i's are not necessarily synchronizing and $(k_1, k_2, \ldots, k_n)$ belong to some subset D of $\mathbb{N}^n$. The marked condition requires that whenever a sequence $\{p_i^\infty\} \subseteq \operatorname{Per} X$ satisfies that (3.3) is a word in X for all $(k_1, k_2, \ldots, k_n)$ in $D \subseteq \mathbb{N}^n$, then there must be sequences $\{q_i^\infty\} \subseteq \operatorname{Per} Y$ and $\{w_i'\} \in \mathbb{W}(Y)$ as well as synchronizing words $u', v' \in S(Y)$, such that

$$u'q_1^{k_1d_1}w_1'q_2^{k_2d_2}w_2'\cdots q_n^{k_nd_n}v',$$

where $d_i = \frac{|p_i|}{|q_i|}$, is a word in $\mathbb{W}(Y)$ for all $(k_1, k_2, \ldots, k_n) \in D$. But more on that in Chapter 5.

By Corollary 3.6.2 and the discussion above, the following result is a noteworthy strengthening of Theorem 7.4 in [T]:

Theorem 3.6.6. *Let X and Y be sofic shift spaces and $\varphi\colon X \to Y$ be a morphism. It follows that there is a map $X_c \to Y_{\varphi(c)}$ from the irreducible components of X to the irreducible components of Y, such that the following holds for all c:*

1. $\varphi(\overline{X_c}) \subseteq \overline{Y_{\varphi(c)}}$,
2. $\varphi_{|\partial\overline{X_c}}\colon \partial\overline{X_c} \to \overline{Y_{\varphi(c)}}$ *is marked,*
3. $E_{\overline{X_c},\overline{Y_{\varphi(c)}}} \subseteq \partial\overline{X_c}$.

Proof. Let X_c be an irreducible component in X. Then $\overline{X_c}$ is an irreducible sofic shift by Proposition 6.16 in [T], and it follows that so is $\varphi(\overline{X_c})$. By Lemma 7.1 in [T], there is therefore a (unique) irreducible component $Y_{\varphi(c)}$ in Y, such that $\varphi(\overline{X_c}) \subseteq \overline{Y_{\varphi(c)}}$ and $\varphi(\overline{X_c})_0 \cap Y_{\varphi(c)} \neq \emptyset$. Thus $\mathbb{W}(\varphi(\overline{X_c})) \cap S(\overline{Y_{\varphi(c)}}) \neq \emptyset$ and $\varphi_{|\partial\overline{X_c}} \colon \partial\overline{X_c} \to \overline{Y_{\varphi(c)}}$ is marked by Lemma 3.5.3 and Lemma 3.5.6. And $E_{\overline{X_c},\overline{Y_{\varphi(c)}}} \subseteq \partial\overline{X_c}$ by Lemma 2.7.11. □

3.7 Recoding Marked Morphisms

Just like 1-block morphisms are preferable over general morphisms, 1-marked morphisms are also preferable over general (m,n)-marked morphisms. It turns out that the exact same idea of recoding morphisms to 1-block morphisms turns marked morphisms into 1-marked morphisms. This follows from the following two lemmas.

Lemma 3.7.1. *Let $S \subseteq X$ and Y be shift spaces. If $\varphi\colon S \to Y$ is (m,n)-marked, then $\varphi \circ \sigma^d$ is* $(\max\{0, m-d\}, n+d)$*-marked for all $d \in \mathbb{N}$.*

Proof. Assume that $\varphi\colon S \to Y$ is (m,n)-marked and let L', α'_n and ω'_n be the ones from the definition of the marked property. Let $d \in \mathbb{N}$ be arbitrary and define $m' = \max\{0, m-d\}$ and $n' = n+d$. I need to show that $\varphi \circ \sigma^d$ is (m', n')-marked. Let therefore $n'' \geq n'$. Since $\varphi\colon S \to Y$ is $(m, n''-d)$-block, $\varphi \circ \sigma^d$ is (m', n'')-block and

$$(\varphi \circ \sigma^d)_{m',n''}(x_{[i,j]}) = \varphi_{m,n''-d}(x_{[i+\max\{0,d-m\},j]}),$$

for all $x_{[i,j]} \in \mathrm{LW}(S, m'+n''+1)$. That is the (m',n'')-word map of $\varphi \circ \sigma^d$ is almost equal to the $(m, n''-d)$-word map of φ. The only difference is that it trims off the first $\max\{0, d-m\}$ symbols.

Thus I can almost use the same α and ω maps that work for φ; I simply define the new α by attaching those symbols at the end of the corresponding words in the image of the original α and keep the ω.

Define $L = \max\{L', m+n''-d-1+\max\{0,d-m\}\}$, $\alpha_{n''}(x_{[-L,L]}) = \alpha'_{n''-d}(x_{[-L',L']})\varphi_{m,n''-d}(x_{[0,m+n''-d-1+\max\{0,d-m\}]})$, for each $x_{[-L,L]} \in A(n'', L)$, and $\omega_{n''}(x_{[-L,L]}) = \omega'_{n''-d}(x_{[-L',L']})$, for each $x_{[-L,L]} \in \Omega(n'', L)$. Then

$$\begin{aligned}\alpha_{n''}(x_{[i-L,i+L]})(\varphi \circ \sigma^d)_{m',n''}(x_{[i,j]})\omega_{n''}(x_{[j-L,j+L]}) = \\ \alpha'_{n''-d}(x_{[i-L',i+L']})\varphi_{m,n''-d}(x_{[i,j]})\omega'_{n''-d}(x_{[j-L',j+L']}) \in \mathbb{W}^*(Y),\end{aligned}$$

for all $x \in X$ and $[i,j] \in \mathrm{MLW}(S, m'+n''+1, x)$. Thus $\varphi \circ \sigma^d$ is (m', n'')-marked. □

Lemma 3.7.2. *Let $S \subseteq X$ and Y be shift spaces. If $\varphi\colon S \to Y$ is m-marked, then $\varphi^d = \varphi \circ \beta_d^{-1} \colon S^{[d]} \to Y$ is* $\max\{1, m-d+1\}$*-marked for all $d \in \mathbb{N}$.*

In order to prove Lemma 3.7.2, I need a connection between the entries and exits of words in X with the entries and exits of words in $X^{[d]}$.

Lemma 3.7.3. *Let $S \subseteq X$ be shift spaces, and $x \in X$. Then the following holds for all $d, n, L \in \mathbb{N}$:*

1. $x^{[d]}_{[-L,L]} \in A_{S^{[d]}}(n, L) \Rightarrow x_{[-L,L]} \in A_S(n + d - 1, L)$.
2. $x^{[d]}_{[-L,L]} \in \Omega_{S^{[d]}}(n, L) \Rightarrow x_{[-L+d-1,L+d-1]} \in \Omega_S(n + d - 1, L)$.
3. $[i, j] \in \mathrm{MLW}(S^{[d]}, n, x^{[d]}) \Leftrightarrow [i, j + d - 1] \in \mathrm{MLW}(S, n + d - 1, x)$.

Proof. Left to the reader. □

Note that both $x_{[-L,L]}$ and $x_{[-L+d-1,L+d-1]}$ can be determined when knowing only $x^d_{[-L,L]}$.

proof of Lemma 3.7.2. Assume that $\varphi\colon S \to Y$ is m-marked and let $d \in \mathbb{N}$. Let $n \geq \max\{1, m - d + 1\}$ and find $L \in \mathbb{N}$ corresponding to $n + d - 1$ and $\alpha_{n+d-1}\colon A(n + d - 1, L) \to S(Y)$ and $\omega_{n+d-1}\colon \Omega(n + d - 1, L) \to S(Y)$, using that φ is marked. Define $\alpha^d_n(x^d_{[-L,L]}) = \alpha_{n+d-1}(x_{[-L,L]})$, for each $x^d_{[-L,L]}$ in $A^d(n, L)$ and $\omega^d_n(x^d_{[-L,L]}) = \omega_{n+d-1}(x_{[-L+d-1,L+d-1]})$, for each $x^d_{[-L,L]}$ in $\Omega^d(n, L)$. This is well defined by Lemma 3.7.3.

Let $x \in X$ and $[i, j] \in \mathrm{MLW}(\partial X^{[d]}, n, x^d)$ and define $n' = n + d - 1$ and $j' = j + d - 1$. Then $[i, j'] \in \mathrm{MLW}(\partial X, n', x)$ and

$$\alpha^d_n(x^d_{[i-L,i+L]})\varphi^d_n(x^d_{[i,j]})\omega^d_n(x^d_{[j-L,j+L]}) = \alpha_{n'}(x_{[i-L,i+L]})\varphi_{n'}(x_{[i,j']})\omega_{n'}(x_{[j'-L,j'+L]}) \in \mathbb{W}^*(Y). \quad \square$$

3.8 Simplification of the Marked Property

When we want to verify that a given morphism $S \to Y$ is marked, the definition of the marked property forces us to check that we can find an $L \in \mathbb{N}$ and define maps α_n and ω_n with the right properties for infinitely many values of n. That can be cumbersome, as the reader may have noticed in Example 3.4.4. It would therefore be nice, if there was a class of shifts for which the marked condition reduces to the following:

Definition 3.8.1 (Simply Marked)**.** Let X and Y be shift spaces and S a subshift of X. A morphism $\varphi\colon S \to Y$ is called m-simply marked, if there is an $L \in \mathbb{N}$ and maps $\alpha\colon A_S(m, L) \to S(Y)$ and $\omega\colon \Omega_S(m, L) \to S(Y)$ such that

$$\alpha(x_{[i-L,i+L]})\varphi_m(x_{[i,j]})\omega(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $x \in X$ and $[i, j] \in \mathrm{MLW}(S, m, x)$.

Remark 3.8.2. To simplify notation, I have assumed that the morphism $S \to Y$ has memory 0. But I could of course define (m, n)-simply marked in the obvious way, and Lemma 3.8.3 below would still hold with m replaced by (m, n).

The following Lemma shows that all that is needed to ensure that

$$\text{'}m\text{-marked} \Leftrightarrow m\text{-simply marked'}$$

is that S has synchronizing edges, X is irreducible and that m is large enough to ensure that $\mathrm{LW}(S, m) \neq \mathbb{W}^*_{\geq m}(X)$.

Lemma 3.8.3. *Let X and Y be shift spaces, X irreducible and S a subshift of X with synchronizing edges. If a morphism $\varphi\colon S \to Y$ is m-simply marked for some $m \in \mathbb{N}$, such that $\mathrm{LW}(S, m) \neq \mathbb{W}^*_{\geq m}(X)$, then it is m-marked.*

Note that if $\mathrm{LW}(S, m) = \mathbb{W}^*_{\geq m}(X)$, then a morphism $S \to Y$ with block length less than or equal to m is m-simply marked if and only if it is a morphism from $\mathrm{L}(S, m) = X$ to Y, since there are no entries nor exits. So since I want to use the marked property as a condition for a morphism $\varphi\colon S \to Y$ to extend to a morphism $X \to Y$, the assumption on m is natural, because otherwise φ would already be a morphism from X to Y.

Proof. The idea of the proof is to exploit the fact that each word corresponding to a maximum local word of order $n \geq m$ is a local word of order m and hence contained in a maximal local word of order m. Thus if $[i, j] \in \mathrm{MLW}(S, n, x)$, then I can find a $y \in X$ and $[i', j'] \in \mathrm{MLW}(S, m, y)$ such that $x_{[i,j]} \subseteq y_{[i',j']}$. This implies that $\alpha(y_{[i'-L,i'+L]})\varphi_m(y_{[i',j']})\omega(y_{[j'-L,j'+L]})$, which has the form $a\varphi_n(x_{[i,j]})b$ for some $a, b \in S(Y)$, is in $\mathbb{W}^*(Y)$. Thus I can simply define α_n and ω_n to be a and b respectively.

The irreducibility of X and the assumption on m are used to ensure that i' and j' can be chosen to be finite if i and j are finite. The synchronizing edges assumption is used to ensure that a and b can be found using only the entry and exit.

Assume that $\varphi\colon S \to Y$ is m-simply marked. It is convenient to assume that the $\{L_n\}_{n\in\mathbb{N}}$ sequence from the definition of synchronizing edges is increasing, that L_m works for L in the definition of φ being simply marked and that $L_n \geq n$ for all $n \in \mathbb{N}$.

Let $n \geq m$ and define $L = L_n$.

In order to define $\alpha_n\colon A_S(n, L) \to S(Y)$ I let $x_{[i-L,i+L]} \in A_S(n, L)$ be arbitrary. The i is of course irrelevant, but it turns out to be convenient during the verification.

By the assumption on m, I can find a word $v \in \mathbb{W}(X)$, which is not in $\mathrm{LW}(S, m)$. And the irreducibility of X implies that I can find a $w \in \mathbb{W}(X)$, such that $vwx_{[i-L,i+L]} \in \mathbb{W}(X)$. The definition of languages enables me to choose a $u \in \mathbb{W}_{L_m}(X)$, such that $uvwx_{[i-L,i+L]} \in \mathbb{W}(X)$. The purpose of

u is to make sure that $uvwx_{[i-L,i+L]}$ contains an entry of order m. I define $y_{[-k,i+L]} = uvwx_{[i-L,i+L]}$, $i' = \max\{a \le i \mid y_{[a,i+L]} \notin \mathrm{LW}(S,m)\} + 1$ and

$$\alpha_n(x_{[i-L,i+L]}) = \alpha(y_{[i'-L_m,i'+L_m]})\varphi_m(y_{[i',i+m-1[}).$$

This is well defined since $[i' - L_m, i' + L_m] \cup [i', i + m - 1[\subseteq [-k, i + L]$, because of my choice of u and because $L \ge L_m \ge m$ by assumption, and since $y_{[i'-L_m,i'+L_m]} \in A_S(n, L_m)$ by my choice of i'.

The definition of ω_n is analogous: Let $x_{[j-L,j+L]} \in \Omega(S,n)$ be arbitrary. By the assumption on m, I can find a word $v \in \mathbb{W}(X)$, which is not in $\mathrm{LW}(S,m)$. And the irreducibility of X implies that I can find a $w \in \mathbb{W}(X)$, such that $x_{[j-L,j+L]}wv \in \mathbb{W}(X)$. I choose a $u \in \mathbb{W}_{L_m}(X)$, such that $x_{[j-L,j+L]}wvu \in \mathbb{W}(X)$, and define $y_{[j-L,k]} = x_{[j-L,j+L]}wvu$, $j' = \max\{a \ge j \mid y_{[j-L,a]} \notin \mathrm{LW}(S,m)\} - 1$ and

$$\omega_n(x_{[j-L,j+L]}) = \varphi_m(y_{]j-n+1,j']})\omega(y_{[j'-L_m,j'+L_m]}).$$

This is well defined since $]j - n + 1, j'] \cup [j' - L_m, j' + L_m] \subseteq [j - L, k]$ and $y_{[j'-L_m,j'+L_m]} \in \Omega_S(n, L_m)$ by my choice of u and j' and since $L = L_n \ge n$ by assumption.

To verify that L, α_n and ω_n work, I take an arbitrary $x \in X$ and $[i, j] \in \mathrm{MLW}(S, n, x)$. Then

$$\begin{gathered}\alpha_n(x_{[i-L,i+L]})\varphi_n(x_{[i,j]})\omega_n(x_{[j-L,j+L]}) = \\ \alpha(y_{[i'-L_m,i'+L_m]})\varphi_m(y_{[i',i+m-1[})\varphi_n(x_{[i,j]})\varphi_m(y_{]j-n+1,j']})\omega(y_{[j'-L_m,j'+L_m]}) = \\ \alpha(y_{[i'-L_m,i'+L_m]})\varphi_m(y_{[i',j']})\omega(y_{[j'-L_m,j'+L_m]}).\end{gathered}$$

So as $y_{[i-L,i+L]}$ and $y_{[j-L,j+L]}$ both are synchronizing by my choice of L, $y_{[i'-L_m,j'+L_m]}$ is in $\mathbb{W}(X)$. I can therefore find a point $y \in X$ such that $[i', j'] \in \mathrm{MLW}(S, n, y)$, by my choice of i' and j'. This implies that

$$\alpha(y_{[i'-L_m,i'+L_m]})\varphi_m(y_{[i',j']})\omega(y_{[j'-L_m,j'+L_m]}) \in \mathbb{W}^*(Y),$$

and hence that

$$\alpha_n(x_{[i-L,i+L]})\varphi_n(x_{[i,j]})\omega_n(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

which is what I needed to prove. □

Example 3.8.4. This example is a continuation of Example 3.4.4 in which I show that for $n = 1$, I can find L, α_n and ω_n, which work in the definition of the marked property for the map $\varphi\colon \partial X \to Y$, which maps $\partial X = \{0^\infty\}$ to itself, and then I leave the verification of the rest of the n's to the reader. Lemma 3.8.3 implies that the reader did not have to do anything, since the argument for $n = 1$ implies that the map $0^\infty \mapsto 0^\infty$ is 1-simply marked and therefore 1-marked.

Chapter 4

The Factor Problem

I am now ready to prove my main result on the lower entropy factor problem. It gives a necessary and sufficient condition for a morphism $\varphi\colon S \to Y$ to extend to a factor map $X \twoheadrightarrow Y$, when X is irreducible sofic, Y is mixing sofic such that $\mathrm{h}(Y) < \mathrm{h}(X)$ and S is a subshift of X with synchronizing edges such that $E_{X,Y} \subseteq S \subseteq \partial X$.

4.1 Extension Results

Recall that

$$E_{X,Y} = \{x \in \operatorname{Per} X \mid \bigcup_{m \| p_x|} Q_m(Y_0^{(\frac{|p_x|}{m})}) = \emptyset\},$$

and that if $X \twoheadrightarrow Y$, then $E_{X,Y} \subseteq \partial X$.

Theorem 4.1.1 (Main Result)**.** *Let X be an irreducible sofic shift, Y be a mixing sofic shift such that $\mathrm{h}(X) > \mathrm{h}(Y)$ and S be a subshift of X such that $E_{X,Y} \subseteq S \subseteq \partial X$. Then a morphism $\varphi_S\colon S \to Y$ extends to a factor map $\varphi\colon X \twoheadrightarrow Y$ if it is marked. And the converse holds if S has synchronizing edges.*

Proof. If φ_S is marked, it extends to a factor map $X \twoheadrightarrow Y$ by Proposition 4.1.3 below. And if φ is a factor map from X to Y and $S \subseteq X$ has synchronizing edges, then $\varphi_{|S}$ is marked by Lemma 3.5.4. □

Corollary 4.1.2. *Let X be an irreducible sofic shift and Y be a mixing sofic shift such that $\mathrm{h}(X) > \mathrm{h}(Y)$. Then a morphism $\varphi_{\partial X}\colon \partial X \to Y$ extends to a factor map $\varphi\colon X \twoheadrightarrow Y$ if and only if it is marked and $E_{X,Y} \subseteq \partial X$.*

Proof. $\Rightarrow$: That $\varphi_{\partial X}$ must be marked follows from Theorem 4.1.1, since ∂X has synchronizing edges by Lemma 3.5.6. And $E_{X,Y} \subseteq \partial X$ follows from Lemma 2.7.11. $\Leftarrow$: Follows from Theorem 4.1.1 since ∂X has synchronized edges by Lemma 3.5.6. □

The following result shows that the existence of a marked map $S \to Y$, for some $E_{X,Y} \subseteq S \subseteq \partial X$, is always sufficient to ensure that X factors onto Y.

Proposition 4.1.3. *Let X be an irreducible sofic shift, Y be a mixing sofic shift such that* $\mathrm{h}(X) > \mathrm{h}(Y)$ *and S be a subshift of X such that $E_{X,Y} \subseteq S \subseteq \partial X$. If a morphism $\varphi_S \colon S \to Y$ is marked, then φ_S extends to a factor map $\varphi \colon X \twoheadrightarrow Y$.*

Proof. I am going to use the marker strategy from Section 2.10, i.e. I want to apply Lemma 2.10.9.

I am going to define five different types of locally recognizable intervals that may occur in a point in X:

1. The W intervals, I_W,
2. the local intervals, I_{LW},
3. the long marker intervals, I_{long},
4. the moderate marker intervals, I_{mod}, and
5. the remaining intervals, I_{rem}.

Define $\tau = \{W, \mathrm{LW}, \mathrm{long}, mod, rem\}$.

The intervals will be constructed in a way such the intervals in

$$\bigcup_{t \in \tau} I_t(x)$$

are disjoint and they cover all of $\mathbb{Z}$ for all $x \in X$. Because then the map

$$I_\tau \colon X \to I_{\mathbb{Z}} \times \tau$$

defined by $I_\tau(x) = \sqcup_{t\in\tau} I_t(x)$ is a splitting map.

For each $(w, t) \in \mathbb{W}_{M,\tau}$, for some M, I define $\overline{\varphi}(w, t)$, such that $\overline{\varphi} \colon \mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$ satisfy the four properties in Lemma 2.10.9. Then the induced map φ is a morphism $X \to Y$. To ensure that φ is a factor map and that it extends φ_S, I find a subshift $W \subseteq X$ and a factor map $\varphi_W \colon W \twoheadrightarrow Y$ using Lemma 2.8.4 and make sure that φ extends both φ_W and φ_S, by defining I_W and I_{LW} such that they send points in W and S to $\mathbb{Z}$ respectively, and defining $\overline{\varphi}$ such that it behaves like φ_W on the W words $\mathbb{W}_W$ and like φ_S on the local words $\mathbb{W}_{\mathrm{LW}}$ using that φ_S is marked.

Set up. Assume that $\varphi_S \colon S \to Y$ is (m,n)-marked.

Let W, D and TL be as in Lemma 2.8.4. and let φ_W be the factor map $W \twoheadrightarrow Y$. By replacing TL by a multiple of TL if necessary I can make sure that TL is a transition length of Y.

Using Lemma 3.7.1 and 3.7.2 I can pass to a higher block shift of X to ensure that W is 1-step, $D = 1$, φ_W is 1-block and φ_S is 1-marked. i.e $W \subseteq X$ is a 1-step SFT with the following three properties:

1. $\Sigma_W \subseteq S(X)$.

2. φ_W is a 1-block factor map from W onto Y.
3. Any two words in $\mathbb{W}(W)$ can be connected by a word in $\mathbb{W}_{TL}(W)$.

Let L, N, $\alpha_{1,k}\colon A(1,k,L) \to S_N(Y)$ and $\omega_1\colon \Omega(1,L) \to S_N(Y)$ be the numbers and functions corresponding to $n = 1$ in Lemma 3.4.7. Then

$$\alpha_{1,k}(x_{[i-L,i+k]})\varphi_S(x_{[i+k,j]})\omega_1(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $k > L$, $x \in X$ and $[i,j] \in \mathrm{MLW}(S,1,x)$.

