ROTATIONAL CROFTON FORMULAE
FOR
FLAGGED INTRINSIC VOLUMES

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PhD Thesis

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Preface

The present thesis consists of four papers, which constitute the result of my PhD study at the Department of Mathematical Sciences, Aarhus University. This study has been conducted under the supervision of Eva B. Vedel Jensen from February 2007 to July 2010.

Summary

The study of stereology can be traced back to the 18th century with the celebrated Buffon's needle problem, which was posed for the first time by the Comte de Buffon and can be described as a method for estimating π by throwing needles on a parquet floor. More generally, stereology is the study of intrinsic geometrical properties of a set through measurements made on lower dimensional sections of that set. For practical reasons, we may require that those sections go through a fixed point in space, instead of being randomly positioned. The study of geometrical properties in that particular set-up is called *local stereology*, the foundations of which were laid by Eva B. Vedel Jensen in 1998, cf. [8].

The focus of interest of the present thesis is integral geometric identities of the type

$$\beta(X) = \int \alpha(X \cap L)\, \mathrm{d}L,$$

where α and β are geometrical quantities of a set X, such as its volume, surface area or, more generally, intrinsic volume, and the integration is over all sections containing the fixed point origo. Our main result is a local stereological analogue to the well-known Crofton formula. More precisely, we derive throughout Paper A, B and C integral geometric formulae that relate new *flagged* intrinsic volumes of a set X with the flagged intrinsic volumes of its sections, $X \cap L$. The development of a potential local stereological analogue of the famous principal kinematic formula will also be discussed. In the last paper, Paper D, we present many new integral geometrical identities that were useful, if not indispensable, for the formulation of our main Theorem. Hopefully, these formulae will prove valuable to the further study of local stereology.

Accompanying papers

[A] Auneau-Cognacq, J., Rataj, J. and Jensen, E.B.V. (2010): Closed form of the rotational Crofton formula. Earlier version appeared as *Thiele Research Report* **2008-13**, Department of Mathematical Sciences, University of Aarhus. Submitted.

[B] Auneau, J. and Jensen, E.B.V. (2010): Expressing intrinsic volumes as rotational integrals. *Adv. Appl. Math.* **45**, 1-11.

[C] Auneau-Cognacq, J. (2010): A rotational Crofton formula for flagged intrinsic volumes of sets of positive reach. To be published as a CSGB preprint.

[D] Auneau-Cognacq, J. (2010): Integral geometric formulae. Manuscript.

Contents

1 Introduction

Imagine an object in space, e.g. a potato. How can we estimate the volume of the potato by placing sticks through it, as shown in Figure 1.1? Is it possible to estimate the surface area of the potato by proceeding similarly? These are some of the questions that can be answered by stereology.

More generally, stereology makes it possible to draw inference on quantitative properties of spatial structures from measurements on randomly positioned and orientated sections through the structure. Consequently, an important role is played by the integral geometric *section formulae*

$$\beta(X) = \int \alpha(X \cap T)\, \mathrm{d}T,$$

where α and β are geometric quantities and $X \subseteq \mathbb{R}^d$ is the spatial object of interest. The integration is over all possible positions of the *probe* T. In the potato example, the probes are one-dimensional sticks in $\mathbb{R}^3$.

An important example of such a section formula is the Crofton formula, where α and β are intrinsic volumes,

$$c_{d,j,k} V_{d-j+k}(X) = \int_{\mathcal{F}_j^d} V_k(X \cap F_j)\, \mathrm{d}F_j^d. \tag{1.1}$$

Here, X is a d-dimensional subset of $\mathbb{R}^d$, $\mathcal{F}_j^d$ denotes the space of j-dimensional affine subspaces of $\mathbb{R}^d$, $\mathrm{d}F_j^d$ is its motion invariant measure, V_k is the kth intrinsic volume and $c_{d,j,k}$ is a known constant of proportionality, $0 \le k \le j \le d$. For specific values of k, the intrinsic volumes have simple interpretations, e.g. $V_d(X)$ is the volume of X, $2V_{d-1}(X)$ is its surface area and $V_0(X)$ is the Euler-Poincaré characteristic. As a useful example, with $d = 3$, $j = k = 1$, the Crofton formula shows how to calculate the volume of a potato using sticks

$$2\pi\, \mathrm{volume(potato)} = \int \mathrm{length\,(potato} \cap \mathrm{stick)\, d\,stick}.$$

Other practical applications of classical stereology can be found in [5] and [8]. In the literature, intrinsic volumes are also called *Minkowski functionals* or *quermass integrals.*

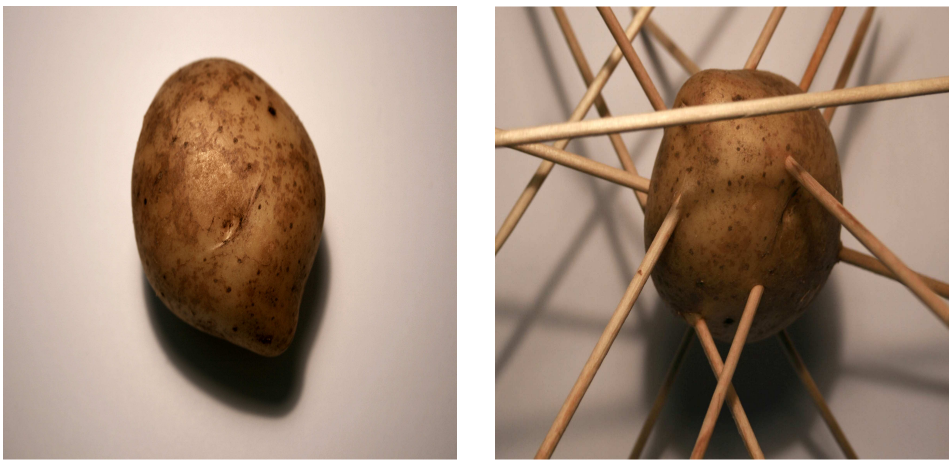

Figure 1.1: Sectioning a spatial object with randomly orientated and positioned linear probes.

Assuming that the boundary of X is sufficiently smooth, the kth intrinsic volume of X is then given by a simple boundary integral, for all $k = 0, \dots, d-1$,

$$V_k(X) := \frac{1}{\sigma_{d-k}} \int_{\partial X} c_k(X,x)\,\mathcal{H}^{d-1}(\mathrm{d}x), \tag{1.2}$$

where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure, σ_{d-k} is the surface area of the unit ball in $\mathbb{R}^{d-k}$ and

$$c_k(X,x) := \sum_{\substack{I \subseteq \{1,\dots,d-1\} \\ |I|=d-1-k}} \prod_{i \in I} \kappa_i(x) \tag{1.3}$$

is the kth symmetric function of the principal curvatures $\kappa_i(x)$ of ∂X at x, $i = 1, \dots, d-1$, cf. [10].

The characterization theorem of Hadwiger states that motion invariant measures on the family of convex bodies in $\mathbb{R}^d$ can be written as a linear combination of the intrinsic volumes $V_0, \dots, V_d$, cf. [7], [11, 14.4.6]. This important result can be used to prove Crofton's formula and the well-known kinematic formula for convex bodies X and Y,

$$\int_{\mathcal{G}_d} V_j(X \cap gY)\,\mathrm{d}g = \sum_{k=j}^{d} k_{d,j,k} V_k(X) V_{d-k+j}(Y), \tag{1.4}$$

where $\mathcal{G}_d$ is the group of rigid motions in $\mathbb{R}^d$, $\mathrm{d}g$ is the element of its normalized Haar measure and $k_{d,j,k}$ is a known constant for $j = 0, \dots, d$, cf. [11].

Note that the Crofton formula requires integration over all j-dimensional *affine* sections of X. For applications, e.g. in microscopy, random affine sections may not be optimal. As an example, by sectioning a biological cell through a centrally placed nucleus, one obtains images with better contrast than in the case of peripheral sections, cf. [9] and the references therein.

In *local* stereology, the focus is on integral geometric identities involving sections through a fixed point, i.e. formulae of the form

$$\beta(X) = \int_{\mathcal{L}_j^d} \alpha(X \cap L_j)\,\mathrm{d}L_j^d, \tag{1.5}$$

where integration is over the set $\mathcal{L}_j^d$ of all j-dimensional *linear* subspaces of $\mathbb{R}^d$ and $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$. As opposed to classical stereology, no equivalent of the Crofton formula, where *both* α and β in (1.5) are intrinsic volumes, exist in local (or rotational) stereology. Nevertheless, as we shall see, rotational version of the Crofton formula can be derived, in particular when either α or β are intrinsic volumes.

A well-known classical result in local stereology is the Blashcke-Petkantschin formula, cf. [8]. Under mild assumptions, the following formula holds for any non-negative measurable function $g \colon \mathbb{R}^d \times \mathcal{L}_j^d \to \mathbb{R}_+$,

$$\int_{\partial X} \int_{\mathcal{L}_{j(1)}^d} g(x, L_j)\,\mathrm{d}L_{j(1)}^d\,\mathcal{H}^{d-1}(\mathrm{d}x) = \int_{\mathcal{L}_j^d} \int_{\partial X \cap L_j} g(x, L_j) \frac{|x|^{d-j}}{\mathcal{G}(\mathrm{Tan}\,(X,x), L_j)}\,\mathcal{H}^{j-1}(\mathrm{d}x)\,\mathrm{d}L_j^d, \tag{1.6}$$

cf. [8, Lemma 5.5] and [11, Section 7.2]. Here, $\mathcal{L}_{j(1)}^d$ denotes the set of j-dimensional subspaces containing the line through the origin spanned by x and $\mathrm{d}L_{j(1)}^d$ is the element of

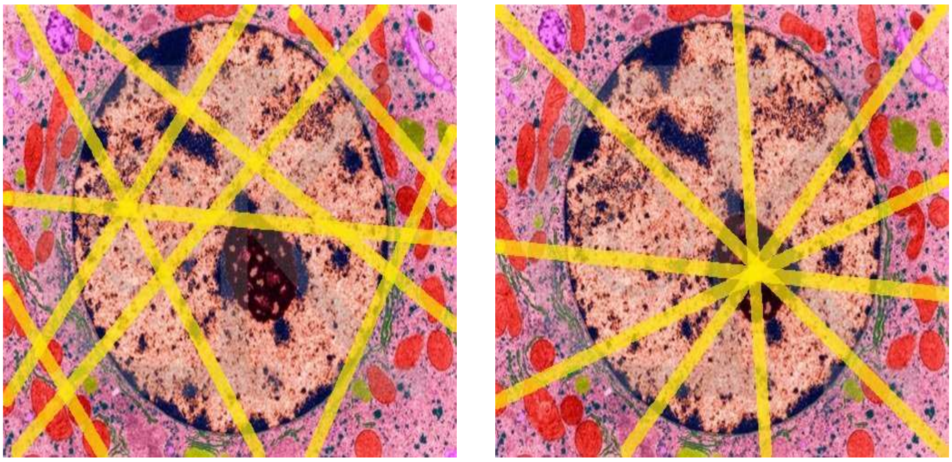

Figure 1.2: (Left) Arbitrary line transects of a liver cell. (Right) Line transects passing through a fixed point of the liver cell.

the rotation invariant measure keeping this line fixed. For any two linear subspaces L and L', $\mathcal{G}(L, L')$ can be regarded as a generalized sinus of the angle between L and L'. A precise definition of $\mathcal{G}$ is provided in Paper A. Recalling the definition of the kth intrinsic volume in (1.2) and inserting $g(x, L_j) = c_k(X \cap L_j, x)\frac{\mathcal{G}(\mathrm{Tan}\,(X,x), L_j)}{|x|^{d-j}}$ in the Blaschke-Petkantschin formula, we obtain an identity involving the *rotational average* $\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\,\mathrm{d}L_j^d$,