Fix an $s \in S(Y)$ and let R be the number from Lemma 2.7.16 corresponding to Y.

Define

$$T = |s| + 3TL + L + 2R + 2N.$$

Construction. When defining $\overline{\varphi}$ I only need to work with $\mathbb{W}_{M,\tau}$ on the local words. In all other cases I don't need the M, and therefore define for example $\overline{\varphi}\colon \mathbb{W}_W \times \{W\} \to \mathbb{W}^*(Y)$. The resulting $\overline{\varphi}$ can of course be seen as a map $\mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$, since it corresponds to ignoring the first and last M symbols of the word w in a point $(w,t) \in \mathbb{W}_{M,\tau}$, if $t \neq \mathrm{LW}$.

The W words. Let $x \in X$ and $[i,j]$ be an interval in $\mathrm{MW}(W,x)$ which is longer than $2T$. Then I call $[i+T+N+TL, j]$ a W interval in x. In other words $I_W\colon X \to I_{\mathbb{Z}}$ is defined by

$$I_W(x) = \{[i+T+N+TL,j] \mid [i,j] \in \mathrm{MW}(W,x), |[i,j]| \geq 2T\}.$$

Note that the W intervals in a point in X are disjoint because W is 1-step. I claim that $N_W = 2T - 1$ works in the definition of locally recognizability for the W intervals (Def. 2.10.3). To see that, let $x \in X$ and $k \in \mathbb{Z}$. Then k occurs in a W interval in x if and only if $x_{[k-N_W,k+N_W]}$ contains a word in $\mathbb{W}_{\geq 2T}(W)$ and $x_{[k-(T+N+TL),k]} \in \mathbb{W}(W)$. And if k occurs in a W interval, then it is an endpoint if and only if $x_{[k-1-(T+N+TL),k-(T+N+TL)]} \notin \mathbb{W}(W)$. Thus N_W works, which proves that I_W is locally recognizable.

Using the surjectivity of φ_W, I find an $s_0 \in \mathbb{W}(W)$, such that $\varphi_W(s_0) = s$. And for each $w \in \mathbb{W}^*(W)$ I find a pair of words $a_-, a_+ \in \mathbb{W}_{TL}(W)$ such that $s_0a_-wa_+s_0 \in \mathbb{W}^*(W)$. This can be done by knowing only the start and end symbol of w respectively, since W is 1-step. Then $\varphi_W(s_0a_-wa_+)s \in \mathbb{W}^*(Y)$ for all $w \in \mathbb{W}^*(W)$.

Let $\mathbb{W}_W$ be the W words:

$$\mathbb{W}_W = \bigcup_{x \in X} \{x_{[i,j]} \mid [i,j] \in I_W(x)\}.$$

Define for each $x_{[i,j]} \in \mathbb{W}_W$:

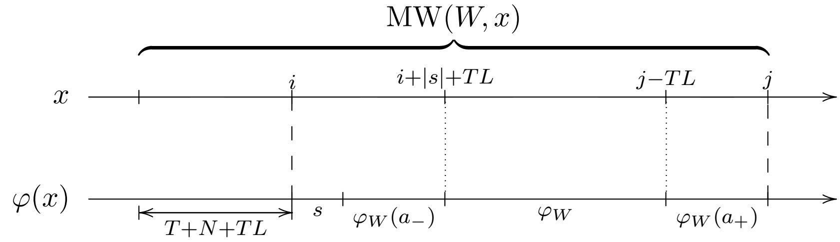
$$\overline{\varphi}(x_{[i,j]}, W) = \varphi_W(s_0a_-x_{[i+|s|+TL,j-TL]}a_+).$$

Then $\overline{\varphi}$ is well-defined on the W words because

$$|x_{[i+|s|+TL,j-TL]}| = |[i,j]| - (|s|+2TL) \geq 2T - (T+N+TL) - (|s|+2TL) \geq 0,$$

and because $s_0 a_- x_{[i+|s|+TL,j-TL]} a_+ \in \mathbb{W}^*(W)$ by definition of the a's.

The definition is illustrated in the following figure:



Now φ extends φ_W because if x is a point in W, then x is a W word, which implies that $\varphi(x) = \overline{\varphi}(x, W) = \varphi_W(x)$.

I will now show that $\overline{\varphi}$ satisfies the four properties from Lemma 2.10.9 on the W words. The first two follows from my choice of a_-, a_+ and s_0. The third follows from the fact that φ_W is 1-block and that $|s_0 a_- x_{[i+|s|+TL,j-TL]} a_+| = |[i,j]|$. And $K_W = |s| + TL + 1$ works in the fourth property: Let $w \in \mathbb{W}_W$ and $k \in [0, |w|[$. I need to show that I can determine $\overline{\varphi}(w, W)_k$ when knowing only $w_{[a,b]} = w_{[0,|w|[\cap[k-K_W,k+K_W]}$.

If $a > k - K_W$, then $a = 0$ and I know $w_{[0,|s|+TL]}$, since $[0, K_W] \subseteq [a, b]$. I can therefore determine the a_- used to define $\overline{\varphi}$ on w, because a_- only depends on $w_{|s|+TL}$. I can therefore determine $\overline{\varphi}(w, W)_k$ as the kth symbol in $\varphi_W(s_0 a_- w_{[|s|+TL]})$.

If $b < k + K_W$, then $b = |w| - 1$ and I can determine the a_+ used to define $\overline{\varphi}$ on w, since I know $w_{|w|-TL-1}$, which is all a_+ depends on. I can therefore determine $\overline{\varphi}(w, W)_k$ as the kth symbol in $\varphi_W(w_{[a,|w|-TL[} a_+)$.

If $[a, b] = [k - K_W, k + K_W]$, then I don't know the endpoints of w, but it doesn't matter since on the center of w, $\overline{\varphi}$ is simply the 1-block map φ_W. Thus $\overline{\varphi}(w)_k = \varphi_W(w_k)$.

The local intervals. Define for each $i, j \in \mathbb{Z}$:

$$i_0 = i + T + N + TL \text{ and } j_0 = j + N + TL.$$

Let $x \in X$ and $[i,j]$ be an interval in $\mathrm{MLW}(S, 1, x)$ which is longer than $2T$. Then I call $[i_0, j_0]$ a local interval. In other words I_{LW} is defined by

$$I_{\mathrm{LW}}(x) = \{[i_0, j_0] \mid [i,j] \in \mathrm{MLW}(S, 1, x), |[i,j]| \geq 2T\}.$$

Note that the local intervals in a point in X are disjoint by Lemma 3.1.10. I claim that $N_{\mathrm{LW}} = 2T + N + TL - 1$ works in the definition of locally recognizable for the local intervals. To see that let $x \in X$ and $k \in \mathbb{Z}$. Then k occurs in a local interval if and only if there is a subinterval $[i,j] \subseteq [k - N_{\mathrm{LW}}, k + N_{\mathrm{LW}}]$ such that $x_{[i,j]} \in \mathrm{LW}(S,1,x)$, $|[i,j]| \geq 2T$ and $k \in [i_0, j_0]$. And if k occurs in a local interval, then it is an endpoint if and only if $x_{k-T-N-TL-1} \notin \mathrm{LW}(S,1,x)$. Thus N_{LW} works, which implies that I_{LW} is locally recognizable.

Find for each $s' \in S_N(Y)$ words $u, v \in \mathbb{W}_{TL}(Y)$ such that $sus', s'vs \in \mathbb{W}(Y)$ and define

$$k = i_0 - i + |s| + TL + N = T + |s| + 2N + 2TL.$$

Then $k > T > L$, which implies that

$$su\alpha_{1,k}(x_{[i-L,i+k)}\varphi_S(x_{[i+k,j]})\omega_1(x_{[j-L,j+L]})vs \in \mathbb{W}^*(Y),$$

for all $x \in X$ and $[i,j] \in \mathrm{MLW}(S,1,x)$ by Lemma 3.4.7.

Let $\mathbb{W}_{\mathrm{LW}}$ be the local words:

$$\mathbb{W}_{\mathrm{LW}} = \bigcup_{x \in X} \{x_{[i_0,j_0]} \mid [i_0,j_0] \in I_{\mathrm{LW}}(x)\}.$$

Define

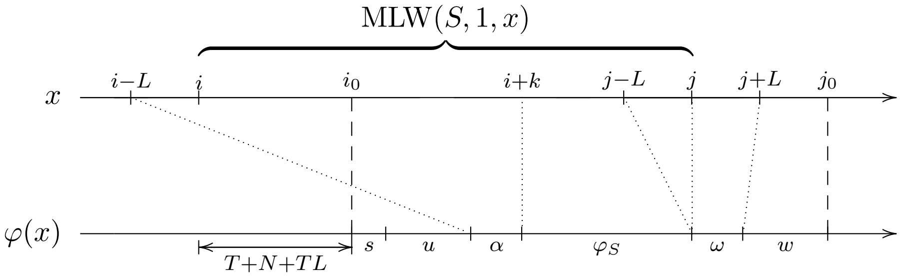
$$M = i_0 - i + L = T + N + TL + L$$

and for each $x_{[i_0,j_0]} \in \mathbb{W}_{\mathrm{LW}}$:

$$\overline{\varphi}(x_{[i_0-M,j_0+M]}, \mathrm{LW}) = su\alpha_{1,k}(x_{[i-L,i+k[})\varphi_S(x_{[i+k,j]})\omega_1(x_{[j-L,j+L]})v.$$

Then $\overline{\varphi}$ is well-defined on the local words because $[i-L, j+L]$ is a subset of $[i_0 - M, j_0 + M]$, by my choice of M.

The definition is illustrated in the following figure:

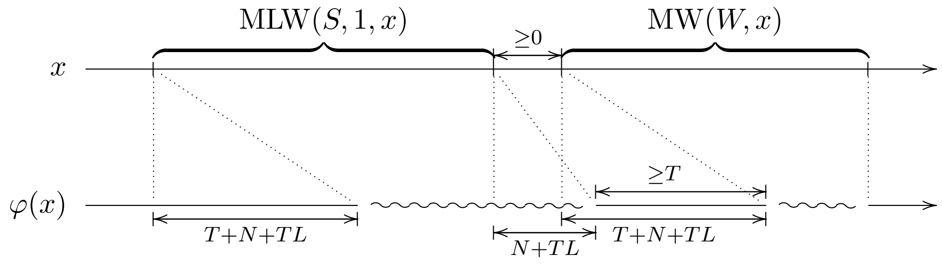


$\overline{\varphi}$ maps the local words into $W^*(Y)$ because of my choice of u and v and because $\alpha_{1,k}(x_{[i-L,i+k[})$ and $\omega_1(x_{[j-L,j+L]})$ are synchronizing words in Y.

Now φ extends φ_S because if x is a point in S, then x is in $\mathbb{W}_{\mathrm{LW}}$, which implies that $\varphi(x) = \overline{\varphi}(x, \mathrm{LW}) = \varphi_S(x)$.

$\overline{\varphi}$ satisfies the four properties from Lemma 2.10.9 on the local words: The first is obviously satisfied. The second follows from my choice of v. The third follows from the choice of k, i_0 and j_0. And with an argument very similar to the one used to prove that K_W works for W words, $K_{\mathrm{LW}} = L + k + 1$ works in the fourth property for local words.

By removing $T+N+TL$ symbols from the beginning of each interval in $\mathrm{MW}(S,x)$ and $\mathrm{MLW}(S,1,x)$, the distance between any two intervals in $I_W(x)\cup I_{\mathrm{LW}}(x)$, is at least T, for any $x\in X$, since words from $\mathrm{MW}(W,x)$ and $\mathrm{MLW}(S,1,x)$ are disjoint because $\Sigma_W\subseteq S(X)$ and $S\subseteq\partial X$. This is illustrated by the following figure, where the wavy stretches correspond to intervals for which $\varphi(x)$ has been defined:



The long marker intervals. There may still be arbitrarily long intervals for which $\varphi(x)$ for a given x is undefined. To remedy this I use Krieger's marker lemma, which for each $k>T$ splits an $x\in X$ into (k,T)-marker intervals. Using Lemma 2.9.7 I choose $k>4T+N+TL$ so large, that when $[i,j]$ is a marker interval which is at least $2T$ long, then there is a unique (up to shifts) $z\in\mathrm{Per}_{<T}(X)$ such that $x_{[i+T-k,j-T+k]}\subseteq z$. Let $|F|$ be the number from Lemma 2.9.8 corresponding to k and T.

I claim that there for any $x\in X$ is a distance of at least T between a marker interval $[i,j]$ in x, which is at least $2T$ long and for which $\varphi(x)_n$ is undefined for some $n\in[i,j]$ and any interval for which $\varphi(x)$ has been defined so far, i.e. any interval in $I_W(x)\cup I_{\mathrm{LW}}(x)$. To see that, assume on the contrary that there is an interval $[i',j']\in I_W(x)\cup I_{\mathrm{LW}}(x)$ such that $[i'-T,j'+T]\cap[i,j]\neq\emptyset$. Then $[i',j']$ and $[i+T-k,j-T+k]$ overlap by at least $k-2T>2T+N+TL$ symbols. This implies that $[i+T-k,j-T+k]$ overlaps an interval in $\mathrm{MLW}(S,1,x)\cup\mathrm{MW}(W,x)$ by at least $2T$ symbols. So as $x_{[i+T-k,j-T+k]}$ is $<T$-periodic, $[i+T-k,j-T+k]$ is a subinterval of one of the intervals in $\mathrm{MLW}(S,1,x)$ or $\mathrm{MW}(W,x)$ by definition of $\mathrm{MLW}(S,1,x)$ and $\mathrm{MW}(W,x)$. But then $\varphi(x)_{[i,j]}$ is defined by the rules above, since $|x_{[i+T-k,j-T+k]}|>2T$ and $k-T>T+N+TL$. This contradicts the assumption that $\varphi(x)_n$ is undefined for some $n\in[i,j]$. I have therefore established the claim.

Let $x\in X$ and $[i,j]$ be a marker interval in x which is longer than $3T$ and for which φ is undefined for some $n\in[i,j]$. Then I call $[i,j]$ a long marker interval. In other words $I_{\mathrm{long}}(x)$ is the set

$$\{[i,j]\in I_{k,T}(x)\mid |[i,j]|\geq 3T,\exists n\in[i,j]\forall[i',j']\in I_W(x)\cup I_{\mathrm{LW}}(x)\colon n\notin[i',j']\}.$$

Then the intervals in $I_{\mathrm{long}}(x)$ are disjoint for all $x\in X$ by the marker lemma. I claim that $N_{\mathrm{long}}=\max\{3T+|F|,N_W,N_{\mathrm{LW}}\}$ works in the definition of locally recognizable for the long marker words. To see that let $x\in X$ and $i\in\mathbb{Z}$.

By inspection of $x_{[i-3T-|F|,i+3T+|F|]}$, I can determine whether i occurs in a marker interval of length at least $3T$. If it does, I look at $x_{[i-N_W,i+N_W]}$ and $x_{[i-N_{\mathrm{LW}},i+N_{\mathrm{LW}}]}$ to check whether i occurs in a W interval or a local interval. If and only if i does not occur in either, then it occurs in a long marker word by definition. Thus I_{long} is locally recognizable.

By the argument above there is now a distance of at least T between any two intervals from $I_W(x) \cup I_{\mathrm{LW}}(x) \cup I_{\mathrm{long}}(x)$ for all $x \in X$.

Since $\operatorname{Per} X \to \operatorname{Per} Y$ by Corollary 3.6.2, there is a shift-commuting map $\lambda\colon\ \operatorname{Per} X - E_{X,Y} \to \operatorname{Per} Y$ such that

$$x \in Q_n(X) \Rightarrow \lambda(x) \in \bigcup_{m|n} Q_m(Y_0^{(\frac{n}{m})}), \tag{4.1}$$

for all $x \in \operatorname{Per} X - E_{X,Y}$. Choose for each $z \in \operatorname{Per} X - E_{X,Y}$ a minimal period $p_z \in \mathbb{W}(X)$ such that $p_z = p_{z'}$ when $z = \sigma^n(z')$ for some $n \in \mathbb{N}$. Choose also $i \in \mathbb{Z}$ such that $z_{[i,i+|p_z|[} = p_z$. Since R is the number from Lemma 2.7.16 corresponding to Y and TL is a transition length for Y, it follows from (4.1), that I for all $n, m \geq R + TL$ can find $u_n \in S_n(Y)$ and $v_m \in S_m(Y)$ depending only on p_z, n and p_z, m respectively such that

$$s u_n \lambda(z)^k_{[i,i+|p_z|[} v_m s \in \mathbb{W}^*(Y),$$

for all $k \in \mathbb{N} \cup \{\infty\}$.

Let $\mathbb{W}_{\mathrm{long}}$ denote the local words:

$$\mathbb{W}_{\mathrm{long}} = \bigcup_{x \in X} \{x_{[i,j]} \mid [i,j] \in I_{\mathrm{long}}(x)\}.$$

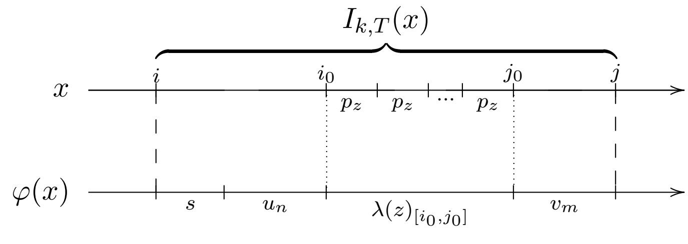
Define for each $x_{[i,j]} \in \mathbb{W}_{\mathrm{long}}$:

$$\overline{\varphi}(x_{[i,j]}, \mathrm{long}) = s u_n \lambda(z)_{[i_0,j_0[} v_m,$$

where $i_0 = \min\{k \geq i + |s| + TL + R \mid p_z = x_{[k,k+|p_z|[}\}$, $j_0 = \max\{k \leq j - (TL + R) | p_z = x_{[k-|p_z|,k[}\}$, $n = i_0 - i - |s|$ and $m = j - j_0 + 1$.

i_0 and j_0 are well-defined because $|x_{[i+|s|+TL+R,j-(TL+R)]}| > 2T > 2|p_x|$, which implies that $p_x \subseteq x_{[i+|s|+TL+R,j-(TL+R)]}$. And u_n, v_m are well-defined because $n, m \geq TL + R$.

The following figure illustrates the definition:



Now $\overline{\varphi}$ satisfies the four properties from Lemma 2.10.9 on the long marker words: The first is obviously satisfied. The second follows from my choice of v_m. The third follows from the choice of i_0 and j_0. And with an argument very similar to the one used to prove that K_W works for W words, $K_{\text{long}} = |s| + TL + R + 3T$ works in the fourth property for long marker words, since $|su_n|, |v_m| < K_{\text{long}} - 2T$ and it requires only knowledge of $2T$ consecutive symbols in a word $w \in \mathbb{W}_{\text{long}}$ to determine z and therefore p_z by Lemma 2.9.5.

The moderate marker intervals. Let $x \in X$ and $[i, j]$ be a marker interval in x whose distance to any of the intervals in $I_W(x) \cup I_{\text{LW}}(x) \cup I_{\text{long}}(x)$ is at least T. Then I call $[i, j]$ a moderate marker interval. In other words $I_{\text{mod}}(x)$ is the set

$$\{[i, j] \in I_{k,T}(x) \mid \forall [i', j'] \in I_W(x) \cup I_{\text{LW}}(x) \cup I_{\text{long}}(x) \colon [i', j'] \cap [i-T, j+T] = \emptyset\}.$$

The intervals in $I_{\text{mod}}(x)$ are disjoint for all $x \in X$ by the marker lemma. I claim that $N_{\text{mod}} = 4T + |F| + \max\{N_W, N_{\text{LW}}, N_{\text{long}}\}$ works in the definition of locally recognizable for the moderate marker intervals. To see that let $x \in X$ and $k \in \mathbb{Z}$. By inspection of $x_{[k-|F|, k+|F|]}$ I can tell whether or not k is a marker in x. And by inspection of $x_{[k-3T-|F|, k+3T+|F|]}$ I can determine whether k occurs in a marker interval $[a, b]$ of length strictly less than $3T$. If it does, I look at $x_{[a-T-N_W, b+T+N_W]}$, $x_{[a-T-N_{\text{LW}}, b+T+N_{\text{LW}}]}$ and $x_{[a-T-N_{\text{long}}, b+T+N_{\text{long}}]}$ to check whether the distance between $[a, b]$ and any interval in $I_W(x) \cup I_{\text{LW}}(x) \cup I_{\text{long}}(x)$ is at least T. If it is, then k occurs in a moderate marker interval. Since the converse is clearly also true, I have established the claim. Hence I_{mod} is locally recognizable.

Let for all $n \in \mathbb{N}$ $\phi \colon \mathbb{W}_n(X) \to \mathbb{W}_n(Y)$ be an arbitrary map. Find for each word $w \in \mathbb{W}(X)$ a pair of words $b_-, b_+ \in \mathbb{W}_{TL}(Y)$, depending only on s and w, such that $sb_-\overline{\varphi}(w)b_+s \in \mathbb{W}(Y)$.

Let $\mathbb{W}_{\text{mod}}$ denote the moderate marker words

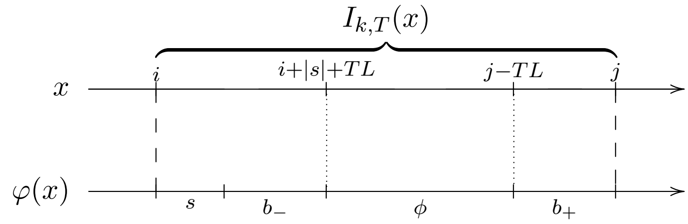
$$\mathbb{W}_{\text{mod}} = \bigcup_{x \in X} \{x_{[i,j]} \mid [i, j] \in I_{\text{mod}}(x)\}$$

and define for each $x_{[i,j]} \in \mathbb{W}_{\text{mod}}$:

$$\overline{\varphi}(x_{[i,j]}, \text{mod}) = \varphi(x)_{[i,j]} = sb_-\phi(x_{[i+|s|+TL, j-TL]})b_+.$$

That is possible because a marker interval is at least T symbols long by definition, which implies that $[i + |s| + TL, j - TL]$ is non-empty. Note that $[i, j]$ is strictly shorter than $3T$, since otherwise $x_{[i,j]}$ would be a long marker word.

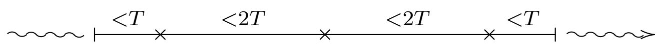
The following figure illustrates the definition:



The argument from the 'long marker words' section of the proof implies that φ has now been defined on all marker intervals of length at least $2T$.

$\overline{\varphi}$ satisfies the four properties from Lemma 2.10.9 on the moderate marker words: The first is obviously satisfied. The second follows from my choice of b_+. The third follows from the fact that ϕ is length preserving. And $K_{\text{mod}} = 3T$ works in the fourth property for moderate marker words, since $w_{[0,|w|[\cap[k-K_W,k+K_W]} = w$ for all $w \in I_{\text{mod}}$ and $k \in [0, |w|[$. So I can see all of w, which means that I can find b_- and b_+ by local inspection.

The remaining intervals. Now there is a gap of at least T symbols and at most $6T-4$ symbols between intervals for which φ is defined on an $x \in X$. This perhaps surprisingly high upper limit is the smallest possible, as illustrated by the following figure, where x's correspond to markers and the wavy stretches correspond to intervals for which $\varphi(x)$ has been defined so far.



Let $x \in X$ and define $I_{\text{rem}}(x)$ to be the set of these remaining intervals. i.e. $I_{\text{rem}}(x)$ is the set

$$\{[i,j] \in I_{\text{undefined}}(x) \mid [i-1,j] \notin I_{\text{undefined}}(x), [i,j+1] \notin I_{\text{undefined}}(x)\},$$

where $I_{\text{undefined}}(x)$ is the set

$$\{[i,j] \mid \forall [i',j'] \in I_W(x) \cup I_{\text{LW}}(x) \cup I_{\text{long}}(x) \cup I_{\text{mod}}(x)\colon [i',j'] \cap [i,j] = \emptyset\}.$$

Then clearly the intervals in $I_{\text{rem}}(x)$ are disjoint for all $x \in X$ and

$$N_{\text{rem}} = \max\{N_W, N_{\text{LW}}, N_{\text{long}}, N_{\text{mod}}\} + 1$$

works in the definition of locally recognizable for the remaining words, since a $k \in \mathbb{Z}$ is in $I_{\text{rem}}(x)$ for some $x \in X$ if and only if k is not in $I_t(x)$ for all $t \in \{W, \text{LW}, \text{long}, mod\}$, and if so it is an endpoint if and only if $k-1 \notin I_{\text{rem}}(x)$. Hence I_{rem} is locally recognizable.

Let $\mathbb{W}_{\text{rem}}$ denote the remaining words:

$$\mathbb{W}_{\text{rem}} = \bigcup_{x \in X} \{x_{[i,j]} \mid [i,j] \in I_{\text{rem}}(x)\}$$

and define for each $x_{[i,j]} \in \mathbb{W}_{\text{rem}}$:

$$\overline{\varphi}(x_{[i,j]}, \text{rem}) = \varphi(x)_{[i,j]} = sb_{-}\phi(x_{[i+|s|+TL, j-TL]})b_{+},$$

where $\phi\colon \mathbb{W}(X) \to \mathbb{W}(Y)$ is an arbitrary length preserving map.

Then $\overline{\varphi}$ is well-defined on the remaining words because $[i+|s|+TL, j-TL]$ is a non-empty subset of $[i,j]$, since $|[i,j]| \geq T$.

$\overline{\varphi}$ satisfies the four properties from Lemma 2.10.9 on the remaining words: The first three and that $K_{\text{rem}} = 6T$ works in the fourth follows in the same way as for I_{mod}.

Conclusion. Let τ be the set $\{W, \text{LW}, \text{long}, \text{mod}, \text{rem}\}$ and define $I_\tau\colon X \to I_{\mathbb{Z}} \times \tau$ by

$$I_\tau(x) = I_W(x) \sqcup I_{\text{LW}}(x) \sqcup I_{\text{long}}(x) \sqcup I_{\text{mod}}(x) \sqcup I_{\text{rem}}(x).$$

Then I_τ is a splitting map thanks to my definition of I_t for each $t \in \tau$.

$\overline{\varphi}\colon \mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$ satisfies the first three properties from Lemma 2.10.9, since it does so on each of the sets $\mathbb{W}_t$, $t \in \tau$. And by defining

$$K = \max\{K_W, K_{\text{LW}}, K_{\text{long}}, K_{\text{mod}}, K_{\text{rem}}\} = 6T$$

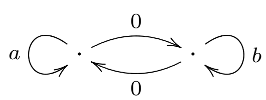
K clearly works in the fourth condition.

Thus I have defined a splitting map I_τ and a map $\overline{\varphi}\colon \mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$ which satisfies all four properties in Lemma 2.10.9. The induced map $\varphi\colon X \to Y$ is therefore a morphism. And since φ extends φ_W, it is a factor map. So because it also extends φ_S, I am done. $\square$

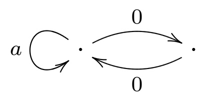
Remark 4.1.4. Let X and Y be sofic shifts, X irreducible and Y mixing such that $\mathrm{h}(X) > \mathrm{h}(Y)$. One advantage of working with general subshifts $S \subseteq X$ in stead of always ∂X is now clear: It may be the case that ∂X is very complicated, which makes finding a morphisms from ∂X to Y difficult, not to mention checking whether such a morphism is marked. But Proposition 4.1.3 says that if there is some (potentially simple) subshift $S \subseteq X$, for which $E_{X,Y} \subseteq S \subseteq \partial X$ and there is a marked morphism $S \to Y$, then X factors onto Y.

If for example $E_{X,Y} = \emptyset$, as in the case handled by Boyle, then Proposition 4.1.3 implies that $X \twoheadrightarrow Y$, since the morphism $\emptyset \to Y$ is trivially marked. But ∂X can be any sofic shift when $E_{X,Y} = \emptyset$; just make sure for example that Y has a fixed point, which is 1-affiliatied to the top component of Y. Thus Corollary 4.1.2 may be difficult to apply.

Example 4.1.5. Proposition 4.1.3 implies that the map $0^\infty \mapsto 0^\infty$ extends to a factor map $X \twoheadrightarrow Y$, where X is the shift presented by



and Y is the shift presented by



That follows from the fact that $E_{X,Y} = \{0^\infty\}$, $\mathrm{h}(X) > \mathrm{h}(Y)$ and that $0^\infty \mapsto 0^\infty$ is marked by Example 3.4.4. Thus Proposition 4.1.3 solves the problem from Example 2.10.1, which we only had ad-hoc solutions for.

The morphism produced by the proof of Proposition 4.1.3 is however far more complicated than the one in Example 2.10.1. So although the proof is constructive, it is far from practical.

4.2 Extending to $X \rightharpoonup Y$.

By ignoring everything concerning W in the proof of Proposition 4.1.3, I get a proof of the following:

Proposition 4.2.1. *Let X and Y be sofic shift spaces. X irreducible and Y mixing. And let S be a subshift of X such that $E_{X,Y} \subseteq S \subset X$. If a morphism $\varphi_S \colon S \to Y$ is marked, then φ_S extends to a morphism $\varphi \colon X \rightharpoonup Y$.*

Which by Lemma 3.5.3 implies

Theorem 4.2.2. *Let X and Y be sofic shift spaces. X irreducible and Y mixing. And let S be a subshift of X with synchronizing edges such that $E_{X,Y} \subseteq S \subset X$. Then a morphism $\varphi_S \colon S \to Y$ extends to a morphism $\varphi \colon X \rightharpoonup Y$ if and only if it is marked.*

Corollary 4.2.3. *Let X and Y be sofic shift spaces. X irreducible and Y mixing. Then a morphism $\varphi_{\partial X} \colon \partial X \to Y$ extends to a morphism $\varphi \colon X \rightharpoonup Y$ if and only if it is marked and $E_{X,Y} \subseteq \partial X$.*

4.3 Existence Results

The results from the previous section clearly imply the following results:

Theorem 4.3.1. *Let X be an irreducible sofic shift, Y be a mixing sofic shift and S be a shift space, such that $\mathrm{h}(X) > \mathrm{h}(Y)$ and $E_{X,Y} \subseteq S \subseteq \partial X$. Then*

$$\exists \varphi \colon S \to Y \text{ marked} \Rightarrow X \twoheadrightarrow Y.$$

And if furthermore S has synchronizing edges in X, then

$$X \twoheadrightarrow Y \Rightarrow \exists \varphi \colon S \to Y \text{ marked}.$$

Corollary 4.3.2. *Let X be an irreducible sofic shift and Y be a mixing sofic shift such that* $\mathrm{h}(X) > \mathrm{h}(Y)$. *Then*

$$X \twoheadrightarrow Y \Leftrightarrow E_{X,Y} \subseteq \partial X, \exists \varphi\colon \partial X \to Y \text{ marked.}$$

Example 4.3.3. Take a look at the shift X defined by the following graph:

$$\begin{tikzcd}
\cdot \arrow[r, bend left, "a"] \arrow[d, "b"] & \cdot \arrow[l, bend left, "a"] \arrow[d, "b"] \\
A \arrow[r, bend left, "b"] \arrow[u, bend left, "c"] & \cdot \arrow[l, bend left, "b"]
\end{tikzcd}$$

where A is an SFT with no fix-points. Then ∂X is the sofic shift presented by:

$$\begin{tikzcd}
\cdot \arrow[loop left, "a"] \arrow[r, "b"] & \cdot \arrow[loop right, "b"]
\end{tikzcd}$$

And ∂X is SFT relative to X with step length 0, since the only extra words we forbid are $\{c\} \cup \Sigma_A$. Thus in order to decide whether X factors onto a mixing sofic shift Y of strictly lower entropy, we need only decide whether there is a simply marked morphism from ∂X to Y.

The $\mathrm{MLW}(\partial X, 1)$-graph looks like

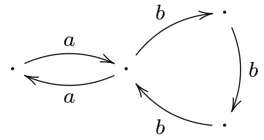
$$\begin{tikzcd}
\odot \arrow[r, bend left, "a"] \arrow[d, "b"] & \cdot \arrow[l, bend left, "a"] \arrow[d, "b"] \\
\otimes \arrow[r, bend left, "b"] & \cdot \arrow[l, bend left, "b"]
\end{tikzcd}$$

We see that both periodic points in this have the same affiliation number (2) to the top component of X and the length of all maximal ∂X words have the same parity (odd). This implies that all that is needed to guarantee a 1-simply marked map from ∂X to a shift space Y, is that Y has a fix-point which is 2-affiliated to the top component[1]. That is however ensured by Thomsen's necessary condition $\mathrm{Per\,X} \overset{(\mathrm{d})}{\twoheadrightarrow} \mathrm{Per\,Y}$. So in this case $\mathrm{Per\,X} \overset{(\mathrm{d})}{\twoheadrightarrow} \mathrm{Per\,Y}$ is both necessary and sufficient for X to factor onto a mixing sofic shift Y of lower entropy by Lemma 7.1.3, corollary 4.3.2 and corollary 3.6.2.

Note that a^∞ is neither 1-affiliated to the top component of X, nor does it have marked exits. Thus X does not have transparent affiliation pattern. The result above therefore doesn't follow from Thomsen's result. And since

[1] Because if $ua^{2d}v \in \mathbb{W}(Y)$ for some $a^\infty \in Q_1(Y)$, $u, v \in S(Y)$ and all $d \in \mathbb{N}$, then we define $\varphi\colon \partial X \to Y$ by $\varphi(x) = a^\infty$ for all x, and $L = 1$, $\alpha(caa) = \alpha(cab) = ua$ and $\omega(abc) = \omega(bbc) = v$.

it is clearly possible to construct a shift space Y such that $Q_1(Y_0^{(2)}) \neq \emptyset$, and $Q_1(Y_0^{(1)}) = \emptyset$, e.g.



it doesn't follow from Boyle's result either.

Theorem 4.3.4. *Let X be an irreducible sofic shift, Y be a mixing sofic shift and S be a shift space, such that $E_{X,Y} \subseteq S \subset X$. Then*

$$\exists \varphi \colon S \to Y \text{ marked} \Rightarrow X \rightharpoonup Y.$$

And if furthermore S has synchronizing edges in X, then

$$X \rightharpoonup Y \Rightarrow \exists \varphi \colon S \to Y \text{ marked.}$$

Corollary 4.3.5. *Let X be an irreducible sofic shift and Y be a mixing sofic shift. Then*

$$X \rightharpoonup Y \Leftrightarrow E_{X,Y} \subseteq \partial X, \exists \varphi \colon \partial X \to Y \text{ marked.}$$

Example 4.3.6. Let X be the shift from Example 4.3.3 and Y be a mixing sofic shift. Then with the same argument as in Example 4.3.3, Corollary 4.3.5 implies that

$$X \rightharpoonup Y \Leftrightarrow Q_1(Y_0^{(2)}) \neq \emptyset.$$

Corollary 4.3.2 and 4.3.5 imply:

Proposition 4.3.7. *Let X be an irreducible sofic shift and Y be a mixing sofic shift such that* $\mathrm{h}(X) > \mathrm{h}(Y)$. *Then*

$$X \twoheadrightarrow Y \Leftrightarrow X \rightharpoonup Y.$$

Thus the lower entropy factor problem, when X is irreducible sofic and Y is mixing sofic, is equivalent to the problem of deciding whether it is possible to find a morphism, which hits a synchronizing word. So instead of checking if it is possible to hit everything, we only need to check if it is possible to hit a single synchronizing word.

This is almost a direct lift of the result for irreducible SFTs:

Proposition 4.3.8. *Let X and Y be irreducible SFTs such that* $\mathrm{h}(X) > \mathrm{h}(Y)$, *then*

$$X \twoheadrightarrow Y \Leftrightarrow X \to Y.$$

Which follows from Boyle's lower entropy factor theorem for SFTs.

Proposition 4.3.9. *Let X and Y be sofic shift spaces. X mixing. Then there is a morphism $X \to Y$ if and only if there is a mixing subshift $Y' \subseteq Y$ such that $E_{X,Y'} \subseteq \partial X$ and there is a marked morphism $\partial X \to Y'$.*

Proof. If there is a morphism $\varphi\colon X \to Y$ then define $Y' = \varphi(X)$. Then Y' is mixing sofic and $E_{X,Y'} \subseteq \partial X$ and there is a marked morphism $\partial X \to Y'$ by Lemma 3.5.3 and 3.5.6.

The converse follows from Corollary 4.3.5. □

Chapter 5

Conditions on the Periodic Points

In this chapter I investigate just how much of the structure of the periodic points that is preserved under morphisms. The first point is a generalization of affiliation which works with several periodic points at the same time:

5.1 Simultaneous Affiliation

Definition 5.1.1 (Simultaneous affiliation)**.** Let X be an irreducible sofic shift space, $n \in \mathbb{N}$, $\boldsymbol{x} \in (\operatorname{Per} X)^n$, $\boldsymbol{d} \in \mathbb{N}^n$ and $F \subseteq \times_{i=1}^{n}\{0, 1, \ldots, d_i - 1\}$. Then $\boldsymbol{x}$ is said to be simultaneously $(\boldsymbol{d}, F)$-affiliated to the top component of X, denoted by $\boldsymbol{x} \in X_0^{(\boldsymbol{d},F)}$, if there are synchronizing words $u, v \in S(X)$ and words $w_1, w_2, \ldots, w_{n-1} \in \mathbb{W}(X)$ such that

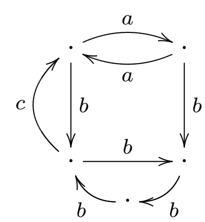
$$up_{x_1}^{k_1 d_1 + f_1} w_1 p_{x_2}^{k_2 d_2 + f_2} w_2 \ldots p_{x_n}^{k_n d_n + f_n} v \in \mathbb{W}(X),$$

for all $\boldsymbol{k} \in \mathbb{N}^n$ and $\boldsymbol{f} \in F$.

Define $X_0^{(\boldsymbol{d})} = X_0^{(\boldsymbol{d},\emptyset)}$

I leave it to the reader to verify that the choice of the p_{x_i}'s is irrelevant.

Example 5.1.2. Let X be the shift:



Then $(a^\infty, b^\infty) \in X_0^{((2,3),\{(0,1),(1,0)\})}$.

By Proposition 6.16 in [T], which says that if X_c is an irreducible component in a sofic shift X, then $\overline{X_c}$ is an irreducible sofic shift space, the following definition of simultaneous affiliation to general irreducible components in a sofic shift makes sense:

Definition 5.1.3. Let X_c be an irreducible component in the sofic shift X. Then
$$X_c^{(\boldsymbol{d},F)} = (\overline{X_c})_0^{(\boldsymbol{d},F)}.$$

And by Lemma 7.3 in [T] Definition 5.1.3 ensures that simultaneous affiliation generalizes the affiliation concept introduced by Thomsen.

In order to work with simultaneous affiliation I need some notation:

Let $k, p \in \mathbb{N}$, $\boldsymbol{x} \in X^k$ and $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^k$. Then I define

$$\begin{aligned}
&Q_{\boldsymbol{n}}(X) = \times_{i=1}^{k} Q_{n_i}(X),\\
&\boldsymbol{x}^{[p]} = (x_1^{[p]}, x_2^{[p]}, \dots, x_k^{[p]}),\\
&\boldsymbol{n} \leq \boldsymbol{m} \Leftrightarrow \forall i \colon n_i \leq m_i,\\
&\frac{\boldsymbol{n}}{\boldsymbol{m}} = \Big(\frac{n_1}{m_1}, \frac{n_2}{m_2}, \dots, \frac{n_k}{m_k}\Big) \text{ and}\\
&\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d}, F) = \Big(\frac{\boldsymbol{n}}{\boldsymbol{m}}\boldsymbol{d}, \frac{\boldsymbol{n}}{\boldsymbol{m}}F\Big),
\end{aligned}$$

where all the m_i's are non-zero in the definitions involving division by $\boldsymbol{m}$ and the following product on vectors in $\mathbb{N}^k$ is used in the last definition:

$$\boldsymbol{n}\boldsymbol{m} = (n_1 m_1, n_2 m_2, \dots, n_k m_k). \tag{5.1}$$

In short: Everything done on vectors is done coordinatewise.

I also introduce the notation:

$$p_{\boldsymbol{x}}^{\boldsymbol{k}\boldsymbol{d}+\boldsymbol{f}}\boldsymbol{w} = p_{x_1}^{k_1 d_1 + f_1} w_1 p_{x_2}^{k_2 d_2 + f_2} w_2 \dots p_{x_n}^{k_n d_n + f_n}, \tag{5.2}$$

which makes sense when one sees $p_{\boldsymbol{x}}^{\boldsymbol{k}\boldsymbol{d}+\boldsymbol{f}}$ as the vector $(p_{x_i}^{k_i d_i + f_i})_{i=1}^n$ and $\boldsymbol{w}$ as the vector $(w_1, \dots, w_{n-1}, \epsilon)$, where ϵ is the empty word, because then it is just the vector product (5.1) and the identificiation $\times_{i=1}^n \Sigma^{a_i} \to \Sigma^{\Sigma_{i=1}^n a_i}$, which maps a vector $(x_1, x_2, \dots, x_n)$ to the word $x_1 x_2 \dots x_n$.

With (5.2) the requirement in the definition of simultaneous affiliation simplifies to
$$u p_{\boldsymbol{x}}^{\boldsymbol{k}\boldsymbol{d}+\boldsymbol{f}} \boldsymbol{w} v \in \mathbb{W}(X),$$
for all $\boldsymbol{k}$ and $\boldsymbol{f} \in F$.

The following lemma shows that the properties of affiliation lifts nicely to simultaneous affiliation.

Lemma 5.1.4. *Let X_c be an irreducible component in the sofic shift space X and $\boldsymbol{1} = (1, \dots, 1)$, $\boldsymbol{d}$, $\boldsymbol{c} \in \mathbb{N}^n$, $F_j \subseteq \times_{i=1}^n \{0, 1, \dots, d_i - 1\}$ and $p \in \mathbb{N}$. Then*

1. $\times_{i=1}^{p} X_c^{(\boldsymbol{d}_i,F_i)} = X_c^{((\boldsymbol{d}_i)_{i=1}^{p},\times_{i=1}^{p}F_i)}$ *in particular* $\times_{i=1}^{n} X_c^{(d_i,F_i)} = X_c^{(\boldsymbol{d},\times_{i=1}^{n}F_i)}$ *and* $\times_{i=1}^{n} X_c^{(d_i)} = X_c^{(\boldsymbol{d})}$,
2. $X_c^{(\boldsymbol{d},F_1)} \subseteq X_c^{(\boldsymbol{d},F_2)}$, *when* $F_2 \subseteq F_1$,
3. $X_c^{(\boldsymbol{d},F)} \subseteq X_c^{(\boldsymbol{cd},\cup_{\boldsymbol{i}<\boldsymbol{c}}F+\boldsymbol{id})}$,
4. $X_c^{(\boldsymbol{d})} \subseteq X_c^{(\boldsymbol{cd})}$,
5. $(\operatorname{Per} X_c)^n \subseteq X_c^{(\boldsymbol{1})}$,
6. $X_c^{(\boldsymbol{1})} \subseteq X_c^{(\boldsymbol{d},F)}$,
7. $\boldsymbol{x} \in X_c^{(\boldsymbol{d},F)} \Leftrightarrow \boldsymbol{x}^{[p]} \in (X^{[p]})_c^{(\boldsymbol{d},F)}$.

Proof. Without loss of generality I can assume that X is irreducible and $X_c = X_0$. In 1. through 6. this follows from Definition 5.1.3. To see that it works in 7. as well I need that $(X_c)^{[p]} = (X^{[p]})_c$, but that follows Lemma 2.2.13 which implies that $\beta_p(\boldsymbol{S}(\partial^n X)) = \boldsymbol{S}(\partial^n X^{[p]})$ for all $n \in \mathbb{N}$. The assumption makes the notation somewhat less cumbersome in what follows.