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\,\mathrm{d}L_j^d \\ = \frac{1}{\sigma_{d-k}} \int_{\partial X} \frac{1}{|x|^{d-j}} \int_{\mathcal{L}_{j(1)}^d} c_k(X \cap L_j, x)\mathcal{G}(\mathrm{Tan}\,(X,x), L_j)\,\mathrm{d}L_{j(1)}^d\,\mathcal{H}^{d-1}(\mathrm{d}x). \tag{1.7}$$

However, both sides of the equality are expressed in terms of curvatures measured on the j-dimensional linear sections of X — this fact renders the above identity unsatisfactory from a stereological point of view. Recently in [9], Jensen and Rataj made a new step toward finding a rotational version of the Crofton formula. When the boundary of X is sufficiently smooth and under mild assumptions on the choice of origo, the rotational average takes the following form,

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\,\mathrm{d}L_j^d = \int_{\partial X} \sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = j-1-k}} w_{I,j,k}^d(x) \prod_{i \in I} \kappa_i(x)\,\mathcal{H}^{d-1}(\mathrm{d}x), \tag{1.8}$$

where $w_{I,j,k}$ is a non-negative function defined on ∂X. Note that the right-hand side involves curvatures measured on the original object X, as opposed to the representation given in (1.7) by the Blaschke-Petkantschin formula. Also, it was shown in [9] that the weight functions $w_{I,j,k}^d$ are given by

$$w_{I,j,k}^d(x) = \frac{1}{\sigma_{j-k}|x|^{d-j}} \int_{\mathcal{L}_{j(1)}^d} \frac{\mathcal{G}(A_I, L_j)^2}{|p(n|L_j)|^{|I|+1}}\,\mathrm{d}L_{j(1)}^d, \tag{1.9}$$

where $n = n(x)$ is the outer unit normal to ∂X at x and $A_I(x)$ is the linear subspace spanned by the principal directions of curvature $a_i(x)$ with $i \notin I$. A closed form of $\omega_{I,j,k}^d$ was derived in [9] for the particular cases $|I| = 0$ and $|I| = d-2$.

In the present thesis, we will derive a closed form of the weight functions $\omega^d_{I,j,k}$, valid for all possible I, j and k (Paper A), address the 'opposite' problem of expressing intrinsic volumes as rotational averages (Paper B), define *flagged* intrinsic volumes that can be espressed as rotational averages of flagged intrinsic volumes defined on sections (Paper C) and present a number of integral geometric formulae that have proved useful in the development of the results in the previous papers (Paper D).

2 Results

The purpose of the present chapter is to give an overview of the main results obtained in this thesis.

2.1 Closed form of the weight functions

In Paper A, we derive an explicit expression for the weight functions $\omega^d_{I,j,k}$ valid for all possible values of the indices j and k. The expression involves *hypergeometric functions* (or Gauss *hypergeometric series*) defined for $a, b, c \in \mathbb{R}$ and $z \in [-1, 1]$ as

$$F(a, b; c; z) = F(b, a; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(x)_k$ is the *rising sequential product* or *Pochhammer symbol* defined for a non-negative integer k and $x \in \mathbb{R}$ by

$$(x)_k := \begin{cases} \frac{\Gamma(x+k)}{\Gamma(x)} & \text{if } x > 0 \\ (-1)^k \frac{\Gamma(-x+1)}{\Gamma(-x-k+1)} & \text{if } x \leq 0. \end{cases}$$

For all integers $0 \leq k < j < d$, $j \geq 2$, it is shown in Paper A, using extensive geometric measure theory, that $\omega^d_{I,j,k}$ can be expressed as

$$\omega^d_{I,j,k}(x) = C_{d,k,j} |x|^{j-d} [f_1(\beta(x)) + f_2(\beta(x)) \cos^2 \alpha_I(x)], \tag{2.1}$$

where $C_{d,k,j}$ is a known constant, $\alpha_I(x) = \angle(x, A_I(x))$ and $\beta(x) = \angle(x, n)$. Whenever x and n are in general position, f_1 and f_2 are defined as

$$f_1(\beta(x)) = (d + k - j) F\left(\frac{j-k}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2 \beta(x)\right)$$

and

$$\begin{aligned} f_2(\beta(x)) = (j - d - (d-1)\cot^2 \beta) F\left(\frac{j-k}{2}, \frac{d-j}{2}; \frac{d+1}{2}; \sin^2 \beta(x)\right) \\ + (d-1)\cot^2 \beta F\left(\frac{j-k}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2 \beta(x)\right). \end{aligned}$$

In other particular cases, e.g. when $j \leq 2$ or when x and n are perpendicular or parallel, the expression (2.1) becomes even simpler, cf. Paper A. Moreover, when extra assumptions are made on the shape of the body X, the rotational integral (1.8) can be simplified even more. At locally spherical boundary points $x \in \partial X$, where $\kappa_i(x) = \kappa(x)$, $i = 1, \ldots, d-1$, the integrand of (1.8) is equal to

$$\kappa(x)^{j-1-k} \sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = j-1-k}} w^d_{I,j,k}(x).$$

In Paper A, it was shown that the above sum has a surprisingly simple expression

$$\sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = j-1-k}} w_{I,j,k}(x) = \frac{c_{d-1,j-1}}{\sigma_{j-k}} \binom{j-1}{k} |x|^{j-d} F\left(\frac{j-k-2}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2 \beta(x)\right),$$

Figure 2.1: The special case where X is a disjoint union of spheres.

where $c_{d-1,j-1}$ is a known constant. Thus, whenever X is a disjoint union of spheres, the rotational average over all j-dimensional sections becomes

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\, \mathrm{d}L_j^d$$
$$= \int_{\partial X} \frac{c_{d-1,j-1}}{\sigma_{j-k}} \binom{j-1}{k} |x|^{j-d} F\left(\frac{j-k-2}{2}, \frac{d-j}{2}; \frac{d-1}{2}; \sin^2\beta\right) \kappa(x)^{j-1-k}\, \mathcal{H}^{d-1}(\mathrm{d}x).$$

However, a more insightful representation of the weight functions $\omega_{I,j,k}^d$ first became apparent under the study of the 'opposite' problem of finding functionals whose rotational averages are equal to intrinsic volumes.