1. $\subseteq$: Let $\boldsymbol{x} = (\boldsymbol{x_i})_{i=1}^{n} \in \times_{i=1}^{n} X_0^{(\boldsymbol{d_i},F_i)}$. By definition of $X_0^{(\boldsymbol{d_i},F_i)}$, there are for each i words $u_i, v_i \in S(X)$ such that $u_i p_{\boldsymbol{x}_i}^{\boldsymbol{k}_i\boldsymbol{d}_i+\boldsymbol{f}_i}\boldsymbol{w}_i v_i \in \mathbb{W}(X)$ for all $\boldsymbol{k}_i$ and $\boldsymbol{f}_i \in F_i$. Use that X is irreducible to find words w_i' such that $v_i w_i' u_{i+1} \in \mathbb{W}(X)$ for all i. Then

   $$u_1 p_{\boldsymbol{x}_1}^{\boldsymbol{k}_1\boldsymbol{d}_1+\boldsymbol{f}_1}\boldsymbol{w}_1 v_1 w_1' u_2 p_{\boldsymbol{x}_2}^{\boldsymbol{k}_2\boldsymbol{d}_2+\boldsymbol{f}_2}\boldsymbol{w}_2 v_2 w_2' u_3 \dots p_{\boldsymbol{x}_n}^{\boldsymbol{k}_n\boldsymbol{d}_n+\boldsymbol{f}_n}\boldsymbol{w}_n v_n \in \mathbb{W}(X),$$

   for all $\boldsymbol{k} = (\boldsymbol{k}_i)_{i=1}^{n}$ and $\boldsymbol{f} = (\boldsymbol{f}_i)_{i=1}^{n} \in \times_{i=1}^{n} F_i$, since the u_i and v_i's are synchronizing for X. Thus $\boldsymbol{x} \in X_0^{(\boldsymbol{d},\times_{i=1}^{n}F_i)}$ by definition.

   $\supseteq$: Let $\boldsymbol{x} \in X_0^{(\boldsymbol{d},\times_{i=1}^{n}F_i)}$. Then by definition, there are words $u, v \in S(X)$ and $w_i \in \mathbb{W}(X)$ such that

   $$u p_{\boldsymbol{x}_1}^{\boldsymbol{k}_1\boldsymbol{d}_1+\boldsymbol{f}_1}\boldsymbol{w}_1 w_1 p_{\boldsymbol{x}_2}^{\boldsymbol{k}_2\boldsymbol{d}_2+\boldsymbol{f}_2}\boldsymbol{w}_2 w_2 \dots p_{\boldsymbol{x}_n}^{\boldsymbol{k}_n\boldsymbol{d}_n+\boldsymbol{f}_n}\boldsymbol{w}_n v \in \mathbb{W}(X),$$

   for all $\boldsymbol{k} = (\boldsymbol{k}_i)_{i=1}^{n} \in \mathbb{N}^n$ and $\boldsymbol{f} = (\boldsymbol{f}_i)_{i=1}^{n} \in \times_{i=1}^{n} F_i$.

   Define for each i,

   $$\begin{aligned} u_i &= u p_{\boldsymbol{x}_1}^{\boldsymbol{f}_1}\boldsymbol{w}_1 w_1 p_{\boldsymbol{x}_2}^{\boldsymbol{f}_2}\boldsymbol{w}_2 w_2 \dots p_{\boldsymbol{x}_{i-1}}^{\boldsymbol{f}_{i-1}}\boldsymbol{w}_{i-1} w_{i-1}, \\ v_i &= w_i p_{\boldsymbol{x}_{i+1}}^{\boldsymbol{f}_{i+1}}\boldsymbol{w}_{i+1} w_{i+1} p_{\boldsymbol{x}_{i+2}}^{\boldsymbol{f}_{i+2}}\boldsymbol{w}_{i+2} w_{i+2} \dots p_{\boldsymbol{x}_n}^{\boldsymbol{f}_n}\boldsymbol{w}_n v \end{aligned}$$

   for some $\boldsymbol{f}_j \in F_j$, $j \neq i$. Then $u_i p_{\boldsymbol{x}_i}^{\boldsymbol{k}_i\boldsymbol{d}_i+\boldsymbol{f}_i}\boldsymbol{w}_i v_i \in \mathbb{W}(X)$ for all $\boldsymbol{k}_i$ and $f_i \in F_i$, which means that $\boldsymbol{x}_i \in X_0^{(\boldsymbol{d}_i,F_i)}$, since u_i and v_i are synchronizing for X because they contain the synchronizing words u and v respectively.

2. If $up_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w}v \in \mathbb{W}(X)$, for all $\boldsymbol{k}$ and $\boldsymbol{f} \in F_1$. Then it is clearly also a word in $\mathbb{W}(X)$ for all $\boldsymbol{k}$ and $\boldsymbol{f} \in F_2$, since $F_2 \subseteq F_1$.

3. Assume that $\boldsymbol{x} \in X_0^{(\boldsymbol{d},F)}$. Let $\boldsymbol{k'} = \boldsymbol{kc} + \boldsymbol{i}$ for some $\boldsymbol{i} < \boldsymbol{c} \in \mathbb{N}^n$. Then $\boldsymbol{k'} \in \mathbb{N}^n$ for all $\boldsymbol{k} \in \mathbb{N}^n$, which by definition of $X_0^{(\boldsymbol{d},F)}$ implies that

$$up_{\boldsymbol{x}}^{\boldsymbol{k}(\boldsymbol{cd})+(\boldsymbol{f}+\boldsymbol{id})}\boldsymbol{w}v = up_{\boldsymbol{x}}^{\boldsymbol{k'd}+\boldsymbol{f}}\boldsymbol{w} \in \mathbb{W}(X),$$

for all $\boldsymbol{k} \in \mathbb{N}^n$ and $\boldsymbol{f} \in F$. Thus $\boldsymbol{x} \in X_0^{(\boldsymbol{cd},\cup_{\boldsymbol{i}<\boldsymbol{c}}F+\boldsymbol{id})}$.

4. Follows from 3. using 2. with $F_2 = \emptyset$.

5. $(\operatorname{Per} X_0)^n \subseteq (X_0^{(1)})^n$ which by 1. equals $X_0^{(\boldsymbol{1})}$.

6. Follows from 3. (with $F = \emptyset$, $\boldsymbol{c} = \boldsymbol{d}$ and $\boldsymbol{d} = \boldsymbol{1}$) and 2. (with $F_2 = F$).

7. By Lemma 2.2.13 $(X^{[p]})_0 = (X_0)^{[p]}$. To finish the proof use β_p and $\beta_{n,1}^{-1}$ on the words on the form:

$$up_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w}v$$

and use that both maps preserve the period of periodic blocks. □

Remark 5.1.5. Note that when $\boldsymbol{x} \in X_0^{(\boldsymbol{d},F)}$ and a synchronizing word occurs somewhere in $p_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w}$, then 1. in Lemma 5.1.4 implies that $\boldsymbol{x}$ splits into a pair $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ such that $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$, $\boldsymbol{x}_1 \in X_0^{(\boldsymbol{d_1},F_1)}$ and $\boldsymbol{x}_2 \in X_0^{(\boldsymbol{d_2},F_2)}$, where $\boldsymbol{d} = (\boldsymbol{d}_1, \boldsymbol{d}_2)$ and $F \subseteq F_1 \times F_2$.

5.2 Preservation of Simultaneous Affiliation

Lemma 5.2.1. *Let X and Y be sofic shift spaces. Assume that a morphism $\varphi\colon \partial X \to Y$ is m-marked. Then*

$$\varphi(Q_{\boldsymbol{n}}(X_0^{(\boldsymbol{d},F)}) \cap \mathrm{L}(\partial X, m)^k) \subseteq \bigcup_{\boldsymbol{m}|\boldsymbol{n}} Q_{\boldsymbol{m}}(Y_0^{\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d},F)}),$$

for all $n, d \in \mathbb{N}^k$ and $F \subseteq \times_{i=1}^k \{0, 1, \ldots, d_i - 1\}$.

Proof. By 7. in Lemma 5.1.4 I can recode φ to ensure that it is 1-marked. Let $\boldsymbol{x} \in Q_{\boldsymbol{n}}(X_0^{(\boldsymbol{d},F)})$, such that $x_i \in \mathrm{L}(\partial X, 1)$ for all i. Then

$$up_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w}v$$

is a word in $\mathbb{W}(X)$ for all $\boldsymbol{k}$ and all $\boldsymbol{f} \in F$. By Remark 5.1.5, I can without loss of generality assume that $p_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w} \notin S(X)$ for all $\boldsymbol{k}$ and $\boldsymbol{f} \in F$. Thus $p_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w} \in \mathrm{LW}(\partial X, 1)$ for all $\boldsymbol{k}$ and $\boldsymbol{f} \in F$.

Extend u and v to ensure that $u'p_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w}v' \in \mathrm{MLW}(\partial X, 1, x)$ for some suffix u' of u, some prefix v' of v and some $x \in X$ in which $up_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w}v$ occurs and that u is an entry and v is an exit.

Then

$$\alpha(u)\varphi_1(u'p_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w}v')\omega(v),$$

has the form

$$ap_{\boldsymbol{y}}^{\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{kd}+\boldsymbol{f})}\boldsymbol{w}'b$$

for some $a, b \in S(Y)$, $w_i' \in \mathbb{W}(Y)$ and $\boldsymbol{y} = \varphi(\boldsymbol{x})$. Which by the definition of the marked property and Lemma 2.3.3 implies that $\boldsymbol{y} \in \bigcup_{\boldsymbol{m}|\boldsymbol{n}} Q_{\boldsymbol{m}}(Y_0^{\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d},F)})$. $\square$

Example 5.2.2. Let X be the shift from Example 5.1.2. Lemma 5.2.1 implies that if there is a marked map $\partial X \to Y$ for some sofic shift Y, then $Q_{1,1}(Y_0^{(2,3),\{(0,1),(1,0)\}}) \neq \emptyset$.

Theorem 5.2.3. *Let X and Y be sofic shift spaces, and $\varphi\colon X \to Y$ a morphism. It follows that there is a map $X_c \to Y_{\varphi(c)}$ from the irreducible components of X to the irreducible components of Y, such that*

$$\varphi(Q_{\boldsymbol{n}}(X_c^{(\boldsymbol{d},F)})) \subseteq \bigcup_{\boldsymbol{m}|\boldsymbol{n}} Q_{\boldsymbol{m}}(Y_{\varphi(c)}^{\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d},F)}).$$

In particular $\varphi(\overline{X_c}) \subseteq \overline{Y_{\varphi(c)}}$.

Proof. If $\boldsymbol{x} \in X_c^{(\boldsymbol{d},F)}$ then, by Remark 5.1.5, I can without loss of generality assume that $\boldsymbol{x}$ is either a single point in $(\overline{X_c})_0$ or else all the x_i's are in $\partial\overline{X_c}$. In the first case $\varphi(\boldsymbol{x}) \in \bigcup_{\boldsymbol{m}|\boldsymbol{n}} Q_{\boldsymbol{m}}(Y_{\varphi(c)}^{\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d},F)}) = \bigcup_{m|n} Q_m(Y_{\varphi(c)}^{\frac{n}{m}(d,F)})$ by Corollary 3.6.2 and Theorem 3.6.6 and in the second case $\boldsymbol{x} \in \bigcup_{\boldsymbol{m}|\boldsymbol{n}} Q_{\boldsymbol{m}}(Y_{\varphi(c)}^{\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d},F)})$ by Theorem 3.6.6 and Lemma 5.2.1. $\square$

Corollary 5.2.4. *Let X and Y be sofic shift spaces, and $\varphi\colon X \rightharpoonup Y$ a morphism, which hits a synchronizing word. Then*

$$\varphi(Q_{\boldsymbol{n}}(X_0^{(\boldsymbol{d},F)})) \subseteq \bigcup_{\boldsymbol{m}|\boldsymbol{n}} Q_{\boldsymbol{m}}(Y_0^{\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d},F)}).$$

Corollary 5.2.5. *Let X and Y be sofic shift spaces, and $\varphi\colon X \hookrightarrow Y$ an embedding. It follows that there is a map $X_c \to Y_{\varphi(c)}$ from the irreducible components of X to the irreducible components of Y, such that*

$$\varphi(X_c^{(\boldsymbol{d},F)}) \subseteq Y_{\varphi(c)}^{(\boldsymbol{d},F)}.$$

Theorem 5.2.3 gives a very strong necessary condition on the periodic points for the existence of a morphism $X \to Y$:

Definition 5.2.6. Let X and Y be shift spaces. We write $\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\to} \mathrm{Per\,Y}$ if there is a shift-commuting map $\lambda\colon \operatorname{Per} X \to \operatorname{Per} Y$ and a map $X_c \mapsto Y_{\lambda(c)}$ from the irreducible components of X to the irreducible components of Y, such that

$$\lambda(Q_{\boldsymbol{n}}(X_c^{(\boldsymbol{d},F)})) \subseteq \bigcup_{\boldsymbol{m}|\boldsymbol{n}} Q_{\boldsymbol{m}}(Y_{\lambda(c)}^{\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d},F)}). \tag{5.3}$$

If $\lambda\colon \operatorname{Per} X \to \operatorname{Per} Y$ is injective we write $\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\hookrightarrow} \mathrm{Per\,Y}$. And if X_0 maps to Y_0 we write $\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \mathrm{Per\,Y}$.

Note that in $\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\hookrightarrow} \mathrm{Per\,Y}$ (5.3) simplifies to

$$\lambda(X_c^{(\boldsymbol{d},F)}) \subseteq Y_{\lambda(c)}^{(\boldsymbol{d},F)}$$

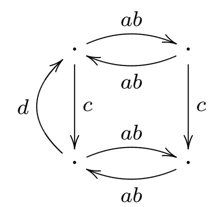
and that

$$\begin{aligned}
X \to Y &\Rightarrow \mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\to} \mathrm{Per\,Y}, \\
X \hookrightarrow Y &\Rightarrow \mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\hookrightarrow} \mathrm{Per\,Y} \text{ and} \\
X \rightharpoonup Y &\Rightarrow \mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \mathrm{Per\,Y}\,.
\end{aligned}$$

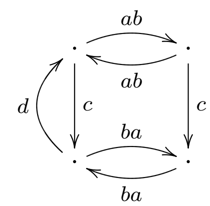
in particular $X \twoheadrightarrow Y \Rightarrow \mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \mathrm{Per\,Y}$.

$\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \mathrm{Per\,Y}$ captures a lot of the information that the marked property does. It has two immediate shortcomings: The first is of the same nature as the one in Remark 3.6.4 in that the condition doesn't require that if the same synchronizing word can be used in two different instances of simultaneous affiliation in X, then the same must be true in Y. The second is that it is missing information about the relationship between the words linking the periodic points in X and the words linking the periodic points in Y as illustrated by the following example:

Example 5.2.7. Let X be the sofic shift presented by the following graph:



And Y be the shift:

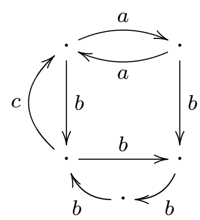


Then X and Y have the same two irreducible components: The top component and $\{(ab)^\infty\}$. And by defining λ to be the identity on both the periodic points and irreducible components $\operatorname{Per}\mathrm{X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \operatorname{Per}\mathrm{Y}$ is satisfied: There is only one set of periodic points, which are non-trivially simultaneously affiliated to X_0, namely $((ab)^\infty, (ab)^\infty)$ which is in $Q_{(2,2)}(X_0^{((2,2),\{(0,0),(1,1)\})})$. And since $\lambda((ab)^\infty, (ab)^\infty)) = ((ab)^\infty, (ab)^\infty) \in Q_{(2,2)}(Y_0^{((2,2),\{(0,0),(1,1)\})})$, because $d(ab)^{2k_1+f_1}cb(ab)^{2k_2+f_2}ad$ for all $\boldsymbol{k} \in \mathbb{N}^2$ and $\boldsymbol{f} \in \{(0,0),(1,1)\}$, the condition holds for that pair.

But in spite of that, there is no morphism from X to Y. In fact there is no morphism from ∂X to Y and therefore of course no marked map $\partial X \to Y$. The reason is that in X, words of the form $(ab)^k$ can only be connected by words of odd length and in Y they can only be connected by words of even length. To see why that is a problem let $x \in \partial X$ be the point such that $x_0 = c$. If $\varphi\colon \partial X \to S$ is a morphism, then, since it is a sliding block code, it works like λ far from index 0. Thus $\varphi(x)$ has to have the form $(ab)^\infty w (ab)^\infty$ for some word $w \in \mathbb{W}(Y)$ of odd length, which is impossible.

In spite of these shortcomings, $\operatorname{Per}\mathrm{X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \operatorname{Per}\mathrm{Y}$ is still sufficient in some cases. It is of course sufficient in the cases handled by Thomsen and Boyle, since $\operatorname{Per}\mathrm{X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \operatorname{Per}\mathrm{Y}$ is stronger than both $\operatorname{Per}\mathrm{X} \overset{(\mathrm{d})}{\twoheadrightarrow} \operatorname{Per}\mathrm{Y}$ and $\operatorname{Per}\mathrm{X} \overset{\sigma}{\to} \operatorname{Per}\mathrm{Y}$. And as illustrated by the following example it is sufficient in more cases.

Example 5.2.8. Let X be the sofic shift from Example 5.1.2:



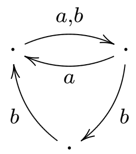
and Y be a mixing sofic shift.

In this example I show that $\operatorname{Per}\mathrm{X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \operatorname{Per}\mathrm{Y}$ is necessary and sufficient for $X \rightharpoonup Y$ and therefore

$$X \twoheadrightarrow Y \Leftrightarrow \operatorname{Per}\mathrm{X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \operatorname{Per}\mathrm{Y},$$

if $\mathrm{h}(X) > \mathrm{h}(Y)$ by Proposition 4.3.7.

So for example $X \rightharpoonup Y$, when Y is the shift:



That $\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \mathrm{Per\,Y}$ is necessary follows from corollary 5.2.4. To prove that it is sufficient, assume that $\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \mathrm{Per\,Y}$. Then $E_{X,Y} \subseteq \partial X$, which by Corollary 4.3.5 implies that I only need to construct a marked morphism $\partial X \to Y$. Note that ∂X is the following shift

$$\begin{tikzcd} \cdot \arrow[loop left, "a"] \arrow[r, "b"] & \cdot \arrow[loop right, "b"] \end{tikzcd}$$

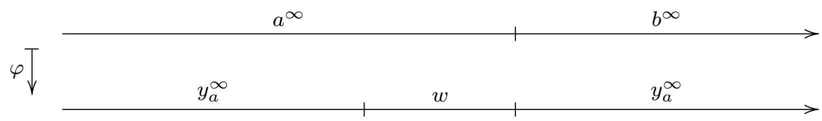
$\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \mathrm{Per\,Y}$ implies that there are words $u, v \in S(Y)$, $y_a, y_b \in \Sigma_Y$ and $w \in \mathbb{W}(Y)$ such that

$$u y_a^{2k_1+f_1} w y_b^{3k_2+f_2} v \in \mathbb{W}^*(Y),$$

for all $\boldsymbol{k} \in \mathbb{N}_\infty^2$ and $\boldsymbol{f} \in \{(0,1),(1,0)\}$ since $(a^\infty, b^\infty) \in X_0^{((2,3),\{(0,1),(1,0)\})}$ as seen in Example 5.1.2. Define a $(0,|w|)$-block map $\Phi\colon \mathbb{W}_{|w|+1}(\partial X) \to \Sigma_Y$ by

$$\begin{aligned} a^{|w|+1} &\mapsto y_a, \\ a^{|w|+1-k} b^k &\mapsto w_{k-1}, \\ b^{|w|+1} &\mapsto y_b. \end{aligned}$$

And let φ be the sliding block code induced by Φ. Then φ is a $|w|+1$-block morphism $\partial X \to Y$, which in a point $a^\infty b^\infty$ removes the last $|w|$ occurrences of a and inserts w in stead and afterwards replaces each a by y_a and each b by y_b.



I claim that φ is $|w|+1$-marked.

To see that let $L = |w|$ and define $\alpha\colon A(|w|+1, L) \to S(Y)$ by

$$\alpha(?ca^{L+1-i}b^i) = \begin{cases} u y_a^{|w|}, & i = 0 \\ u y_a^{L+1-i} w_{[0,i-1[}, & i \in [1,|w|] \\ u, & i = |w|+1 \end{cases}$$

and $\omega\colon \Omega(|w|+1, L) \to S(Y)$ by

$$\omega(a^{L+1-i}b^i c?) = \begin{cases} v, & i = 0 \\ w_{[i,|w|[} y_b^i, & i \in [1,|w|] \\ y_b^{|w|} v, & i = |w|+1 \end{cases}$$

Let $[i,j] \in \mathrm{MLW}(\partial, |w|+1, x)$ for some $x \in X$. Then $x = ca^p b^q c$ for some $p, q \in \mathbb{N}_\infty$ such that $([p]_2, [q]_3) \in \{(0,1),(1,0)\}$ and

$$\alpha(x_{[i-L,i+L]})\varphi_{|w|+1}(x_{[i,j]})\omega(x_{[j-L,j+L]}) = \\ \alpha(x_{[i-L,-1]}ca^{i_a}b^{i_b})\varphi_{|w|+1}(a^p b^q)\omega(a^{j_a}b^{j_b}) = uy_a^p w y_b^q v,$$

where

$$i_a = \min\{p, |w|+1\}, \qquad i_b = |w|+1-i_a,$$
$$j_a = |w|+1-j_b, \qquad j_b = \min\{q, |w|+1\}.$$

So as $uy_a^p w y_b^q v \in \mathbb{W}^*(Y)$ by definition of u, v and w, φ is $|w|+1$-simply marked and therefore $|w|+1$-marked by Lemma 3.8.3.

Hence Corollary 4.3.5 implies that $X \rightharpoonup Y$.

To formulate a condition on the periodic points, which does not have those two shortcomings, I need to add three maps to the $\mathrm{Per\,X} \overset{(\mathbf{d},\mathrm{F})}{\twoheadrightarrow} \mathrm{Per\,Y}$ condition; two maps similar to α and ω in the marked condition to take care of the first and a map from the words linking periodic points in X to the words linking periodic points in Y, which preserves some information about the length, to take care of the complication in Example 5.2.7. Obviously that makes the condition more complicated.

Through the rest of this section the p_x's are fixed for all $x \in X$ such that $p_x = p_{x'}$, when $x' = \sigma^n(x)$ for some $n \in \mathbb{N}$. And when $\lambda\colon \operatorname{Per} S \to \operatorname{Per} Y$ is shift-commuting, then for each $y \in \lambda(\operatorname{Per} S)$, p_y is chosen such that if $p_x = x_{[i,i+|p_x|[}$ for some $i \in \mathbb{Z}$, then $p_{\lambda(x)} = \lambda(x)_{[i,i+|p_{\lambda(x)}|[}$.

Condition 5.2.9. There are maps
$\lambda\colon \operatorname{Per} X \to \operatorname{Per} Y$ shift-commuting,
$\alpha, \omega\colon S(X) \times \operatorname{Per} X \to S(Y)$,
$L\colon \operatorname{Per} X \times \mathbb{W}(X) \times \operatorname{Per} X \to \mathbb{W}(Y)$
with the following properties:

1. For all $u, v \in S(X)$, $\boldsymbol{x} \in \operatorname{Per} X^n, \boldsymbol{d} \in \mathbb{N}^n$, $F \subseteq \times_{i=1}^n \{0, 1, \ldots, d_i - 1\}$ and $\boldsymbol{w} \in L(X)^n$ the following holds:

   If
   $$up_{\boldsymbol{x}}^{\boldsymbol{dk}+\boldsymbol{f}}\boldsymbol{w}v \in \mathbb{W}^*(X),$$
   for all $\boldsymbol{k} \in \mathbb{N}^n$ and $\boldsymbol{f} \in F$, then
   $$\alpha(u, x_1)p_{\lambda(\boldsymbol{x})}^{\frac{|p_{\boldsymbol{x}}|}{|p_{\lambda(\boldsymbol{x})}|}(\boldsymbol{dk}+\boldsymbol{f})} L(\boldsymbol{w}')\omega(v, x_n) \in \mathbb{W}^*(Y),$$
   for all $\boldsymbol{k} \in \mathbb{N}^n$ and $\boldsymbol{f} \in F$, where $\boldsymbol{w}' = ((x_i, w_i, x_{i+1}))_{i=1}^{n-1}$.

2. For all $(x, w, y) \in \operatorname{Per} X \times \mathbb{W}(X) \times \operatorname{Per} X$ there are $a, b \in \mathbb{N}$ such that:

$$|L(x, w, y)| = |w| + a|p_x| + b|p_y|.$$

Remark 5.2.10. The condition can of course be extended to take care of simultaneous affiliation to all the irreducible components in X, but I think that it is complicated enough as it is.

Note that when Y is mixing, then the condition simplifies a bit. Because then we only need to define L on the points in $\operatorname{Per} \partial X \times \mathbb{W}(\partial X) \times \operatorname{Per} \partial X$, because the existence of L is automatic on the remaining points. Because if x or y lies in the top component of X or $w \notin \mathbb{W}(\partial X)$, then the definition of $L(x, w, y)$ is simple:

For all $x \in X_0$ there is an $n \in \mathbb{N}$ such that p_x^n is synchronizing for X. Hence the existence of α and ω imply that there are words $u_x, v_x \in S(Y)$ such that $u_x p_{\lambda(x)}^{\frac{|p_x|}{|p_{\lambda(x)}|}k} v_x \in \mathbb{W}(Y)$ for all $k \in \mathbb{N}$. And if both x and y are in ∂X but $w \notin \mathbb{W}(\partial X)$ then by Lemma 3.5.5 there are $i, j \in \mathbb{N}$ such that $p_x^i w p_y^j \in S(X)$.

Define L on $\operatorname{Per} X \times \mathbb{W}(X) \times \operatorname{Per} X - \operatorname{Per} \partial X \times \mathbb{W}(\partial X) \times \operatorname{Per} \partial X$ by

$$L(x, w, y) = \begin{cases} v_x w' u_y, & \text{if } x, y \in X_0 \\ v_x w' \alpha(p_x^n w, y), & \text{if } x \in X_0, y \in \partial X \\ \omega(w p_y^n, x) w' u_y, & \text{if } x \in \partial X, y \in X_0 \\ \omega(p_x^i w p_y^j, x) w' \alpha(p_x^i w p_y^j, y), & \text{if } x, y \in \partial X \end{cases}$$

where in each case $w' \in \mathbb{W}(Y)$ is chosen of appropriate length, such that property 2. of Condition 5.2.9 is satisfied, using that Y is mixing.

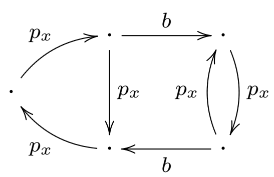
I leave it to the reader to verify that these definitions of L work.

To show that Condition 5.2.9 is necessary for $X \rightharpoonup Y$, when X and Y are shift spaces the following definition is convenient:

Definition 5.2.11 (d_x)**.** Let X be irreducible sofic and $x \in \operatorname{Per} X$. Define

$$d_x = \frac{\operatorname{lcm}\{\operatorname{period}(p) \mid p \in \pi^{-1}(x)\}}{|p_x|}$$

Example 5.2.12. Let X be the irreducible sofic shift presented by the following graph:



Then there are essentially two different paths presenting x; one with minimal period $2|p_x|$ and one with minimal period $3|p_x|$. Hence $d_x = 6$.

My reason for introducing d_x is that it has the property that if $\boldsymbol{x} \in X_0^{(\boldsymbol{d},F)}$ then $\boldsymbol{x}$ is also in $X_0^{(\boldsymbol{d},F+\boldsymbol{c}d_{\boldsymbol{x}})}$, for all $\boldsymbol{c}$, where $d_{\boldsymbol{x}}$ is the vector whose ith coordinate is d_{x_i}. Or in other words if $up_{\boldsymbol{x}}^{\boldsymbol{dk}+\boldsymbol{f}}\boldsymbol{w}v \in \mathbb{W}^*(X)$ for some $u, v, \boldsymbol{x}, \boldsymbol{d}, \boldsymbol{w}, F$ and all $\boldsymbol{k}$ and $\boldsymbol{f} \in F$, then $up_{\boldsymbol{x}}^{\boldsymbol{dk}+\boldsymbol{c}d_{\boldsymbol{x}}+\boldsymbol{f}}\boldsymbol{w}v \in \mathbb{W}^*(X)$, for all $\boldsymbol{k}, \boldsymbol{c}$ and $\boldsymbol{f} \in F$. That follows easily from the fact that the minimal period of any path presenting a periodic point x divides d_x.

Lemma 5.2.13. *Let X and Y be shift spaces, X irreducible sofic. Then*

$$X \rightharpoonup Y \Rightarrow X, Y \text{ satisfy Condition } 5.2.9.$$

Proof. Let X and Y be shift spaces and $\varphi\colon X \to Y$ be an (m,n)-block morphism, which hits the synchronizing word $s \in S(Y)$. Choose $s_0 \in \mathbb{W}(X)$ such that $\varphi_{m,n}(s_0) = s$ and use the irreducibility of X to find, for each $u \in S(X)$, words $u_\pm \in \mathbb{W}(X)$ such that $s_0u_-uu_+s_0 \in \mathbb{W}(X)$.

Choose for each $x \in \operatorname{Per} X$ numbers $a_x, b_x \in \mathbb{N}$ such that $|p_x^{a_xd_x}| \geq m$ and $|p_x^{b_xd_x}| \geq n$.

λ: Define $\lambda\colon \operatorname{Per} X \to \operatorname{Per} Y$ by $\lambda = \varphi_{|\operatorname{Per}(X)}$. Then clearly λ is shift-commuting.

α: Let $u \in S(X)$ and $x \in \operatorname{Per} X$. Without loss of generality I can assume that $up_x^\infty \in \mathbb{W}^*(X)$, because otherwise $\alpha(u,x)$ can be any word in $S(Y)$ since it has no influence on whether α works in condition 1. Define

$$\alpha(u,x) = \varphi_{m,n}(s_0u_-up_x^{a_xd_x}(p_x^\infty)_{[0,n[}).$$

ω: The definition of ω is analogous: Let $v \in S(X)$ and $x \in \operatorname{Per} X$. Assume that $p_x^\infty v \in \mathbb{W}^*(X)$. Define

$$\omega(v,x) = \varphi_{m,n}((p_x^\infty)_{[-m,0[}p_x^{b_xd_x}vv_+s_0).$$

L: Let $(x,w,y) \in \operatorname{Per} X \times \mathbb{W}(X) \times \operatorname{Per} X$. Define

$$L(x,w,y) = \varphi_{m,n}((p_x^\infty)_{[-m,0[}p_x^{b_xd_x}wp_y^{a_yd_y}(p_y^\infty)_{[0,n[}).$$

Then $|L(x,w,y)| = |w| + a|p_x| + b|p_y|$ with $a = a_xd_x$ and $b = b_yd_y$. Thus condition 2. is satisfied.

To verify that condition 1. is also satisfied let $u, v \in S(X)$, $\boldsymbol{x} \in \operatorname{Per} S^q$, $\boldsymbol{w} \in (\operatorname{Per} X \times \mathbb{W}(X) \times \operatorname{Per} X)^{q-1}$, $\boldsymbol{d} \in \mathbb{N}^q$ and $F \subseteq \times_{i=1}^q\{0,1,\dots,d_i-1\}$ and assume that

$$up_{\boldsymbol{x}}^{\boldsymbol{dk}+\boldsymbol{f}}\boldsymbol{w}v \in \mathbb{W}^*(X),$$

for all $\boldsymbol{k} \in \mathbb{N}_\infty^q$ and $\boldsymbol{f} \in F$. Then

$$s_0u_-up_{\boldsymbol{x}}^{\boldsymbol{dk}+(a_{\boldsymbol{x}}+b_{\boldsymbol{x}})d_{\boldsymbol{x}}+\boldsymbol{f}}\boldsymbol{w}vv_+s_0 \in \mathbb{W}^*(X),$$

for all $\boldsymbol{k} \in \mathbb{N}_\infty^q$ and $\boldsymbol{f} \in F$ by definition of $d_{\boldsymbol{x}}$. Hence

$$\alpha(u, x_1) p_{\lambda(\boldsymbol{x})}^{\frac{|p_{\boldsymbol{x}}|}{|p_{\lambda(\boldsymbol{x})}|}(\boldsymbol{dk}+\boldsymbol{f})} L(\boldsymbol{w}') \omega(v, x_n) = \\ \varphi_{m,n}(s_0 u_- u p_{\boldsymbol{x}}^{\boldsymbol{dk}+(a_{\boldsymbol{x}}+b_{\boldsymbol{x}})d_{\boldsymbol{x}}+\boldsymbol{f}} \boldsymbol{w} v v_+ s_0) \in \mathbb{W}^*(Y),$$

for all $\boldsymbol{k} \in \mathbb{N}_\infty^q$ and $\boldsymbol{f} \in F$, where $\boldsymbol{w}' = ((x_i, w_i, x_{i+1}))_{i=1}^{n-1}$, which is what I needed to prove. $\square$

The added complexity of Condition 5.2.9 makes it hard to work with, when trying to prove sufficiency. In fact in all examples that I have found of shifts X and Y, for which Condition 5.2.9 ensures that $X \rightharpoonup Y$, X and Y are so simple that it is just as easy to verify that there is a marked map from ∂X to Y and that $E_{X,Y} \subseteq \partial X$.

All my attempts to turn Condition 5.2.9 into a necessary and sufficient condition have only resulted in a more complicated formulation of the marked condition. But that is in fact not very surprising. Because if you want to use Condition 5.2.9 to construct a morphism, then it is apparent that the map L has to work like a sliding block code, because the w's may be of arbitrary length in general. And if X is irreducible, any word in $\mathbb{W}(X)$ may occur between two periodic points, which means that L has to be the word map of some morphism $X \to Y$. So Condition 5.2.9 involves a morphism from $X \to Y$ and not just the periodic points. Even the weaker condition mentioned in Remark 5.2.10 requires the existence of a morphism from $\partial X \to Y$ in general. Hence Condition 5.2.9 brings us back to extending morphisms.

In conclusion: A lot more of the structure of the periodic points in a shift space is preserved under morphisms than was previously known. And it seems that there is no simple tractable condition on the periodic points, which is both necessary and sufficient for a factor map or embedding to exist between sofic shifts.

Chapter 6

The Embedding Problem

The purpose of this chapter is to investigate if the ideas used on the factor problem can be used on the embedding problem as well.

6.1 Extension Results

Similar to Krieger, I want to define the word map $\overline{\varphi}$ to be injective on each interval and I want to somehow code the transition between different intervals in such a way that given a point in $\varphi(X)$, it is possible to find the intervals $[i_k, i_{k+1}[$ used to define it.

Because then φ is injective, since each $y = \varphi(x) \in \varphi(X)$ can be the image of at most one point in X, since I can reconstruct x from y by finding the intervals $[i_k, i_{k+1}[$ used to define it, and use the inverse of $\overline{\varphi}$ on each $y_{[i_k,i_{k+1}[}$.

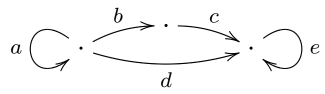
The coding is done by inserting different powers, depending on the type of the interval, of some fixed word $s \in S(Y)$ at the beginning of the image of each interval, and then making sure that s is not used anywhere but at the transitions.

When working with only marker intervals it is relatively easy to make sure that the word map is injective, since I can do almost what I want on the short intervals, and the long intervals correspond to periodic points of period $< T$, for some $T \in \mathbb{N}$. So by assuming that I have an injective map $\lambda\colon \operatorname{Per}_{<T} X \to \operatorname{Per}_{<T} Y$, I can find z when knowing only a subword of length $2T + 1$ of $\lambda(z)$ by Lemma 2.9.5. Thus by making sure that the image of each long marker interval contains a word of form $\lambda(z)_{[i,i+2T]}$, the word map becomes injective.

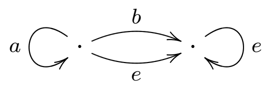
In my approach I am however forced to define the word map on arbitrarily long non-periodic words from $\mathrm{MLW}(S, n, x)$. So how do I make the word map injective on those words? It would of course be natural to assume that the map $\varphi\colon S \to Y$ is injective, since I want to extend it to an injective map. But does that imply that the corresponding word map φ_n is injective on $\mathrm{LW}(S, n)$?

The following example show that the answer is 'No', even on 'long' words:

Example 6.1.1. Let $X = S$ be the sofic shift presented by the following labeled graph:



Let Y be the sofic shift presented by the following labeled graph:



And let $\varphi\colon X = \mathrm{L}(S,1) \to Y$ be the 1-block morphism induced by the block map:

$$\begin{aligned} a &\mapsto a \\ b &\mapsto b \\ c, d, e &\mapsto e. \end{aligned}$$

Then $\varphi_1(ce^k) = \varphi_1(de^k)$ for all $k \in \mathbb{N}$ even though φ is clearly injective.

By adding an edge labeled f from the c-cycle to the a-cycle in the first graph and similarly an edge labeled f from the e-cycle to the a-cycle in the second and extending φ by defining $f \mapsto f$, I see that it is still false even if I assume that S is mixing.

The word map will however be injective on the center of words in the following sense:

Lemma 6.1.2. *Let X be a sofic shift space, $S \subseteq X$ and Y be shift spaces and $\varphi\colon \mathrm{L}(S,1) \to Y$ be an injective 1-block morphism. Then there exists a $\gamma \in \mathbb{N}$, such that the following holds for all $x, y \in \mathrm{LW}_{\geq 2\gamma+1}(S,1)$:*

$$\varphi_1(x) = \varphi_1(y) \Rightarrow x_{[\gamma,|x|-\gamma]} = y_{[\gamma,|y|-\gamma]}.$$

Proof. Let G be a presentation of X and let E be the set of edges in G. Define $\gamma = E^2 + 1$. Let $x, y \in \mathrm{LW}_{\geq 2\gamma+1}(S,1)$ such that $\varphi_1(x) = \varphi_1(y)$. Let p and q be paths in G which present x and y respectively. By the pigeon hole principle I can find $a, b, c, d \in \mathbb{N}$ such that

$$\begin{gathered} 0 \leq a \leq b < \gamma, \\ |x| - \gamma \leq c \leq d < |x| \end{gathered}$$

and such that $p_a = p_b$, $q_a = q_b$ and $p_c = p_d$, $q_c = q_d$. This implies that both $x^{\infty}_{[a,b[} x_{[b,c]} x^{\infty}_{]c,d]}$ and $y^{\infty}_{[a,b[} y_{[b,c]} y^{\infty}_{]c,d]}$ are in $\mathrm{L}(S,1)$. And because $\varphi_1(x) = \varphi_1(y)$ and φ is 1-block $\varphi(x^{\infty}_{[a,b[} x_{[b,c]} x^{\infty}_{]c,d]})$ must be equal to $\varphi(y^{\infty}_{[a,b[} y_{[b,c]} y^{\infty}_{]c,d]})$. Thus the injectivity of φ implies that $x^{\infty}_{[a,b[} x_{[b,c]} x^{\infty}_{]c,d]} = y^{\infty}_{[a,b[} y_{[b,c]} y^{\infty}_{]c,d]}$, which by my choice of a, b, c and d implies that $x_{[\gamma,|x|-\gamma]} = y_{[\gamma,|y|-\gamma]}$. $\square$

By letting $S = X$ in Lemma 6.1.2 I get:

Corollary 6.1.3. *Let X be a sofic shift, Y be a shift space and $\varphi\colon X \to Y$ be an injective 1-block morphism. Then there exists a $\gamma \in \mathbb{N}$, such that the following holds for all $x, y \in \mathbb{W}_{\geq 2\gamma+1}(X)$:*

$$\varphi_1(x) = \varphi_1(y) \Rightarrow x_{[\gamma,|x|-\gamma]} = y_{[\gamma,|y|-\gamma]}.$$

Lemma 6.1.2 is all I need to use the idea from the factor problem on the embedding problem:

Proposition 6.1.4. *Let X and Y be sofic shift spaces, X irreducible and Y mixing, such that $\mathrm{h}(X) < \mathrm{h}(Y)$. Let S be a subshift of X such that $E_{X,Y} \subseteq S$. Let $\varphi_S\colon S \to Y$ be an injective morphism and $\lambda\colon \operatorname{Per} X \to \operatorname{Per} Y$ be an injective shift-commuting map with the following properties:*

1. *$\lambda(X_0^{(n)}) \subseteq Y_0^{(n)}$, for all $n \in \mathbb{N}$,*
2. *φ_S is marked,*
3. *$\widetilde{\varphi}_S\colon \mathrm{L}(S, n) \to Y$ is injective for some $n \in \mathbb{N}$,*
4. *$\widetilde{\varphi}_S$ and λ agree on their common domain.*

Then φ_S extends to an injective morphism $X \hookrightarrow Y$.

Proof. The proof is very similar to the proof of Proposition 4.1.3 in that I apply Lemma 2.10.9 and more or less reuse the different types of intervals. The main difference is that the problem of extending two morphisms at the same time is exchanged with the problem of making sure that the resulting map is injective. I solve that by ensuring that $\overline{\varphi}$ is injective on each type of word and that I, given a point in $\varphi(X)$, can find the intervals that were used to define it. Because then I can reconstruct x, when given $\varphi(x)$.

Set up. Recode to ensure that φ_S is 1-block, 1-marked and injective on $\mathrm{L}(S, 1)$. Let L, N, α_k and ω be the numbers and functions corresponding to $n = 1$ in Lemma 3.4.7 such that

$$\alpha_k(x_{[i-L,i+k[})\varphi_S(x_{[i+k,j]})\omega(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $k > L$, $x \in X$ and $[i, j] \in \mathrm{MLW}(S, 1, x)$.

Let TL be a transition length for Y and let R be the number from Lemma 2.7.16 corresponding to Y.

By Lemma 5.7 in [T] there is a strictly increasing sequence of mixing SFTs $\{A_n\}_{n\in\mathbb{N}}$, such that $A_n \subseteq Y$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathrm{h}(A_n) = \mathrm{h}(Y)$. Choose a $K \in \mathbb{N}$ such that $\mathrm{h}(A_K) > \mathrm{h}(X)$.

If $A_K = Y$, then Y is a mixing SFT and the result follows from Theorem 8.5 in [T]. Assume therefore that $A_K \neq Y$. I claim that I can find a word s

in $\mathbb{W}(Y)$, which is in neither $\varphi_S(\mathrm{LW}(S,1))$, $\mathbb{W}(\lambda(\operatorname{Per} X))$ nor $\mathbb{W}(A_K)$. To see that note that $\mathrm{LW}(S,1) \cup \operatorname{Per} X \subseteq X$. So as $\mathrm{h}(X) < \mathrm{h}(Y)$, φ_S and λ cannot hit every word in Y. I can therefore choose a word $s' \in \mathbb{W}(Y)$ which is in neither $\varphi_S(\mathrm{LW}(S,1))$ nor $\mathbb{W}(\lambda(\operatorname{Per} X))$, and a word $s'' \in \mathbb{W}(Y)$ which is not in $\mathbb{W}(A_K)$. Since Y is mixing I can find an $x \in \mathbb{W}(Y)$ such that $s = s'xs''$ is a word in $\mathbb{W}(Y)$. This s is clearly not in $\varphi_S(\mathrm{LW}(S,1))$, $\mathbb{W}(\lambda(\operatorname{Per} X))$ nor $\mathbb{W}(A_K)$, which establishes the claim.

Since Y is mixing, I can extend s to ensure that it satisfies the following:

1. s is synchronizing,
2. $s^n \in \mathbb{W}(Y)$ for all $n \in \mathbb{N}$,
3. s longer than $\max\{2(TL+R), N\}$,
4. $s \notin \varphi_S(\mathrm{LW}(S,1)) \cup \mathbb{W}(\lambda(\operatorname{Per} X) \cup A_K)$ and
5. s occurs exactly once in a word of the form ws or sw, when $|w| < |s|$.

Let γ be the number from Lemma 6.1.2.

Since $\mathrm{h}(X) < \mathrm{h}(A_K)$, I can choose constants $A, W \in \mathbb{N}$, such that

$$\mathbb{W}_{|s|+A+2TL+N+\gamma}(X) \leq \mathbb{W}_A(A_K) \text{ and}$$
$$\mathbb{W}_{\gamma+N+W+2TL}(X) \leq \mathbb{W}_W(A_K).$$

Again using that $\mathrm{h}(X) < \mathrm{h}(A_K)$, I choose a constant $M \in \mathbb{N}$, such that for all $n \geq M$:

$$\mathbb{W}_n(X) \leq \mathbb{W}_{n-3|s|-2TL}(A_K)$$

Define

$$T = 10|s| + 8TL + A + W + 2\gamma + L + M + 2N + 2R.$$

Using Lemma 2.9.7 I choose $k > 5T$ so large, that when $[i,j]$ is a (k,T)-marker interval, which is longer than $2T$, then I can find a $z \in \operatorname{Per}_{<T}(X)$, such that $x_{[i+T-k,j-T+k]} \subseteq z$.

Construction. I am going to use the following four types of locally recognizable intervals that may occur in a point x in X:

1. The local intervals,

$$I_{\mathrm{LW}}(x) = \{[i_0, j_0] \mid [i,j] \in \mathrm{MLW}(S,1,x), |[i,j]| \geq 2T\}.$$

where $i_0 = i + T + N + W + 2TL$ and $j_0 = j + N + W + 2TL$.