2.2 Expressing intrinsic volumes as rotational integrals

As we have seen, the rotational average $\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\, \mathrm{d}L_j^d$ is a complicated geometric measure depending on the principal curvatures and their principal directions of the set X. In contrast to Crofton's classical formula, it has no simple interpretation as an intrinsic volume of X in general.

In Paper B, we address the problem of finding functionals defined on $X \cap L_j$ with rotational average equal to the intrinsic volumes of X. More specifically, the problem is to find, for each $j = 0, 1, \ldots, d$ and $k = 0, 1, \ldots, j$, a functional $\alpha_{d,k}^j$ satisfying the geometric equation

$$\int_{\mathcal{L}_j^d} \alpha_{d,k}^j(X \cap L_j)\, \mathrm{d}L_j^d = V_{d-k}(X). \tag{2.2}$$

Once again, the Blashcke-Petkantschin formula suggests a solution to this equation. By inserting $g(x, L_j) = c_k(X, x)$ in formula (1.6) and combining with (1.2), we obtain

$$\sigma_{d-k}\, c_{d-1,j-1} V_k(X) = \int_{\mathcal{L}_j^d} \int_{\partial X \cap L_j} \frac{|x|^{d-j}}{\mathcal{G}(\mathrm{Tan}\,(X,x), L_j)}\, c_k(X,x)\, \mathcal{H}^{j-1}(\mathrm{d}x)\, \mathrm{d}L_j^d.$$

Sadly, the inner integral in the last expression depends on information outside the intersecting subspace L_j and therefore, it cannot be considered, from a stereological perspective, as a viable solution to (2.2). We showed in Paper B that a more satisfactory solution to (2.2) is the functional $\alpha^j_{d,k}$ defined for all $j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$, by

$$\alpha^j_{d,k}(Y) = \frac{1}{c_{d,j-1,j-k-1}} \int_{\mathcal{F}^j_{j-1}} V_{j-k-1}(Y \cap F_{j-1})\, d(O, F_{j-1})^{d-j}\, \mathrm{d}F^j_{j-1}. \tag{2.3}$$

Here, Y is a compact subset of $\mathbb{R}^j$ satisfying some regularity conditions and d is the distance function. Furthermore, in Paper B, a more explicit expression of $\alpha_{j,k}$ was obtained for $k = 0$,

$$\alpha^j_{d,0}(Y) = \frac{1}{c_{d-1,j-1}} \int_Y |z|^{d-j}\, \mathrm{d}z^j$$

and for $k = 1$,

$$\alpha^j_{d,1}(Y) = \frac{1}{2c_{d-1,j-1}} \int_{\partial Y} |z|^{d-j} F\left(-\frac{1}{2}, -\frac{d-j}{2}; \frac{j-1}{2}; \sin^2(n, z)\right) \mathcal{H}^{j-1}(\mathrm{d}z). \tag{2.4}$$

Motivated by the results obtained in Paper B for $\alpha^j_{d,0}$ and $\alpha^j_{d,1}$, our new goal was to derive, for *all* $\alpha^j_{d,k}$, the corresponding integral representation over the boundary of X.

2.3 A rotational Crofton formula for flagged intrinsic volumes of sets of positive reach

In Paper C, we give an explicit expression for $\alpha^j_{d,k}(X)$ as an integral over the boundary of the section $X \cap L_j$, for all $1 \leq k < j$, or, as an integral over $X \cap L_j$, when $k = 0$. If the boundary of X is smooth, we obtain, under mild assumptions on the choice of origo,

$$\alpha^j_{d,k}(X \cap L_j) = \int_{\partial X \cap L_j} \sum_{\substack{I \subseteq \{1,\ldots,j-1\} \\ |I| = k-1}} w^j_{I,d,d-k}(x) \prod_{i \in I} \kappa_i(x)\, \mathcal{H}^{j-1}(\mathrm{d}x), \tag{2.5}$$

where the weight functions $w^j_{I,d,d-k}$ are similar to those found above. This connection between (1.8) and (2.5) led us to define *flagged* intrinsic volumes.

Flagged intrinsic volumes: *Let $Y \in \mathbb{R}^r$ be a compact set with smooth boundary. Define for all $k = 1, \ldots, r$, $r \geq 1$ and $j \geq k$,*

$$\alpha^r_{j,0}(Y) := \frac{1}{c_{j-1,r-1}} \int_Y |x|^{j-r}\, \mathrm{d}y^r$$

and

$$\alpha^r_{j,k}(Y) := K^r_{j,k} \int_{\partial Y} |x|^{j-r} \sum_{\substack{|I|=k-1 \\ I \subset \{1,\ldots,r-1\}}} \prod_{i \in I} \kappa_i(x,n)\, Q^r_{j,k}(x,n,A_I)\, \mathcal{H}^{r-1}(\mathrm{d}x),$$

where

$$\begin{aligned} Q^r_{j,k}(x,n,A_I) := {} & F\left(-\frac{j-r}{2}, \frac{k}{2}; \frac{r+1}{2}; \sin^2(x,n)\right) \\ & + \frac{(j-r)(r-k+1)}{r+1} \frac{\cos^2(x,A_I)}{r-k} F\left(-\frac{j-r}{2}+1, \frac{k}{2}; \frac{r+3}{2}; \sin^2(x,n)\right) \end{aligned}$$

and

$$K^r_{j,k} := \frac{1}{\sigma_k\, c_{j-1,r-1}} \frac{\Gamma(r-k+1)\,\Gamma(j)}{\Gamma(r)\,\Gamma(j-k+1)}.$$