2. The long marker intervals,

$$I_{\mathrm{long}}(x) = \{[i,j] \in I_{k,T}(x) \mid |[i,j]| \geq 5T, \exists n \in [i,j] \forall [i',j'] \in I_{\mathrm{LW}}(x) \colon n \notin [i',j']\}.$$

3. The moderate marker intervals,

$$I_{\mathrm{mod}}(x) = \{[i,j] \in I_{k,T}(x) \mid \forall [i',j'] \in I_{\mathrm{LW}}(x) \cup I_{\mathrm{long}}(x) \colon [i',j'] \cap [i-T, j+T] = \emptyset\}.$$

4. The remaining intervals,

$$I_{\mathrm{rem}}(x) = \{[i,j] \in I_{undef}(x) \mid [i-1,j] \notin I_{undef}(x), [i,j+1] \notin I_{undef}(x)\},$$

where $I_{undef}(x)$ is the set

$$\{[i,j] \mid \forall [i',j'] \in I_W(x) \cup I_{\mathrm{LW}}(x) \cup I_{\mathrm{long}}(x) \cup I_{\mathrm{mod}}(x) \colon [i',j'] \cap [i,j] = \emptyset\}.$$

Let $\tau = \{\mathrm{LW}, \mathrm{long}, \mathrm{mod}, \mathrm{rem}\}$. The same argument used in the proof of Proposition 4.1.3 shows that I_τ defined by $I_\tau(x) = \sqcup_{t\in\tau} I_t(x)$ is a splitting map. So all that remains is to define a map $\overline{\varphi}\colon \mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$ with the four properies in Lemma 2.10.9, with the extra property that it is almost injective in the sense that $\overline{\varphi}(x_{[i-M,j+M]})$ determines $x_{[i,j]}$.

The local intervals. Let w be a word in X. By Lemma 6.1.2 the information about $w_{[0,\gamma[}$ and $w_{]|w|-\gamma,|x|[}$ is lost after applying φ_S on w. So I need a way of encoding that information into $\overline{\varphi}(w)$, when $w \in \mathbb{W}_{\mathrm{LW}}$. I do that by defining two injective maps a and w.

Choose for each pair $(v, s_N) \in \mathbb{W}(A_K) \times S_N(Y)$ words $w_-, w_+, w_\alpha, w_\omega$ in $\mathbb{W}_{TL}(Y)$ such that both $sw_+vw_\alpha s_N$ and $s_N w_\omega v w_- s$ are words in Y. I have chosen not to include v and s_N in the notation, since it will be obvious from the context which words that w_+, w_-, w_α and w_ω are supposed to connect.

By my choice of A and W I can define injective maps

$$\begin{aligned} a'&\colon \mathbb{W}_{|s|+A+2TL+N+\gamma}(X) \hookrightarrow \mathbb{W}_A(A_K) \text{ and} \\ w'&\colon \mathbb{W}_{\gamma+N+W+2TL}(X) \hookrightarrow \mathbb{W}_W(A_K). \end{aligned}$$

Define $|a| = A + 2TL$, $|w| = W + 2TL$ and

$$\begin{aligned} a\colon \mathbb{W}_{|s|+|a|+N+\gamma}(X) \times S_N(Y) &\hookrightarrow \mathbb{W}_{|a|}(Y) \\ (v, s_N) &\mapsto w_- a'(v) w_\alpha \end{aligned}$$

and

$$\begin{aligned} w\colon \mathbb{W}_{\gamma+N+|w|}(X) \times S_N(Y) &\hookrightarrow \mathbb{W}_{|w|}(Y) \\ (v, s_N) &\mapsto w_\omega w'(v) w_+. \end{aligned}$$

Then a and w have the property, that both $a(v, s_N)$ and $w(v, s_N)$ determine v, and that $sa(v,\alpha)\alpha \in \mathbb{W}(Y)$ and $\omega w(v,\omega)s \in \mathbb{W}(Y)$ for long enough $v \in \mathbb{W}(X)$ and $\alpha, \omega \in S_N(Y)$.

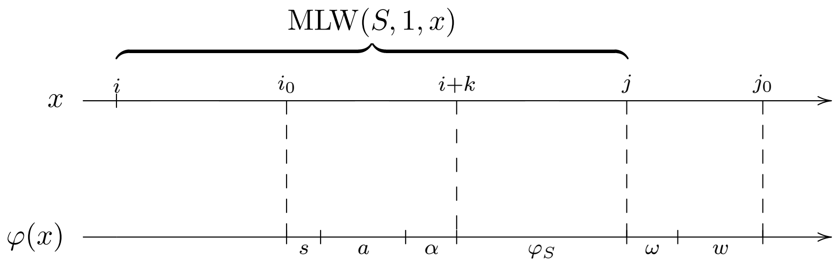
Let $M = i_0 - i + L$ and $k = i_0 - i + |s| + |a| + N$.

Define for each $x_{[i_0,j_0]} \in \mathbb{W}_{\mathrm{LW}}$:

$$\overline{\varphi}(x)_{[i_0-M,j_0+M],\mathrm{LW}} = sa\alpha\varphi_S(x_{[i+k,j]})\omega w.$$

Then $\overline{\varphi}$ maps the local words to $\mathbb{W}^*(Y)$ by Lemma 3.4.7 since $k > T$.

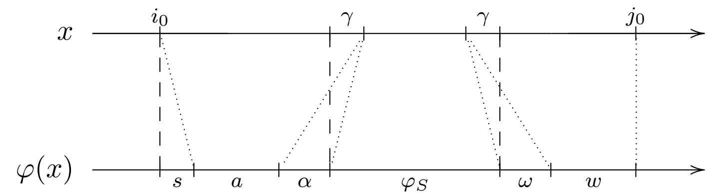
The definition is illustrated in the following figure:



I have omitted the arguments for the functions involved in order to make it easier to read. The functions with arguments:

$$\begin{aligned}
&a(x_{[i_0,i+k+\gamma[}, \alpha(x_{[i-L,i+k[}))\\
&\alpha(x_{[i-L,i+k[})\\
&\omega(x_{[j-L,j+L]})\\
&w(x_{]j-\gamma,j_0]}, \omega(x_{[j-L,j+L]})).
\end{aligned}$$

Thanks to Lemma 6.1.2 and my definition of a and w, I can now reconstruct $x_{[i_0,j_0]}$ when knowing $\overline{\varphi}(x_{[i_0-M,j_0+M]})$. The idea is illustrated in the following figure, where dotted lines indicate which word is given by which:



In symbols:

$$\begin{aligned}
x_{[i_0,i+k+\gamma[} &= a^{-1}(\varphi(x)_{[i_0+|s|,i+k-N[}),\\
x_{[i+k+\gamma,j-\gamma]} &= \varphi_S^{-1}(\varphi(x)_{[i+k,j]})_{[i+k+\gamma,j-\gamma]},\\
x_{]j-\gamma,j_0]} &= w^{-1}(\varphi(x)_{]j+N,j_0|]}).
\end{aligned}$$

Thanks to my choice of k and w, $\overline{\varphi}$ satisfies the first three properties of Lemma 2.10.9. I leave it to the reader to verify that $K_{\mathrm{LW}} = L + k + \gamma$ works in the fourth.

The long marker intervals. Choose for each $z \in \operatorname{Per} X$ a minimal period $p_z \in \mathbb{W}(X)$ such that $p_z = p_{z'}$ when $z = \sigma^n(z')$ for some $n \in \mathbb{N}$. Choose also $i \in \mathbb{Z}$ such that $z_{[i,i+|p_z|[} = p_z$.

Similar to the proof of Proposition 4.1.3, I need sequences of words $\{u_n\}$ and $\{v_n\}$, such that $u_n, v_n \in S_n(Y)$ and

$$s u_n \lambda(z)^p_{[i,i+|p_z|[} v_m s \in \mathbb{W}^*(Y),$$

for all $z \in \operatorname{Per} X - E_{X,Y}$, $p \in \mathbb{N}$ and large large enough n and m. But in order to make sure that I can recognize the endpoints, I need to be able to recognize u_n and v_n, which makes their definition more complicated. The idea is to make sure that all u_n and v_n have the form $w_1 s^p w_2$, for some $0 < |w_1|, |w_2| \leq |s|$ and $p \geq 4$.

Since R is the number from Lemma 2.7.16 corresponding to Y, it follows that I for each $n, m \geq TL+R$ can find $u'_n \in S_n(Y)$ and $v'_m \in S_m(Y)$ depending only on p_z and n and m respectively, such that $su'_n\lambda(z)^k_{[i,i+|p_z|[}v'_m s \in \mathbb{W}^*(Y)$ for all $k \in \mathbb{N}$. Choose for each $n \geq TL$ a word $s_n \in \mathbb{W}(Y)$, such that $ss_n s \in \mathbb{W}(Y)$.

Define for each $n \geq 2TL + R + 4|s|$:

$$\begin{aligned} p &= \left\lfloor \frac{n - R - 2TL}{|s|} \right\rfloor, \\ n' &= \left\lfloor \frac{n - k|s| - R}{2} \right\rfloor + R \text{ and} \\ n'' &= \left\lceil \frac{n - k|s| - R}{2} \right\rceil. \end{aligned}$$

Then

$$\begin{aligned} TL + R \leq n' < TL + R + \frac{|s|}{2} < |s|, \\ TL \leq n'' < TL + \frac{|s|}{2} < |s| \end{aligned}$$

and $n' + n'' = n - p|s|$, which makes the following well-defined and ensures that $u_n, v_n \in S_n(Y)$ and that $p \geq 4$:

$$\begin{aligned} u_n &= s_{n''} s^p u'_{n'}, \\ v_n &= v'_{n'} s^p s_{n''}. \end{aligned}$$

In an attempt to make the upcoming definition easier to understand I introduce the name $\min_{|u|,|v|}$ for the minimal length of u_n and v_n, i.e.

$$\min_{|u|,|v|} = 2TL + R + 4|s|.$$

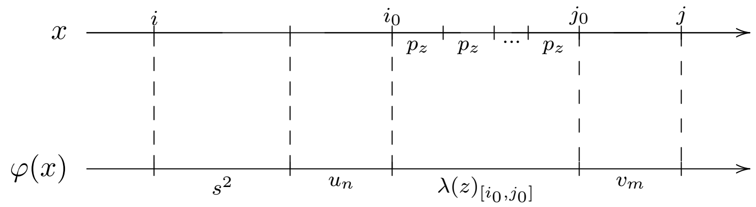
Define for each $x_{[i,j]} \in \mathbb{W}_{\text{long}}$:

$$\overline{\varphi}(x_{[i,j]}, \text{long}) = s^2 u_n \lambda(z)_{[i_0,j_0]} v_m,$$

where

$$\begin{aligned} i_0 &= \min\{k \geq i + |s^2| + \min_{|u|,|v|} \mid p_z = x_{[k,k+|p_z|[}\}, \\ j_0 &= \max\{k \leq j - \min_{|u|,|v|} \mid p_z = x_{]k-|p_z|,k]}\}, \end{aligned}$$

$n = i_0 - i - |s^2|$ and $m = j - j_0$.



This is well defined since $n, m \geq \min_{|u|,|v|}$ and $|[i_0, j_0]| \geq 0$, because $[i_0, j_0]$ is contained in $[i + |s^2| + \min_{|u|,|v|} + |p_z|, j - (\min_{|u|,|v|} + |p_z|)]$, which contains at least one occurrence of p_z, since $|[i+|s^2|+\min_{|u|,|v|} + |p_z|, j-(\min_{|u|,|v|} + |p_z|)]| = |[i, j]| - (2|s| + 2\min_{|u|,|v|} + 2|p_z|) > 2T > 2|p_z|$.

$\overline{\varphi}$ maps the long marker words to words in $\mathbb{W}^*(Y)$ because the z in the definition has to be in $\operatorname{Per} X - E_{X,Y}$ because otherwise the marker interval would be part of a local word interval by my choice of k.

Note that $\varphi(x)_{[i+|s^2|+\min_{|u|,|v|} + T, j-(\min_{|u|,|v|} + T)]}$ determines $\lambda(z)$ and hence z by Lemma 2.3 in [B], since $[i+|s^2|+\min_{|u|,|v|} + T, j-(\min_{|u|,|v|} + T)] \subseteq [i_0, j_0]$, and $|[i + |s^2| + \min_{|u|,|v|} + T, j - (\min_{|u|,|v|} + T)]| > 2T$. Hence φ is injective on the long marker intervals.

Thanks to my choice of n, m and v_m, $\overline{\varphi}$ satisfies the first three properties of Lemma 2.10.9. I leave it to the reader to verify that $K_{\text{long}} = 3T$ works in the fourth.

The moderate marker intervals. By my choice of M I can for each $n \geq M$ define an injective map $\phi\colon \mathbb{W}_n(X) \hookrightarrow \mathbb{W}_{n-3|s|-2TL}(A_K)$.

Choose for each $v \in \mathbb{W}(Y)$ a pair of words $b_-, b_+ \in \mathbb{W}(Y)$ such that $sb_-vb_+s \in \mathbb{W}(Y)$.

For each moderate marker word $x_{[i,j]} \in \mathbb{W}_{\text{mod}}$ I define:

$$\overline{\varphi}(x_{[i,j]}) = s^3 b_- \phi(x_{[i,j]}) b_+.$$

This is possible because a marker interval is at least $T > M$ symbols long by definition. Note that $5T$ is an upper limit on the length of these intervals.

The first three properties of Lemma 2.10.9 are clearly satisfied for $\overline{\varphi}$ on the moderate marker words. And $K_{\text{mod}} = 5T$ works in the fourth because $5T$ is an upper bound on the length of the moderate marker intervals.

The remaining intervals. For each remaining word $x_{[i,j]} \in \mathbb{W}_{\text{rem}}$ I define:

$$\overline{\varphi}(x_{[i,j]}) = s^3 w_- \phi(x_{[i,j]}) w_+.$$

Note that $\overline{\varphi}$ is injective on the moderate marker words and the remaining words, since $w = \phi^{-1}(\overline{\varphi}(w)_{[3|s|+TL, |w|-TL[})$.

The first three properties of Lemma 2.10.9 are clearly satisfied for $\overline{\varphi}$ on $\mathbb{W}_{\text{rem}}$ and just like in the proof of Proposition 4.1.3, my choice of k implies that any marker interval of length at least $2T$ is a subinterval of either a local word interval, a long maker interval or a moderate marker interval. Hence $6T - 4$ is an upper bound on the length of the remaining intervals, which implies that $K_{\text{rem}} = 6T - 4$ works in the fourth property.

Conclusion Since I_τ is a splitting map and $\overline{\varphi}\colon \mathbb{W}_{M,\tau} \to \mathbb{W}^*(Y)$ satisfies the four properties on each type of interval the induced map φ is a morphism from X to Y by Lemma 2.10.9.

All that remains is to prove that φ is injective. To see that, I take an arbitrary point $\varphi(x) \in \varphi(X)$. Thanks to the third, fourth and fifth property of s and my definition of u_n, v_n the locations of words of the form s^k for $k \in [1,3]$ give away the left endpoints of the intervals used to define $\varphi(x)$. So if I could figure out the type of each interval, I could use the injectivity of the map on each interval to recover x, which would complete the proof.

The type of each left finite interval is given away by the number of s's in the beginning of it, so all that remains are the left infinite intervals. By the definition of v_m, an interval which is left infinite and right finite is a long marker interval if and only if s^4 occurs in the last $2T$ symbols. And if s^4 does not occur, the interval is a local word interval, since infinite intervals are either local or long marker intervals. Finally, if the interval is both right and left infinite, then it is a local word interval if it is not $<T$ periodic, and if it is $<T$ periodic, I can't tell its type, but it doesn't matter since I can tell which periodic point it is the image of, since $\widetilde{\varphi}_S(x) = \lambda(y) \Rightarrow x = y$ by assumption. □

Note that when S is almost SFT relative to X, then the third requirement is superfluous.

6.2 Existence Results

Definition 6.2.1. Let X and Y be sofic shift spaces. X and Y are called correlated if there exists injective maps, $\varphi_\partial\colon \partial X \to Y$ and $\lambda\colon \operatorname{Per} X \to \operatorname{Per} Y$ with the following properties:

1. $\lambda(X_0^{(n)}) \subseteq Y_0^{(n)}$, for all $n \in \mathbb{N}$,

2. φ_∂ is marked,

3. $\widetilde{\varphi}_\partial\colon \operatorname{L}(\partial X, n) \to Y$ is injective for some $n \in \mathbb{N}$,

4. $\widetilde{\varphi}_\partial$ and λ agree on their common domain.

Theorem 6.2.2. *Let X and Y be sofic shift spaces, X mixing. If X embeds into Y, then there is an irreducible component Y_c in Y, such that*

1. X and $\overline{Y_c}$ are correlated and

2. $\mathrm{h}(X) \leq \mathrm{h}(\overline{Y_c})$.

Conversely, if there is an irreducible component Y_c in Y, such that 1. holds and

2'. $\mathrm{h}(X) < \mathrm{h}(\overline{Y_c})$,

then X embeds into Y.

Proof. Assume that $\varphi\colon X \to Y$ is an embedding. Then by Lemma 7.1 in [T] I can find an irreducible component Y_c in Y, such that $\varphi(X) \subseteq \overline{Y_c}$, $\varphi(X) \cap S(\overline{Y_c}) \neq \emptyset$ and $\varphi(X_0^{(n)}) \in Y_0^{(n)}$, for all $n \in \mathbb{N}$. Define $\lambda = \varphi_{|\operatorname{Per} X}$ and $\varphi_\partial = \varphi_{|\partial X}$. Then

1. $\lambda(X_0^{(n)}) \subseteq Y_0^{(n)}$, for all $n \in \mathbb{N}$ by definition,
2. φ_∂ is marked by Lemma 3.5.3,
3. $\varphi_\partial\colon \mathrm{L}(\partial X, n) \to Y$ is injective, when n is the block length of φ, since $\mathrm{L}(\partial X, n) \subseteq X$ and φ is injective,
4. φ_∂ and λ agree on their common domain, since they are restrictions of the same function.

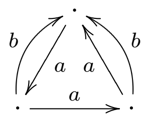
For the converse assume that there is an irreducible component Y_c in Y, such that $\mathrm{h}(X) < \mathrm{h}(\overline{Y_c})$ and X and $\overline{Y_c}$ are correlated. All I need to prove is that $\overline{Y_c}$ is mixing, because then X embeds into $\overline{Y_c} \subseteq Y$ by Proposition 6.1.4. To see that $\overline{Y_c}$ is mixing I observe that the existence of λ implies that $\operatorname{period}(Y_c^{(1)})$ divides $\operatorname{period}(X_0^{(1)})$, which is 1, since X is mixing. Thus $\operatorname{period}(Y_c^{(1)}) = 1$ and $\overline{Y_c}$ is mixing by Lemma 6.2.3 below. □

The following Lemma is due to Thomsen.

Lemma 6.2.3. *If* $\operatorname{period}(Y_c^{(1)}) = 1$*, then* $\overline{Y_c}$ *is mixing.*

Proof. Lemma 5.7 in [T] provides an increasing sequence of irreducible SFTs $\{A_n\}$ in $\overline{Y_c}$, such that $\overline{\cup_{n\in\mathbb{N}} A_n} = \overline{Y_c}$ and $\operatorname{period}(A_n) = \operatorname{period}(Y_c^{(1)}) = 1$, for all $n \in \mathbb{N}$. Since each A_n is an SFT of period 1, they are all mixing by Lemma 3.6 in [T]. I claim that the fact $\overline{\cup_{n\in\mathbb{N}} A_n} = \overline{Y_c}$ implies that $\overline{Y_c}$ is mixing. To see that, observe that $\mathbb{W}(\overline{Y_c}) = \mathbb{W}(\overline{\cup_{n\in\mathbb{N}} A_n}) = \mathbb{W}(\cup_{n\in\mathbb{N}} A_n)$. Thus any pair of words in $\mathbb{W}(\overline{Y_c})$ lie in the same mixing *SFT* within Y. □

Example 6.2.4. Let X be the mixing sofic shift presented by the following graph:



And let Y be a sofic shift. Then Theorem 6.2.2 implies that if X embeds into Y then there is an irreducible component of Y, Y_c, for which $\mathrm{h}(\overline{Y_c}) \geq \mathrm{h}(X)$, and an injective, shift-commuting and affiliation preserving map $\lambda\colon \operatorname{Per} X \to \operatorname{Per} \overline{Y_c}$.

And conversely if there is such an irreducible component Y_c which has strictly larger entropy than X and a map λ, then X embeds into Y.

Hence in this example it is possible to tell whether X embeds into a sofic shift Y by looking only at periodic points.

Note that X does not have transparent affiliation pattern and that Y doesn't have to contain a fixed point which is 1-affiliated to Y_0, which means that this example does not follow from Thomsen's, Boyle's or Krieger's results.

Chapter 7

Special Cases

Chapter 4 and 6 indicate that understanding the marked property is critical when trying to solve the lower entropy problem and higher entropy embedding problem.

In this chapter I investigate the marked condition under different simplifying assumptions on S. In Section 3.8 I showed that under mild assumptions, the marked condition is equivalent to the simpler condition called 'simply marked'. In the next section I show that if S is SFT relative to X, then the marked condition simplifies further and there is even a finite time algorithm, which decides whether a given morphism $S \to Y$ is marked. And when S contains only finitely many points the marked condition involves only periodic points and there is a finite time algorithm, which decides whether $X \twoheadrightarrow Y$.

7.1 When S is SFT-like.

When a shift S is SFT-like with respect to a shift X and $L \in \mathbb{N}$, I denote the set $A(\mathrm{SL}_X(S), L)$ by $A(L)$ and the set $\Omega(\mathrm{SL}_X(S), L)$ by $\Omega(L)$, for all $L \in \mathbb{N}$. The reader easily verifies the following lemma:

Lemma 7.1.1. *Let S be a subshift of a shift space X. If S is SFT-like with respect to X, then $A(n, L) = A(L)$, $\Omega(n, L) = \Omega(L)$ and $\mathrm{MLW}(S, n, x) = \mathrm{MW}_{\geq n}(S, x)$, for all $L \geq n \geq \mathrm{SL}_X(S)$ and $x \in X$.*

Thus when S is SFT-like, the definition of m-simply marked from page 63 reduces to the following:

Definition 7.1.2. Let X be a shift space and S be a SFT-like subshift of X. A morphism $\varphi\colon S \to Y$, is called m-simply marked, for some $m \geq \mathrm{SL}_X(S)$, if there exists an $L \in \mathbb{N}$ and maps $\alpha\colon A(L) \to S(Y)$ and $\omega\colon \Omega(L) \to S(Y)$, such that

$$\alpha(x_{[i-L,i+L]})\varphi_m(x_{[i,j]})\omega(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $x \in X$ and $[i,j] \in \mathrm{MW}_{\geq m}(S, x)$.