Here, $A_I = \mathrm{span}\{a_i \,:\, i \notin I\}$ *and, for the special case* $r = k$*, we set* $\frac{\cos^2(x, A_{\{1,\ldots,r-1\}})}{0} := 1$*. Note that* $c_{j-1,r-1} := \frac{1}{c_{r-1,j-1}}$ *for* $j < r$*.*

The appellation 'flagged intrinsic volumes' is motivated by the concept of *flag spaces*, cf. [11], and the fact that flagged intrinsic volumes are identical to classical intrinsic volumes for particular values of their indices:

$$\alpha^r_{r,k}(Y) = \frac{1}{\sigma_k} \int_{\partial Y} \sum_{\substack{|J|=k-1 \\ J\subset\{1,\ldots,r-1\}}} \prod_{j\in J} \kappa_j \,\mathrm{d}\mathcal{H}^{r-1} = V_{r-k}(Y),$$

for $k = 1, \ldots, r$, and

$$\alpha^r_{r,0}(Y) = \int_Y \mathcal{H}^r(\mathrm{d}x) = V_r(Y),$$

for any compact set $Y \in \mathbb{R}^r$ with smooth boundary. Note that in Paper C, our results are formulated and proved in the more general setup of sets having positive reach. As a consequence, the combination of the two key results (1.8) and (2.5) yields the following proposition, under mild assumptions on the choice of origo.

Rotational Crofton Formula: *Let* $X \subset \mathbb{R}^d$ *be a compact subset of positive reach. Then,*

$$\alpha^d_{j,k}(X) = c_{d-r,j-r} \int_{\mathcal{L}^d_r} \alpha^r_{j,k}(X \cap L_r)\,\mathrm{d}L_r,$$

for all $0 \le k < r \le j \le d$*.*

Notice that the last statement is a natural generalization of the results obtained in [9] and in Paper B.

2.4 Integral geometric formulae

The resolution of the geometric problems mentioned above involves computation of complicated geometric integrals, and the task of expressing those integrals in explicit forms is indispensable for making the formulae manageable for applications. Moreover, the complexity of the integral formulae obtained makes it difficult and time-consuming to check their validity. With that in mind, we have developed integral geometric tools that have proved useful for deriving the results in Paper A, B and C. Paper D contains several such useful original identities, involving the area and co-area formulae. One of the main results is a relation between integrals over the unit sphere contained in different linear subspaces of $\mathbb{R}^d$. It is formulated below.

Let $B_p \in \mathcal{L}_p^d$ and $A_q \in \mathcal{L}_q^d$ such that $A_q \cap (B_p^\perp \cap A_q)^\perp = p$. Define the mapping

$$\begin{aligned} \psi \colon S^{q-1}(A_q) &\to S^{p-1}(B_p) \\ x &\mapsto \frac{p(x|B_p)}{|p(x|B_p)|} = x_0. \end{aligned}$$

Let $a_1, \ldots, a_{d-q}$ be an orthonormal basis of $A_q^\perp$. Then, for almost all $x \in S^{q-1}(A_q)$, the $(p-1)$-dimensional Jacobian of ψ is

$$J_{p-1}\psi(x) = \frac{\prod_{i=1}^{d-q} \sqrt{\sin^2\theta_i + \cos^2\theta_i \cos^2(\pi(x|B_p), \pi(a_i|B_p))}}{|p(u|B_p)|^{p-1}},$$

where $\theta_i = \angle(a_i, B_p)$ (set $\angle(\pi(x|B_p), \pi(a_i|B_p)) = 0$ when $a_i \perp B_p$).

This result is a general formulation of several integral geometric identities which were used extensively in [9] and also in Paper A. The second important result in Paper D is again an integral geometric formula, which has been the key to our boundary integral representation of a solution to the aforementioned 'opposite' problem, cf. Paper B.

Let $x, y \in S^{d-1}$ and $m, n \in \mathbb{N}$. Then,

$$\begin{aligned} &\int_{S^{d-1}} \sqrt{1-(x\cdot\omega)^2}^m |y\cdot\omega|^n \mathrm{d}\omega \\ &\qquad = \sigma_{d-1} B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{d-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{d-1}{2}; \sin^2\angle(x,y)\right). \end{aligned}$$

Later, we found a generalized version that proved to be an indispensable tool for the definition of the *flagged* intrinsic volumes.

Let x, y and z be unit vectors in $\mathbb{R}^d$ with $y \perp z$ and let $a, b, c \in \mathbb{Z}$. Then, if $x \neq y$ and $x \notin y^\perp$, the following identity holds,

$$\begin{aligned} &\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \, \mathrm{d}\omega^{d-1} \\ &\quad = \sigma_{d-2} \, |x\cdot y|^a B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+c+d-1}{2}\right) B\left(\frac{c}{2}+\frac{1}{2}, \frac{d-2}{2}\right) \\ &\qquad \times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(\frac{b+c+d-1}{2}\right)_s}{\left(\frac{c+d-1}{2}\right)_s} \frac{(-1)^s}{s!} \tan^{2s}(x,y) F\left(-s, -\frac{c}{2}; \frac{1}{2}; \frac{\cos^2(x,z)}{\sin^2(x,y)}\right), \end{aligned}$$

whenever both sides of the equation converge.