The following Lemma proves that when S is SFT-like with respect to X, then a morphism $S \to Y$ is marked if and only if it is simply marked. The point being that S does not have to have synchronizing edges as in Lemma 3.8.3.

Lemma 7.1.3. *Let S be a subshift of a shift space X, and let Y be a shift space. If S is SFT relative to X and $m \geq \mathrm{SL}_X(S)$, then a morphism $\varphi\colon S \to Y$ is m-simply marked if and only if it is m-marked.*

Proof. $\Leftarrow$: Follows directly from definition 3.4.3 and Lemma 7.1.1.
$\Rightarrow$: Assume that $\varphi\colon S \to Y$ is m-simply marked, for some $m \geq \mathrm{SL}_X(S)$. Let L', α and ω be the ones from the definition of simply marked. Let $n \geq m$ and define $L = \max\{L', n-2\}$. Then $x_{[-L',L']} \in A(L')$ for each $x_{[-L,L]} \in A(n,L) = A(L)$ and $x_{[-L',L']} \in \Omega(L')$ for each $x_{[-L,L]} \in \Omega(n,L) = \Omega(L)$. Define $\alpha_n\colon A(n,L) \to S(Y)$ by $\alpha_n(x_{[-L,L]}) = \alpha(x_{[-L',L']})$ and $\omega_n\colon \Omega(n,L) \to S(Y)$ by $\omega_n(x_{[-L,L]}) = \varphi_m(x_{[-n+2,0]})\omega(x_{[-L',L']})$. Then ω_n is well-defined because $L \geq n-2$.

Let $x \in X$ and $[i,j] \in \mathrm{MLW}(S,n,x)$. Then

$$\alpha_n(x_{[i-L,i+L]})\varphi_n(x_{[i,j]})\omega_n(x_{[j-L,j+L]}) = \\ \alpha(x_{[i-L',i+L']})\varphi_m(x_{[i,j]})\omega(x_{[j-L',j+L']}) \in \mathbb{W}^*(Y),$$

since $\mathrm{MLW}(S,n,x) = \mathrm{MW}_{\geq n}(S,x) \subseteq \mathrm{MW}_{\geq m}(S,x)$. □

Remark 7.1.4. To simplify notation I have assumed that the morphism $S \to Y$ has memory 0. But I could of course define (m,n)-simply marked in the obvious way, and Lemma 7.1.3 would still hold with m replaced by (m,n).

7.1.1 Decidability of the Marked Condition

In this section I will show that, when S is SFT relative to X, then there is a finite time algorithm, which decides whether or not a given morphism $S \to Y$ is marked. In order to do that I show that it is possible to find upper bounds on each variable in the marked condition.

Definition 7.1.5. Let X and Y be shift spaces and S be an SFT-like subshift of X. A morphism $\varphi\colon S \to Y$, is called $(k,L,1)$-marked, if there exists maps $\alpha\colon A(L) \to S(Y)$ and $\omega\colon \Omega(L) \to S(Y)$, such that

$$\alpha(x_{[i-L,i+L]})\varphi_1(x_{[i+k,j-k]})\omega(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $x \in X$ and $[i,j] \in \mathrm{MW}_{\geq 2k+1}(S,x)$.

Lemma 7.1.6. *Let X and Y be shift spaces and S be an SFT-like subshift of X with $\mathrm{SL}_X(S) = 0$. If a morphism $S \to Y$ is $(k,L,1)$-marked, for some $k, L \in \mathbb{N}$, then it is marked.*

Proof. Note that $\varphi_1(x_{[i+k,j-k]}) = \varphi_{k,k}(x_{[i,j]})$ and apply Lemma 7.1.3 (and Remark 7.1.4) to prove that φ is (k,k)-marked. □

Thus $(k,L,1)$-marked is a sufficient condition for a morphism $\varphi\colon S \to Y$ to be marked. The following lemma shows that it is also necessary for some recoding of φ. It even gives (weak) upper bounds on k and L, which can be determined by knowing only X, Y, S and φ_S.

Lemma 7.1.7. *Let X be an irreducible sofic shift, Y be a shift space, $S \subseteq X$ be SFT-like and $\varphi\colon S \to Y$ be marked. Assume that*

1. $\mathrm{SL}_X(S) = 0$,
2. $\varphi\colon S \to Y$ *is* 1*-block,*
3. *The Fischer cover of X has V vertices.*

Then φ is $(V^V+1, V^V, 1)$-marked.

Proof. Define $k = V^V + 1$ and $L = V^V$. And assume that φ is (m,n)-marked with $L_{m,n}$, $\alpha_{m,n}$ and $\omega_{m,n}$.

Let $x = x_{[-L,L]} \in A(L)$. Using the pigeon hole principle, I can pump both the right and left half of x to obtain a word $u = u_{[-L_{m,n},k]} \in \mathbb{W}(X)$ with the following properties:

1. $k \geq \max\{L_{m,n}, m\}$,
2. $u_{[-L_{m,n},L_{m,n}]} \in A(L_{m,n})$ and $u_{[0,k]} \in \mathbb{W}(S)$,
3. for each path $p \in \pi^{-1}(x)$, there is a path in $\pi^{-1}(u)$ with the same terminal vertex as p.

Define $u_A = u_{[-L_{m,n},L_{m,n}]}$, $u_S = u_{[0,k]}$ and

$$\alpha(x) = \alpha_{m,n}(u_A)T_m^-(\varphi_1(u_S)),$$

where T_m^- is the map $\mathbb{W}_{\geq m}(X) \to \mathbb{W}(X)$, which removes the first m symbols.

The definition of ω is analogous:

Let $x = x_{[-L,L]} \in \Omega(L)$. Using the pigeon hole principle, I can pump both the right and left half of x to obtain a word $v = v_{[-k,L_{m,n}]} \in \mathbb{W}(X)$ with the following properties:

1. $k \geq \max\{L_{m,n}, m\}$,
2. $v_{[-L_{m,n},L_{m,n}]} \in \Omega(L_{m,n})$ and $v_{[-k,0]} \in \mathbb{W}(S)$,
3. for each path $p \in \pi^{-1}(x)$, there is a path in $\pi^{-1}(v)$ with the same initial vertex as p.

Define $v_\Omega = v_{[-L_{m,n},L_{m,n}]}$, $v_S = v_{[-k,0]}$ and

$$\omega(x) = T_n^+(\varphi_1(v_S))\omega_{m,n}(v_\Omega),$$

where T_n^+ is the map $\mathbb{W}_{\geq n}(X) \to \mathbb{W}(X)$, which removes the last n symbols.

To verify that α and ω work, I let $x \in X$ and $[i,j] \in \mathrm{MW}_{\geq 2k+1}(S,x)$. Then

$$\begin{aligned}
&\alpha(x_{[i-L,i+L]})\varphi_1(x_{[i+k,j-k]})\omega(x_{[j+L,j+L]}) \\
&\quad = \alpha_{m,n}(u_A)T_m^-(\varphi_1(u_S))\varphi_1(x_{]i+L,j-L[})T_n^+(\varphi_1(v_S))\omega_{m,n}(v_\Omega) \\
&\quad = \alpha_{m,n}(u_A)(T_m^- \circ T_n^+)(\varphi_1(u_S x_{]i+L,j-L[} v_S)\omega_{m,n}(v_\Omega) \\
&\quad = \alpha_{m,n}(u_A)\varphi_{m,n}(u_S x_{]i+L,j-L[} v_S)\omega_{m,n}(v_\Omega),
\end{aligned}$$

which I claim is in $\mathbb{W}^*(Y)$, since φ is (m,n)-marked. To see that, observe that $ux_{]i+L,j-L[}v \in \mathbb{W}^*(X)$, by property 3 of u and v, and that $u_S x_{]i+L,j-L[} v_S$ corresponds to an interval in $\mathrm{MW}_{\geq m+n+1}(S,x')$, for some point $x' \in X$ in which $ux_{]i+L,j-L[}v$ occurs, by property 1 and 2, since $\mathrm{SL}_X(S) = 0$. $\square$

If I had an upper bound on the length of the words in the image of α and ω in the definition of $(k,L,1)$-marked, i.e. if i could find an $R \in \mathbb{N}$ depending only on X, Y, S and φ, such that $\alpha\colon A(L) \to S_{\leq R}(Y)$ and $\omega\colon \Omega(L) \to S_{\leq R}(Y)$, then, since $A(L)$ and $\Omega(L)$ are finite sets, there would be only finitely many possibilities for defining α and ω. So when trying to decide if a morphism is $(k,L,1)$-marked, I could simply try each map from $A(L)$ to $S_R(Y)$ for α and each map $\Omega(L) \to S_R(Y)$ for ω. And if one combination works, then the map is marked and if not, then it is not marked.

Fortunately Lemma 2.7.17 provides such an R.

Proposition 7.1.8. *Let X be an irreducible sofic shift, Y be a mixing sofic shift of strictly lower entropy than X, $S \subseteq X$ be SFT relative to X and φ be a morphism from S to Y. Then there is a finite time algorithm, which decides whether or not φ is marked.*

Proof. The algorithm in pseudo code:

1. Recode φ to ensure that $\mathrm{SL}_X(S) = 0$ and φ is 1-block.

2. Let V_X and V_Y be the number of vertices in the Fischer cover of X and Y respectively. Define $L = V_X^{V_X}$.

3. Find an $s \in S(X)$ and define $R = V_Y^{V_Y} + 2V_Y + |s|$. [1]

4. Find all elements in the sets $A(L), \Omega(L)$ and $S_{\leq R}(Y)$.

[1] To find s, pick a word in $\mathbb{W}(Y) - \mathbb{W}(\partial Y)$ and extend it by V_Y^2 symbols to the right and left to a word in $\mathbb{W}(Y)$. Then the resulting word is in $S(Y)$ by Lemma 3.5.5.

5. Define maps $t\colon A(L) \to 2^{V_X}$ and $i\colon \Omega(L) \to 2^{V_Y}$ by $t(a) = t(\pi_X^{-1}(a))$ and $i(w) = i(\pi_X^{-1}(w))$. [2]

6. Construct the local word graph, by deleting all edges, whose labels are not in Σ_S.

7. Relabel the graph by using φ_S on each label.

8. **for each** pair of maps $\alpha\colon A(L) \to S_{\leq R}(Y)$ and $\omega\colon \Omega(L) \to S_{\leq R}(Y)$ **:**
    - **for each** $a \in A(L)$ **:**
        - Attach a path with label $\alpha(a)$ at each vertex in $t(a)$.
    - **for each** $w \in \Omega(L)$ **:**
        - Attach a path with label $\omega(w)$ at each vertex in $i(w)$.
    - **if** the language presented by the graph is a sublanguage of Y **then**
        - **return** YES.
    - **else**
        - Delete all the added paths.

    return NO.

Since there are only finitely many possible choices for α and ω, the last for loop takes only finite time, because the sublanguage problem is decidable. □

Theorem 4.3.1 together with Proposition 7.1.8 implies:

Corollary 7.1.9. *Let S be a SFT-like subshift of an irreducible sofic shift X, Y be a mixing sofic shift of strictly lower entropy than X such that $E_{X,Y} \subseteq S \subseteq \partial X$, and φ be a morphism from S to Y. If S has synchronizing edges then there is a finite time algorithm, which decides whether or not φ extends to a factor map $X \twoheadrightarrow Y$.*

Corollary 7.1.10. *Let X be an irreducible sofic shift such that ∂X is SFT-relative to X, Y be a mixing sofic shift of strictly lower entropy than X, and φ be a morphism from ∂X to Y. Then there is a finite time algorithm, which decides whether or not φ extends to a factor map $X \twoheadrightarrow Y$.*

Proof. Given the algorithm for checking whether φ is marked, Corollary 4.3.2 implies that all we need is an algorithm for checking whether $E_{X,Y} \subseteq \partial X$. But since Y is mixing there are only finitely many candidates and for each candidate there are only finitely many points in Y we need to inspect. Thus $E_{X,Y} \subseteq \partial X$ is also decidable. □

[2] i.e. t finds the set of terminal vertices of the paths in the Fischer cover, which present a given entry, and i finds the set of initial vertices of the paths in the Fischer cover, which present a given exit.

7.1.2 When S is SFT-like and X is AFT

In this section I show that if we assume something more about X, then the upper limits from the previous section can be improved significantly. The extra assumption is that X is an AFT, which is defined by:

Definition 7.1.11 (AFT)**.** An irreducible sofic shift is said to have almost finite type if the Fischer cover of $X^{[n]}$ is left-resolving for some $n \in \mathbb{N}$.

Lemma 7.1.12. *Let X be an irreducible sofic shift, Y be a shift space, $S \subseteq X$ be SFT-like and $\varphi\colon S \to Y$ be marked. Assume that*

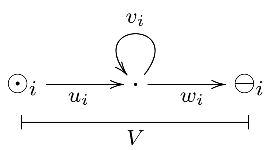
1. *The Fischer cover of X is left-resolving and has V vertices,*
2. $\mathrm{SL}_X(S) = 0$ *and* $A(L_s), \Omega(L_s) \subseteq S(X)$ *for some* $L_s \in \mathbb{N}$*,*
3. $\varphi\colon S \to Y$ *is* 1*-block.*

Then φ is $(V, L, 1)$-marked, where $L = \max\{L_s, V\}$*.*

Proof. Denote the Fischer cover of X by F_X. Construct the first local word graph of ∂X. Assume that φ is l-block and (m, n)-marked with L_l, α_l and ω_l, and that $A(L_l), \Omega(L_l) \subseteq S(X)$.

Definition of α:

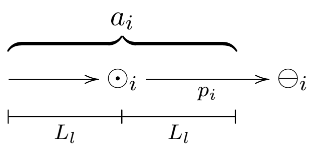
Put a $\ominus$ at each vertex to which there is a path of length V from a $\odot$, and enumerate these $\ominus$'s in some arbitrary way $\{\ominus_i\}_{i\in I}$. Then by the pigeon hole principle, I can to each such $\ominus_i$ assign a $\odot_i$ and a word $u_i v_i w_i \in \mathbb{W}(S)$ of length V, such that there is a path from $\odot_i$ to $\ominus_i$, with label $u_i v_i^k w_i$, for each $k \in \mathbb{N}$ and $i \in I$.



Define $p_i = u_i v_i^{\max\{m, L_l\}} w_i$ and $S_-\colon I \to S(Y)$ by

$$S_-(i) = \alpha_l(a_i) T_m^-(\varphi_1(p_i)),$$

where T_m^- is the map $\mathbb{W}_{\geq m}(X) \to \mathbb{W}(X)$, which removes the first m symbols, and $a_i = (a_i)_{[-L_l, L_l]}$ is an entry in $A(L_l)$, such that $(a_i)_{[0, L_l]}$ is a prefix of p_i and such that the path presenting a_i in F_X goes through $\odot_i$ at index 0.



All I need now is a map $I\colon A(L) \to I$.

In order to define I let $x_{[-L,L]} \in A(L)$ be arbitrary. Then $x_{[-L,L]}$ is synchronizing for X, since $L \geq L_s$. Thus there is only one path in F_X, $p_{[-L,L]}$, with label $x_{[-L,L]}$, since F_X is left-resolving. By definition of the $\ominus$'s, I can therefore find an $i \in I$, such that $\ominus_i$ is the terminal vertex of p_V. Define $I(x_{[-L,L]}) = i$.

Define

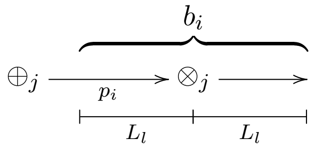
$$\alpha = S_- \circ I.$$

The definition of ω is analogous:

Put a $\oplus$ at each vertex from which there is a path of length V to a $\otimes$, and enumerate these $\oplus$'s in some arbitrary way $\{\oplus_j\}_{j\in J}$. Then by the pigeon hole principle, I can to each such $\oplus_j$ assign a $\otimes_j$ and a word $u_j v_j w_j \in \mathbb{W}(S)$ of length V, such that there is a path from $\oplus_j$ to $\otimes_j$, with label $u_j v_j^k w_j$, for each $k \in \mathbb{N}$ and $j \in J$. Define $p_j = u_j v_j^{\max\{n,L_l\}} w_j$ and $S_+ \colon J \to S(Y)$ by

$$S_+(j) = T_n^+(\varphi_1(p_j))\omega_l(b_j),$$

where T_n^+ is the map $\mathbb{W}_{\geq n}(X) \to \mathbb{W}(X)$, which removes the last n symbols, and $b_j = (b_j)_{[-L_l,L_l]}$ is an exit in $\Omega(L_l)$, such that $(b_j)_{[-L_l,0]}$ is a suffix of $u_j v_j^{\max\{n,L_l\}} w_j$ and such that the path presenting b_j in F_X goes through $\otimes_j$ at index 0.



All I need now is a map $J \colon \Omega(L) \to J$.

In order to define J let $x_{[-L,L]} \in \Omega(L)$ be arbitrary. Then $x_{[-L,L]}$ is synchronizing for X, since $L \geq L_s$. Thus there is only one path in F_X, $p_{[-L,L]}$, with label $x_{[-L,L]}$, since F_X is left-resolving. By definition of the $\oplus$'s, we can therefore find a $j \in J$, such that $\oplus_j$ is the terminal vertex of p_{-V}. Define $J(x_{[-L,L]}) = j$.

Define

$$\omega = S^+ \circ J.$$

In order to check that α and ω work, I let $x \in X$ and $[i,j] \in \mathrm{MW}(S,x)$ be arbitrary. And to simplify the expressions involved, I define $i' = I(x_{[i-L,i+L]})$ and $j' = J(x_{[j-L,j+L]})$. Then

$$\begin{aligned}
&\alpha(x_{[i-L,i+L]})\varphi_1(x_{[i+V,j-V]})\omega(x_{[j+L,j+L]}) \\
&\quad = \alpha_l(a_{i'})T_m^-(\varphi_1(p_{i'}))\varphi_1(x_{[i+V,j-V]})T_n^+(\varphi_1(p_{j'}))\omega_l(b_{j'}) \\
&\quad = \alpha_l(a_{i'})(T_m^- \circ T_n^+)(\varphi_1(p_{i'})\varphi_1(x_{[i+V,j-V]})\varphi_1(p_{j'}))\omega_l(b_{j'}) \\
&\quad = \alpha_l(a_{i'})\varphi_{m,n}(p_{i'}x_{[i+V,j-V]}p_{j'})\omega_l(b_{j'}),
\end{aligned}$$

$$\begin{tikzcd}
 & \odot \arrow[r, "x_{[i,i+V[}"] & \ominus_{i'} \arrow[r, "x_{[i+V,j-V]}"] & \oplus_{j'} \arrow[r, "x_{]j-V,j]}"] \arrow[dr, "p_{j'}"] & \otimes & \\
{} \arrow[r, "(a_{i'})_{[-L_l,-1]}"] & \odot_{i'} \arrow[ur, "p_{i'}"] & & & \otimes_{j'} \arrow[r, "(b_{i'})_{[1,L_l]}"] & {}
\end{tikzcd}$$

which is in $W^*(Y)$, since φ is (m,n)-marked and since $p_{i'}x_{[i+V,j-V]}p_{j'}$ is in $\mathrm{MW}_{\geq m+n+1}(S,x)$, because there is a path in the local word graph from a $\odot$ to a $\otimes$ with that label. $\square$

Let S be an SFT-like subshift of an irreducible AFT, X, and let Y be an irreducible sofic shift. If S has synchronizing edges, then Lemma 7.1.12 and Lemma 2.7.17 give a finite time algorithm, which decides whether or not a given morphism $\varphi_S\colon S \to Y$ is marked. So if $E_{X,Y} \subseteq S \subseteq \partial X$ and Y is mixing sofic with strictly lower entropy than X, the algorithm tells us whether or not φ_S extends to a factor map $X \twoheadrightarrow Y$, by Theorem 4.1.1. Algorithm:

1. Recode φ to ensure that the conditions in Lemma 7.1.12 are satisfied.
2. Find L_s, V_Y and an $s \in S(X)$. Define $R = 2V_Y + |s|$.
3. Construct the local word graph and insert $\ominus$'s and $\oplus$'s as in the proof of Lemma 7.1.12.
4. Relabel the resulting graph by using φ_S on each label.
5. Try all combinations of words in $S_{\leq R}(Y)$ on the $\ominus$'s and $\oplus$'s and check whether the resulting regular language is a sublanguage of $W(Y)$.

Since there are only finitely many words in $S_{\leq R}(Y)$ and only finitely many vertices in the local word graph, 5. can be done in finite time, because the sublanguage problem is decidable.

The small size of R compared to the one in the proof of Lemma 2.7.17 is possible because when the Fischer cover is both left- and right-resolving, there is only one path p presenting a given synchronizing word $w \in S(Y)$. This makes it easy to replace w by a synchronizing word w' of length at most $2V_Y + |s|$: Choose a path of length at most V_Y from the initial vertex of p to the initial vertex of the path presenting s and a path of length at most V_Y from the terminal vertex of the path presenting s to the terminal vertex of p. Then the concatenation of these paths present a synchronizing word w', with the property from Lemma 2.7.17.

7.2 When S is Finite

If S is a finite subshift of X, then S must be a subset of $\operatorname{Per} X$, since it is shift invariant. Thus there are only finitely many morphisms from S into a shift space Y, since morphisms respect periods. And since Lemma 2.4.21 implies that S is an SFT with $\mathrm{SL}(S) = 2\max\{|p_x| \mid x \in S\} - 1$ Proposition 7.1.8 implies that if X is irreducible sofic, then there is a finite time algorithm which decides whether X factors onto a mixing sofic shift of strictly lower entropy.

The algorithm is however not very practical due to the size of the upper bounds involved. I therefore investigate what the marked property looks like when S is a finite set.

Since $S \subseteq \operatorname{Per} X$ the S entries and exits are entries and exits of periodic points.

Definition 7.2.1 (Periodic entries and exits)**.** Let X be a shift space and $x \in \operatorname{Per} X$. A word $y_{[-L,L]} \in W(X)$ is called an L-entry of x, if $y_{[0,L]} \subseteq x$ and $y_{[-1,L]} \nsubseteq x$. And it is called an L-exit of x, if $y_{[-L,0]} \subseteq x$ and $y_{[-L,1]} \nsubseteq x$. The set of all L-entries in x is denoted by $A_{\mathrm{Per}}(x, L)$ and the L-exits by $\Omega_{\mathrm{Per}}(x, L)$. The set of all L-entries of periodic points is denoted by $A_{\mathrm{Per}}(L)$ and the exits by $\Omega_{\mathrm{Per}}(L)$.

And since S is an SFT, $A_S(n, L) \subseteq A_{\mathrm{Per}}(L)$, $\Omega_S(n, L) \subseteq \Omega_{\mathrm{Per}}(L)$ and $\mathrm{MLW}(S, n, x) = \mathrm{MW}_{\geq n}(S, x)$, when $L \geq n \geq \mathrm{SL}(S)$, by Lemma 7.1.1, since $\mathrm{SL}(S) \geq \mathrm{SL}_X(S)$. The following is therefore a special case of Lemma 7.1.3.

Lemma 7.2.2. *Let X, Y and S be shift spaces and $S \subseteq X$ be a finite set. Then a map $\varphi\colon S \to Y$ is m-marked for some $m \geq \mathrm{SL}(S)$ if and only if $\exists L \in \mathbb{N}, \alpha\colon A_{\mathrm{Per}}(L) \to S(Y), \omega\colon \Omega_{\mathrm{Per}}(L) \to S(Y)$, such that*

$$\alpha(x_{[i-L,i+L]})\varphi_m(x_{[i,j]})\omega(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $x \in X$ and $[i, j] \in \mathrm{MW}_{\geq m}(S, x)$.

Definition 7.2.3. Let X be a shift space. A point $x \in \operatorname{Per} X$ is said to have synchronizing entries if there exists an $L \in \mathbb{N}$, such that $A_{\mathrm{Per}}(x, L) \subseteq S(X)$. And it is said to have synchronizing exits if there is an $L \in \mathbb{N}$, such that $\Omega(x, L) \subseteq S(X)$. x is said to have synchronizing edges if it has both synchronizing entries and exits.

Lemma 7.2.4. *Let X be a shift space and $S \subseteq X$ be finite. Then S has synchronizing edges if and only if each point in S has synchronizing edges.*

Proof. Left to the reader. □

Definition 7.2.5. Let X and Y be shift spaces. We say that $\operatorname{Per} X \twoheadrightarrow \operatorname{Per} Y$ if $E_{X,Y} \subseteq \partial X$ and there is a morphism $\varphi\colon E_{X,Y} \to \operatorname{Per} Y$, and $m \geq \mathrm{SL}(E_{X,Y})$, $L \in \mathbb{N}, \alpha\colon A_{\mathrm{Per}}(L) \to S(Y), \omega\colon \Omega_{\mathrm{Per}}(L) \to S(Y)$ such that

$$\alpha(x_{[i-L,i+L]})\varphi_m(x_{[i,j]})\omega(x_{[j-L,j+L]}) \in \mathbb{W}^*(Y),$$

for all $x \in X$ and $[i, j] \in \mathrm{MW}_{\geq m}(E_{X,Y}, x)$.

Remark 7.2.6. Despite its immediate appearance $\operatorname{Per} X \twoheadrightarrow \operatorname{Per} Y$ is in fact quite similar to $\operatorname{Per} X \to \operatorname{Per} Y$. $\operatorname{Per} X \twoheadrightarrow \operatorname{Per} Y$ is more or less $\operatorname{Per} \to \operatorname{Per} Y$ extended with the property from Remark 3.6.4.

Using Lemma 7.2.2 and Theorem 4.3.1, I obtain:

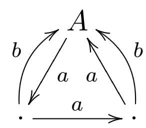
Theorem 7.2.7. *Let X and Y be sofic shift spaces. X irreducible and Y mixing. If $E_{X,Y}$ has synchronizing edges and $\mathrm{h}(X) > \mathrm{h}(Y)$, then*

$$X \twoheadrightarrow Y \Leftrightarrow \operatorname{Per} X \twoheadrightarrow \operatorname{Per} Y.$$

Corollary 7.2.8. *Let X and Y be sofic shift space. X irreducible and Y mixing. If ∂X is a finite set and $\mathrm{h}(X) > \mathrm{h}(Y)$, then*

$$X \twoheadrightarrow Y \Leftrightarrow \operatorname{Per} X \twoheadrightarrow \operatorname{Per} Y.$$

Example 7.2.9. Let X be the irreducible sofic shift presented by the labeled graph



where A is a graph, which presents an irreducible SFT with no fixed points. Let Y be a mixing sofic shift with lower entropy than X. Then $\partial X = \{a^\infty\}$ and Theorem 7.2.8 implies that $X \twoheadrightarrow Y$ if and only if $Q_1(Y_0^{(3,\{0,1\})}) \neq \emptyset$ and $\bigcup_{m|n} Q_m(Y_0^{(\frac{n}{m})}) \neq \emptyset$, for all $n > 1$.

I claim that Theorem 7.2.7 extends Theorem 1.3.6 in the sense that it handles more shift spaces.

To verify that it handles more shift spaces, I have to show that if the points in $E_{X,Y}$ have marked entries and exits, then $E_{X,Y}$ has synchronizing edges. But the definition of marked entries and exits implies, that all sufficiently long periodic entries and exits are magic for π_X, and therefore synchronizing for X by Proposition 2.6.22. This means that $E_{X,Y}$ has synchronizing edges by definition 3.5.1.

But how does one decide whether Theorem 7.2.7 applies to a given pair of shift spaces X and Y? I.e. how does one decide whether $E_{X,Y}$ has synchronizing edges? First we need to find the points in it. But since Y is mixing, there are only finitely many candidates. And checking whether a point x is in $E_{X,Y}$ is easy, since we only need to check the points in Y, whose period divides $|p_x|$, which is clearly a finite set.

The points in $E_{X,Y} \cap X_0$ have synchronizing edges by definition of X_0. So we need only a way of checking the points in $E_{X,Y} \cap \partial X$.

Definition 7.2.10 (Isolated component)**.** Let X be a shift space and x be a periodic point in X. $\{\sigma^n(x)\}_{n\in\mathbb{Z}}$ is called an isolated component in X if

$$\exists K \in \mathbb{N} \forall y \in X : p_x^K \subseteq y \Rightarrow y \in \{\sigma^n(x)\}_{n\in\mathbb{Z}}.$$

The reason for calling sets of the form $\{\sigma^n(x)\}_{n\in\mathbb{Z}}$, which satisfy the condition in the definition, for isolated components, is that any component reduced graph presenting a sofic shift X would have to contain a subgraph, which presents $\{\sigma^n(x)\}_{n\in\mathbb{Z}}$ and is not connected to anything.

Lemma 7.2.11. *Let X be an irreducible sofic shift. An $x \in \mathrm{Per}(\partial X)$ has synchronizing edges if and only if $\{\sigma^n(x)\}_{n\in\mathbb{Z}}$ is an isolated component in ∂X.*

Proof. $\Rightarrow$: Since $x \in \partial X$ has synchronizing edges, there is a $K \in \mathbb{N}$, such that $A_{\mathrm{Per}}(x, L) \subseteq S(X)$ and $\Omega_{\mathrm{Per}}(x, L) \subseteq S(X)$ for all $L \geq K$. I claim that the same K can be used to show that $\{\sigma^n(x)\}_{n\in\mathbb{Z}}$ is an isolated component in ∂X. Assume that $p_x^K \subseteq y$ for some $y \in \partial X$. If $y \notin \{\sigma^n(x)\}_{n\in\mathbb{Z}}$, then y contains an element in $\Omega_{\mathrm{Per}}(x, K|p_x|) \cup A_{\mathrm{Per}}(x, K|p_x|)$. So as $K|p_x| \geq K$, y contains a synchronizing word. Thus $y \notin \partial X$, which contradicts our assumption on y. Thus $y \in \{\sigma^n(x)\}_{n\in\mathbb{Z}}$.
$\Leftarrow$: Assume that $\{\sigma^n(x)\}_{n\in\mathbb{Z}}$ is an isolated component in ∂X. Choose $K \in \mathbb{N}$ such that $p_x^K \subseteq y \Rightarrow y \in \{\sigma^n(x)\}_{n\in\mathbb{Z}}$ for all $y \in \partial X$. Let $y_{[-k,k]}$ be a $|p_x|(K+1) + V^2$-entry of x, where V is the number of vertices in the Fischer of X. I claim that $y_{[-|p_x|(K+1),|p_x|(K+1)]} \notin \mathbb{W}(\partial X)$.

Assume that $y_{[-|p_x|(K+1),|p_x|(K+1)]} \in \mathbb{W}(\partial X)$. Then we can find a $y' \in \partial X$, such that $y_{[-|p_x|(K+1),|p_x|(K+1)]} \subseteq y'$. So since $p_x^K \subseteq y_{[0,|p_x|(K+1)]}$, y' must be in $\{\sigma^n(x)\}_{n\in\mathbb{Z}}$ by our choice of K. But that contradicts the fact that $y_{[-1,|p_x|(K+1)]} \notin x$, since $y_{[-k,k]}$ is an entry of x. Thus $y_{[-|p_x|(K+1),|p_x|(K+1)]} \notin \mathbb{W}(\partial X)$ as claimed.

This implies that $y_{[-k,k]}$ is synchronizing by Lemma 3.5.5. The case when $y_{[-k,k]}$ is an exit is handled analogously. Thus x has synchronizing edges. $\square$

This makes it easy to check whether the points in $E_{X,Y} \cap \partial X$ have synchronizing edges, when X is irreducible sofic. Simply use the algorithm described after definition 2.7.3 to construct a graph presenting ∂X. After deleting superfluous components, check whether the points in $E_{X,Y} \cap \partial X$ correspond to isolated components.

Chapter 8

Eventual Factors

In this chapter, I prove that if X and Y are mixing sofic shifts such that $\mathrm{h}(X) > \mathrm{h}(Y)$, then Y is a eventual factor of X. This result is not new. The new thing is that my proof does not use Krieger's marker lemma.

Definition 8.1.1 (Eventual Factor)**.** Let X and Y be shift spaces. Y is called an eventual factor of X, if there exists an $N \in \mathbb{N}$, such that $X^n \twoheadrightarrow Y^n$ for all $n \geq N$.

Lemma 8.1.2. *Let X be a mixing shift space. Then X^n is mixing for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{W}(X^n)$. Since X is mixing, we can for each $k \geq \mathrm{TL}(X)$ find an $x \in \mathbb{W}_k(X)$ such that $\lambda^{-1}(a)x\lambda^{-1}(b) \in \mathbb{W}(X)$. This implies that $\forall k \geq \lceil \frac{\mathrm{TL}(X)}{n} \rceil \exists x \in \mathbb{W}_{kn}(X)\colon \lambda^{-1}(a)x\lambda^{-1}(b) \in \mathbb{W}(X)$. Thus $\lceil \frac{\mathrm{TL}(X)}{n} \rceil$ is a transition length for X^n. $\square$

Lemma 8.1.3. *Let X be a shift space and $x \in \mathbb{W}(X^n)$ for some $n \in \mathbb{N}$. Then*

$$x \in S(X^n) \Leftrightarrow \lambda_n^{-1}(x) \in S(X).$$

Proof. Left to the reader. $\square$

Lemma 8.1.4. *Let X be a mixing sofic shift. Then*

$$\lambda_n(\mathrm{Per}_n(X_0^{(1)})) \subseteq Q_1(X^n)_0^{(1)},$$

for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and $x \in \mathrm{Per}_n(X_0^{(1)})$. Then $x \in Q_m(X_0^{(1)})$ for some $m|n$. By Lemma 2.7.16 we can therefore find $u, v \in S_{kn}(X)$, for some $k \in \mathbb{N}$, such that $up_x^i v \in \mathbb{W}(X)$, for all $i \in \mathbb{N}$, where $p_x = x_{[0,m[}$. Define $p = \lambda_n(p_x^{\frac{n}{m}})$. Then $\lambda_n(up_x^{\frac{n}{m}i}v) = u'p^i v' \in \mathbb{W}(X^n)$ for all $i \in \mathbb{N}$. So as $u', v' \in S(X^n)$ by lemma 8.1.3, $\lambda_n(x) = p^\infty \in Q_1(X^n)_0^{(1)}$. $\square$

Lemma 8.1.5. *Let X be a shift space and $n \in \mathbb{N}$. Then*

$$\mathrm{h}(X^n) = n\,\mathrm{h}(X).$$

Proof. As $|\mathbb{W}_i(X^n)| = |\mathbb{W}_{in}(X)|$, we get that

$$\mathrm{h}(X^n) = \lim_{i\to\infty} \frac{1}{i}|\mathbb{W}_i(X^n)| = \lim_{i\to\infty} \frac{1}{i}|\mathbb{W}_{in}(X)| = \lim_{j\to\infty} \frac{n}{j}|\mathbb{W}_j(X)| = n\,\mathrm{h}(X),$$

because $\{\frac{1}{i}|\mathbb{W}_{in}(X)|\}_{i\in\mathbb{N}}$ is a subsequence of $\{\frac{n}{j}|\mathbb{W}_j(X)|\}_{j\in\mathbb{N}}$. $\square$

Definition 8.1.6. A pair of sofic shift spaces X, Y is said to be SFT-related if there exists a $W \in \mathrm{SFT}$ with the following properties:

1. $W \subseteq X$.
2. $W \twoheadrightarrow Y$.
3. $\exists L \in \mathbb{N} \forall a, b \in \mathbb{W}(W) \exists x \in \mathbb{W}_L(W)\colon axb \in \mathbb{W}(W)$.

Remark. Y is irreducible, since it is a factor of W, which is irreducible by 3.

Theorem 8.1.7. *Let X and Y be* SFT*-related sofic shift spaces. Then*

$$Q_1(Y_0^{(1)}) \neq \emptyset \Rightarrow X \twoheadrightarrow Y.$$

Proof. Let W be a SFT with the properties in definition 8.1.6. As passing to a higher block presentation of X is a conjugacy we can assume that that W is 1-step and the factor map $\varphi\colon W \to Y$ is 1-block. The idea of the proof is to extend φ to a factor map $\varphi\colon X \to Y$.

Pick a $y^\infty \in Q_1(Y_0^{(1)})$ and find $u, v \in S(Y)$ so that $uy^kv \in \mathbb{W}(Y)$, for all $k \in \mathbb{N}$. Choose $s \in S(Y)$ and $s_0 \in \varphi^{-1}(s) \cap \mathbb{W}(W)$. Find $u', v' \in \mathbb{W}(Y)$ such that $su'uy^kvv's \in \mathbb{W}(Y)$, for all $k \in \mathbb{N}$. Such a pair exists because Y is irreducible and $u, v \in S(Y)$. Find also for each $\omega \in \mathbb{W}(W)$ a pair $u_\omega, v_\omega \in \mathbb{W}_L(W)$, such that $s_0u_\omega\omega v_\omega s_0 \in \mathbb{W}(W)$. u_ω and v_ω kan be chosen such that u_ω depends only on the first symbol and v_ω on the last symbol in ω, since W is 1-step.

Define $T = 2L + |s| + |u'uvv'|$.

Let $x \in X$. An interval $[i, j] \subseteq \mathbb{Z}$, such that $j - i \geq T$ will be called a W-interval if $x_{[k,k+T]} \in \mathbb{W}(W)$, for all $k \in [i, j]$, and $[i, j]$ is maximal with respect to inclusion. Note that the distance between two different W-intervals must be at least T because two words in W, which both are maximal with respect to inclusion, must be disjoint as W is 1-step. For W-intervals we define:

$$\varphi(x)_{[i,j]} = \varphi(s_0u_\omega x_{[i+|s|+L,j-L]}v_\omega).$$

For the rest of x, which is a union of intervals of length at least T, we define

$$\varphi(x)_{[i,j]} = su'uy^{|I|-|s|-|uu'v'v|}vv'.$$

Now $\varphi(x)_i$ depends only on $x_{[i-4T,i+4T]}$, because all you need to know, is if your distance to the edge of the nearest W-interval of length at least T, is more than T. Thus φ is continuous. So since φ obviously commutes with the shift map and is an extension of the factor map φ, $\varphi\colon X \to Y$ is a factor map. □

Corollary 8.1.8. *Let X and Y be sofic shift spaces. X irreducible, Y mixing and $\mathrm{h}(X) > \mathrm{h}(Y)$. Then*

$$Q_1(Y_0^{(1)}) \neq \emptyset \Rightarrow X \twoheadrightarrow Y.$$

Proof. All I have to do is to show that X and Y are SFT-related. But this follows from lemma 2.8.4. □

Lemma 8.1.9. *Let Y be a mixing sofic shift space. Then*

$$\exists N \in \mathbb{N} \forall n \geq N\colon Q_1(Y^n)_0^{(1)} \neq \emptyset.$$

Proof. Let $s \in S(Y)$ and define $N = \mathrm{TL}(Y) + |s|$. Since Y is mixing, we can for each $n \geq \mathrm{TL}(Y)$ find a $y_n \in \mathbb{W}_n(Y)$, such that $sy_n \in S(Y)$ and $(sy_n)^\infty \in Y$. So as $(sy_n)^\infty \in \mathrm{Per}_{|s|+n}(Y)$, $\lambda_{n+|s|}((sy)^\infty) \in Q_1(Y^{n+|s|})_0^{(1)}$ by lemma 8.1.4. □

Theorem 8.1.10. *Let X and Y be mixing sofic shift spaces, such that $\mathrm{h}(X) > \mathrm{h}(Y)$. Then Y is an eventual factor of X.*

Proof. Lemma 8.1.9 gives an $N \in \mathbb{N}$ such that $Q_1(Y^n)_0^{(1)} \neq \emptyset$, for all $n \geq N$. So since X^n and Y^n both are mixing for all $n \in \mathbb{N}$, by lemma 8.1.2, the result follows from corollary 8.1.8. □

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Notation Index

Non-Alphabetizeable Symbols

$[i]_n$	i modulo n.
$\lceil x \rceil$	$\min\{n \in \mathbb{N} \mid n \geq x\}$.
$\lfloor x \rfloor$	$\max\{n \in \mathbb{N} \mid n \leq x\}$.
x_I	$\{x_i\}_{i \in I}$.
$a \vert b$	a divides b
$A - B$	$A \cap B^c$
$\frac{n}{m}(d, F)$	$(\frac{n}{m}d, \{\frac{n}{m}f \mid f \in F\})$
$\bigsqcup_{i \in I} A_i$	$\cup_{i \in I}\{(a, i) \mid a \in A_i\}$
$\vert X \vert$	The cardinality of the set X
$\subseteq$	When between words: 'occurs in'. p. 8
$\vert w \vert$	The length of the word w. p. 8
2^A	The subsets of set A.
2^G	The subset graph of G. p. 23
$\mathbb{N}_\infty$	$\mathbb{N} \cup \{0, \infty\}$.
$ax_{[i,j]}b$	$ax_{[i,j]}b$, when $i, j \in \mathbb{Z}$.
	$ax_{[i,\infty[}$, when $i \in \mathbb{Z}$ and $j = \infty$.
	$x_{]\infty,j]}b$, when $i = \infty$ and $j \in \mathbb{Z}$.
	$x_{]\infty,\infty[}$, when $i, j = \infty$.
	$Q_{\boldsymbol{n}}(X) = \times_{i=1}^{k} Q_{n_i}(X)$,
$\boldsymbol{x}^{[p]}$	$(x_1^{[p]}, x_2^{[p]}, \ldots, x_k^{[p]})$, p. 82
$\boldsymbol{n} \leq \boldsymbol{m}$	$\forall i \colon n_i \leq m_i$, p. 82
$\frac{\boldsymbol{n}}{\boldsymbol{m}}$	$(\frac{n_1}{m_1}, \frac{n_2}{m_2}, \ldots, \frac{n_k}{m_k})$, p. 82
$\frac{\boldsymbol{n}}{\boldsymbol{m}}(\boldsymbol{d}, F)$	$(\frac{\boldsymbol{n}}{\boldsymbol{m}}\boldsymbol{d}, \frac{\boldsymbol{n}}{\boldsymbol{m}}F)$, p. 82
$\boldsymbol{nm}$	$(n_1 m_1, n_2 m_2, \ldots, n_k m_k)$, p. 82.
$p_{\boldsymbol{x}}^{\boldsymbol{kd}+\boldsymbol{f}}\boldsymbol{w}$	$p_{x_1}^{k_1 d_1 + f_1} w_1 p_{x_2}^{k_2 d_2 + f_2} w_2 \ldots p_{x_n}^{k_n d_n + f_n}$, p. 82

Alphabetized List

V	The set of vertices. p. 17, 19
$\mathbb{W}(G)$	G a graph: The set of paths in G. p. 17
	G a labeled graph: The labels of paths in G. p. 20
$\mathbb{W}(x)$	The set of subwords in x. p. 8
$\mathbb{W}(X)$	The language of X. p. 8
$\mathbb{W}^*(X)$	The possibly infinitely long words occurring in points in X. p. 8
$\mathbb{W}_{M,\tau}$	p. 40
$\mathbb{W}_n, \mathbb{W}_{\leq n}, \mathbb{W}_{\geq n}$	The words of length n, less than n and more than n, respectively. p. 8
$\mathbb{W}_n(X)^c$	$\mathbb{W}_n(\Sigma^{\mathbb{Z}}) - \mathbb{W}_n(X)$.
word	p. 8
word map	p. 12
$\mathbb{W}_t, \mathbb{W}_\tau$	The t and τ words. p. 39
x^n	$\lambda_n(x)$. p. 13
$x^{[n]}$	$\beta_n(x)$. p. 13
$X_{(\alpha,n)}$	An irreducible component in X. p.27
X_0	The top component of X. p. 27
$X_{2\mathrm{n}}$	The even shift. p. 9
$X_0^{(d,F)}$	The points who are (d,F)- affiliated to the top component of X. p. 27
$X_c^{(\boldsymbol{d},F)}$	The vectors of periodic points who are simultaneously $(\boldsymbol{d},F)$- affiliated to the irreducible component X_c. p. 81, 82
$X^{[n]}$	The nth higher block shift of X. p. 13
X^n	The nth higher power shift of X. p. 13
X_F	$\{x \in X \mid \mathbb{W}(x) \cap F = \emptyset\}$. p. 9
X_{gm}	The golden mean shift. p. 9
$X \to Y$	There is a morphism from X to Y. p. 11
$X \rightharpoonup Y$	There is a morphism from X to Y, which hits a synchronizing word. p. 28
$X \twoheadrightarrow Y$	X factors onto Y. p. 11
$X \hookrightarrow Y$	X embeds into Y. p. 11