2.5 Discussion on a rotational kinematic formula

As mentioned earlier, a well-known result of classical integral geometry, the kinematic formula, states, for any two convex bodies K and M, that

$$\int_{\mathcal{G}_d} V_j(K \cap gM)\,\mathrm{d}g = \sum_{k=j}^{d} k_{d,j,k}\, V_k(K)\, V_{d-k+j}(M), \tag{2.6}$$

where $\mathcal{G}_d$ is the group of rigid motions in $\mathbb{R}^d$ and $k_{d,j,k}$ is a known constant. The kinematic formula can easily be proven using the Crofton formula and Hadwiger's characterization theorem. Is it possible to derive a *rotational* kinematic formula? A characterization theorem such as the one formulated by Hadwiger does not yet exist in rotational integral geometry, even though Alesker came very close to such a result in [2] and the erratum to match, cf. [3]. Thus, for the time being, other strategies must be employed in order to prove a potential rotational kinematic formula. For the rest of this section, we derive such a formula in two special cases.

Let K and M be d-dimensional compact convex sets in $\mathbb{R}^d$. We shall examine integrals of the type

$$\int_{\mathrm{SO}_d} V_k(K \cap \rho M)\,\mathrm{d}\rho, \tag{2.7}$$

where SO_d is the group of rotations in $\mathbb{R}^d$. For simplicity, we set the total measure of SO_d to be 1. In the special case $k = d$, a simple expression for (2.7) can be derived. First, note that

$$\int_{SO_d} V_d(K \cap \rho M)\,\mathrm{d}\rho = \int_{SO_d}\int_{\mathbb{R}^d} 1_{K\cap\rho M}(x)\,\mathrm{d}x^d\,\mathrm{d}\rho = \int_{\mathbb{R}^d} 1_K(x)\int_{SO_d} 1_{\rho M}(x)\,\mathrm{d}\rho\,\mathrm{d}x^d.$$

Recalling that the group of rotations acts transitively on the unit sphere and that the surface area of a d-dimensional convex body is $2V_{d-1}$, we obtain

$$\begin{aligned}\int_{SO_d} 1_{\rho M}(x)\,\mathrm{d}\rho &= \int_{SO_d} 1_M(\rho^{-1}x)\,\mathrm{d}\rho\\ &= \int_{SO_d} 1_{\frac{1}{|x|}M\cap S^{d-1}}\left(\rho\frac{x}{|x|}\right)\mathrm{d}\rho\\ &= 2|x|^{-(d-1)}V_{d-1}\left(M\cap|x|S^{d-1}\right),\end{aligned}$$

Hence,

$$\int_{SO_d} V_d(K\cap\rho M)\,\mathrm{d}\rho = 2\int_K |x|^{-(d-1)}V_{d-1}\left(M\cap|x|S^{d-1}\right)\mathrm{d}x^d$$

and an application of the coarea formula with $g(x) = |x|$ yields

$$\begin{aligned}\int_{SO_d} V_d(K\cap\rho M)\,\mathrm{d}\rho &= 2\int_0^\infty\int_{K\cap g^{-1}(r)} |x|^{-(d-1)}V_{d-1}\left(M\cap|x|S^{d-1}\right)\mathrm{d}x^{d-1}\,\mathrm{d}r\\ &= 4\int_0^\infty r^{-(d-1)}V_{d-1}\left(K\cap rS^{d-1}\right)V_{d-1}\left(M\cap rS^{d-1}\right)\mathrm{d}r. \end{aligned}\tag{2.8}$$

Assuming $\partial K \cap \rho M$ has $(d-1)$-dimensional Hausdorff measure 0 for almost all $\rho \in \mathrm{SO}_d$, a similar result can be obtained in the case $k = d-1$. We find

$$\begin{aligned}
\int_{SO_d} V_{d-1}(K \cap \rho M) &= \int_{SO_d} \left[\int_{\partial K \cap \rho M} \mathrm{d}x^{d-1} + \int_{\partial M \cap \rho^{-1} K} \mathrm{d}x^{d-1} \right] \mathrm{d}\rho \\
&= 2 \int_{\partial K} |x|^{-(d-1)} V_{d-1}(M \cap |x| S^{d-1}) \, \mathrm{d}x^{d-1} \\
&\quad + 2 \int_{\partial M} |x|^{-(d-1)} V_{d-1}(K \cap |x| S^{d-1}) \, \mathrm{d}x^{d-1}.
\end{aligned}$$

and, once again, the coarea formula implies

$$\begin{aligned}
\int_{SO_d} V_{d-1}(K \cap \rho M) = 4 \int_0^\infty r^{-(d-1)} \Big[& V_{d-2}(\partial K \cap r S^{d-1}) V_{d-1}(M \cap r S^{d-1}) \\
& + V_{d-2}(\partial M \cap r S^{d-1}) V_{d-1}(K \cap r S^{d-1}) \Big] \, \mathrm{d}r. \qquad (2.9)
\end{aligned}$$

Thus, in the very special case $d = 2$,

$$\int_{SO_2} V_1(K \cap \rho M) \, \mathrm{d}\rho \;\; = 4 \int_0^\infty \frac{\chi(\partial K \cap r S^1) V_1(M \cap r S^1) + \chi(\partial M \cap r S^1) V_1(K \cap r S^1)}{r} \, \mathrm{d}r.$$

Similar formulae for $k < d-1$ remain to be studied. Notice that the two identities (2.8) and (2.9) suggest that results from spherical integral geometry may be helpful in such an endeavour, cf. [11, Section 6.5].

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Paper A

Closed form of the rotational Crofton formula

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Submitted

Closed form of the rotational Crofton formula

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Abstract

The closed form of a rotational version of the famous Crofton formula is derived. In the case where the sectioned object is a compact d-dimensional C^2 manifold with boundary, the rotational average of intrinsic volumes measured on sections passing through a fixed point can be expressed as an integral over the boundary involving hypergeometric functions. In the more general case of a compact subset of $\mathbb{R}^d$ with positive reach, the rotational average also involves hypergeometric functions. For convex bodies, we show that the rotational average can be expressed as an integral with respect to a natural measure on supporting flats. It is an open question whether the rotational average of intrinsic volumes studied in the present paper can be expressed as a limit of polynomial rotation invariant valuations.

Keywords: Geometric measure theory, hypergeometric functions, integral geometry, intrinsic volume, stereology

MSC: 60D05; 53C65; 52A22

1 Introduction

Local stereology is a collection of sampling designs based on sections through a reference point of the structure under study, cf. [14]. The majority of the local stereological methods has been derived in the eighties and the nineties, including methods of estimating number, length, surface area and volume. These methods have found numerous applications, in particular in the microscopic analysis of tissue samples, cf. [8, 13, 15, 16, 20, 26, 28] and references therein. As pointed out in [11], local stereology is closely related to geometric tomography, especially to central concepts of the dual Brunn-Minkowski theory, see also [10]. Up-to-date monographs on stereology are Baddeley and Jensen [6] and Beneš and Rataj [7].

Rotational integral formulae are the fundamental tool of local stereology. A theory of rotational integral geometry, dual to the theory of translative integral geometry [25], has evolved, including rotational integral formulae for number, length, surface area and volume [14]. A basic tool in these developments has been the generalized Blaschke-Petkantschin formula, see [17] and [31]. Only very recently, rotational integral formulae have been derived for intrinsic volumes in general, cf. [4, 12, 18]. These new formulae open up the possibility for developing local stereological methods of estimating curvature (for instance, integral of mean curvature).

One of these formulae shows how rotational averages of intrinsic volumes measured on sections are related to the geometry of the sectioned object $X \subset \mathbb{R}^d$. The rotational average considered is of the following form

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j) \mathrm{d}L_j^d, \tag{1}$$

$0 \leq k \leq j \leq d$, where $\mathcal{L}_j^d$ is the set of $j-$dimensional *linear* subspaces in $\mathbb{R}^d$, V_k is the kth intrinsic volume and $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$ with total measure

$$\int_{\mathcal{L}_j^d} \mathrm{d}L_j^d = c_{d,j}.$$

Here,

$$c_{d,j} = \frac{\sigma_d \sigma_{d-1} \cdots \sigma_{d-j+1}}{\sigma_j \sigma_{j-1} \cdots \sigma_1},$$

where $\sigma_k = 2\pi^{\frac{k}{2}}/\Gamma(\frac{k}{2})$ is the surface area of the unit sphere in $\mathbb{R}^k$.

The rotational average (1) is an example of a rotational invariant valuation. Such valuations have been studied in recent years by Alesker [2] among others. For $k = j$, (1) is the jth dual elementary mixed volume, cf. e.g. [19], and we have

$$\int_{\mathcal{L}_j^d} V_j(X \cap L_j) \mathrm{d}L_j^d = c_{d-1,j-1} \int_X |x|^{-(d-j)} \mathrm{d}x^d,$$

where $\mathrm{d}x^d$ is the element of the $d-$dimensional Lebesgue measure, cf. e.g. [18, (9)].

The situation is more complicated for $k < j$. Assume (for simplicity) that $X \subset \mathbb{R}^d$ is a compact $d-$dimensional C^2 manifold with boundary. For a boundary point $x \in \partial X$, let $n(x)$ be the unit outer normal vector to X at x, let $\kappa_i(x)$, $i = 1, \ldots, d-1$, be the principal curvatures at $x \in \partial X$ and $a_i(x)$, $i = 1, \ldots, d-1$, the corresponding principal directions. In [18], it was shown under mild regularity conditions ($O \notin \partial X$ and for almost all $L_j \in \mathcal{L}_j^d$, there is no $x \in \partial X \cap L_j$ with $n(x) \perp L_j$) that the rotational average (1) is of the following form

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j) \mathrm{d}L_j^d = \int_{\partial X} \sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = j-1-k}} w_{I,j,k}(x) \prod_{i \in I} \kappa_i(x) \, \mathcal{H}^{d-1}(\mathrm{d}x), \tag{2}$$

provided the integral exists. In (2), $\mathcal{H}^k$ denotes the $k-$dimensional Hausdorff measure. The weight functions $w_{I,j,k}$ are non-negative functions defined on ∂X. The function $w_{I,j,k}(x)$ depends on the linear subspace spanned by the principal directions $a_i(x)$, $i \in I$.

If X is a ball, the function $w_{I,j,k}$ is constant and the rotational average is therefore proportional to the $(d-j+k)$th intrinsic volume of X which has the following integral representation

$$V_{d-j+k}(X) = \frac{1}{\sigma_{j-k}} \int_{\partial X} \sum_{|I| = j-1-k} \prod_{i \in I} \kappa_i(x) \, \mathcal{H}^{d-1}(\mathrm{d}x),$$

cf. [24, Section 13.6] or [27, Section V.3].

In the present paper, we derive a simple closed form expression of $w_{I,j,k}$, involving hypergeometric functions. We show that $w_{I,j,k}(x)$ depends on the norm of x and of two angles: the angle $\beta(x)$ formed by x and $n(x)$, and the angle $\alpha_I(x)$ formed by x and span$\{a_i(x) : i \notin I\}$. This expression allows us to understand the geometric structure of the rotational average and derive a simplified form of the integrand at the right-hand side of (2) at locally spherical boundary points. Furthermore, it will be shown that for convex bodies the rotational average can be expressed as an integral with respect to a natural measure on supporting $(j-1-k)-$dimensional flats. This result gives new insight concerning the question of characterizing rotation invariant valuations [2, 3].

The paper is organized as follows. In Section 2, we provide background knowledge on hypergeometric functions and angles of subspaces. In Section 3, the closed form expression of $w_{I,j,k}$ is presented. The proof of the result is deferred to the Section 7. In Section 4, further simplifications are derived for locally spherical boundary points. In Section 5, a reformulation of (2) is derived in terms of an integral with respect to a natural measure on supporting flats. In Section 6, we discuss the possibilities for expressing the rotation average as a limit of polynomial rotation invariant valuations.

2 Preliminaries

2.1 Hypergeometric functions

A hypergeometric function can be represented by a series of the following form

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}.$$

When $a=0$ or $b=0$, the hypergeometric function is identically equal to 1. The series converges absolutely for $|z|<1$. In case $0<b<c$, we can also represent the hypergeometric series by an integral

$$F(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 (1-zy)^{-a} y^{b-1} (1-y)^{c-b-1} \mathrm{d}y. \tag{3}$$

Here $B(s,t)=\Gamma(s)\Gamma(t)/\Gamma(s+t)$ is the Beta function. When $z=1$, the extra assumption $c-a-b>0$ is necessary. Transformation formulae for hypergeometric functions are often useful. In particular, we shall use the following formulae, cf. [1, (15.2.17), (15.2.20) and (15.2.24)],

$$(c-a-1)F(a,b;c;z) + aF(a+1,b;c;z) = (c-1)F(a,b;c-1;z), \tag{4}$$

$$c(1-z)F(a,b;c;z) + (c-b)zF(a,b;c+1;z) = cF(a-1,b;c;z), \tag{5}$$

$$bF(a,b+1;c;z) = (c-1)F(a,b;c-1;z) - (c-b-1)F(a,b;c;z). \tag{6}$$

We shall also use the following integral representation which can be obtained from (3), using the substitution $y = r^2/(1+r^2)$

$$\int_0^\infty \left(1-z+\frac{z}{1+r^2}\right)^{-a} (r^2)^{b-\frac{1}{2}} (1+r^2)^{-c} \mathrm{d}r = \frac{1}{2} B(b,c-b) F(a,b;c;z). \tag{7}$$

This integral representation is valid under the same assumptions as in (3).

2.2 Angle of subspaces

For $x_1, \dots, x_p \in \mathbb{R}^d$, $p \le d$, we let $P(x_1, \dots, x_p)$ be the parallelotope spanned by $x_1, \dots, x_p$,

$$P(x_1, \dots, x_p) = \{\lambda_1 x_1 + \dots + \lambda_p x_p : 0 \le \lambda_i \le 1, i = 1, \dots, p\}.$$

We denote its p-dimensional volume by

$$\nabla_p(x_1, \dots, x_p) = \mathcal{H}^p(P(x_1, \dots, x_p)).$$

This quantity equals the norm of the corresponding p-vector $x_1 \wedge \dots \wedge x_p$.

Definition [30, p. 532]. Let $L_p \in \mathcal{L}_p^d$ and $L_q \in \mathcal{L}_q^d$. Choose an orthonormal basis of $L_p \cap L_q$ and extend it to an orthonormal basis of L_p and an orthonormal basis of L_q. Then, $\mathcal{G}(L_p, L_q)$ is the d-dimensional volume of the parallelotope spanned by these vectors. □

For any two linear subspaces L_p and L_q, $\mathcal{G}(L_p, L_q)$ can be regarded as a generalized sinus of the angle between L_p and L_q. In particular, for $d = 3$ and $0 < p, q < d$, it is easy to show that $\mathcal{G}(L_p, L_q)$ is simply $|\sin \alpha|$ where α is the angle between L_p and L_q.

If $\dim(L_p + L_q) < d$ then $\mathcal{G}(L_p, L_q) = 0$. In the case $\dim(L_p + L_q) = d$ and either $p = 0$ or $q = 0$, then $\mathcal{G}(L_p, L_q) = 1$. Finally, if $\dim(L_p + L_q) = d$ and $0 < p, q < d$, we can choose orthonormal bases for

$$\begin{aligned} L_p \cap L_q &: a_1, \dots, a_{p+q-d} \\ L_p \cap (L_p \cap L_q)^\perp &: b_1, \dots, b_{d-q} \\ L_q \cap (L_p \cap L_q)^\perp &: c_1, \dots, c_{d-p}. \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{G}(L_p, L_q) &= \nabla_d\left(a_1, \dots, a_{p+q-d}, b_1, \dots, b_{d-q}, c_1, \dots, c_{d-p}\right) \\ &= \nabla_{d-q}\left(p(b_1|L_q^\perp), \dots, p(b_{d-q}|L_q^\perp)\right) \\ &= \nabla_{d-p}\left(p(c_1|L_p^\perp), \dots, p(c_{d-p}|L_p^\perp)\right), \end{aligned}$$

cf. [14, Proposition 2.13 and 2.14].

If both L_p and L_q are contained in a subspace L_r of $\mathbb{R}^d$, we can consider the $\mathcal{G}$-function relatively in L_r. This will be denoted by $\mathcal{G}^{(L_r)}(L_p, L_q)$. If $L_p \perp x$ ($x \in \mathbb{R}^d, x \neq O$), then the following identity holds, cf. [14, Proposition 5.1],

$$\mathcal{G}(L_p, L_q) = \cos \angle(x, L_q) \mathcal{G}^{(x^\perp)}(L_p, L_q \cap x^\perp), \tag{8}$$

where $x^\perp$ is the orthogonal complement to the linear subspace spanned by x.

The integral over the whole Grassmannian of the squared $\mathcal{G}$-function is constant

$$\int_{\mathcal{L}_i^d} \mathcal{G}(L_i, L_j)^2 \mathrm{d}L_i^d = K_{ij}^d c_{d,i}, \tag{9}$$

where $K_{ij}^d = \frac{i!j!}{d!(i+j-d)!}$ if $i + j \ge d$ and 0 otherwise, see e.g. [21, Lemma 4.3].

3 The closed form of $w_{I,j,k}$

We shall formulate the main result, the closed form of $w_{I,j,k}$, for a compact set X with positive reach, in order to cover both important applications, convex bodies and sets with C^2 smooth boundary. The reader can for his/her convenience always imagine one of these two particular cases.

Let $X \subset \mathbb{R}^d$ be a compact set with positive reach and let $\operatorname{nor} X$ denote its unit normal bundle. Let $\kappa_1(x,n), \ldots, \kappa_{d-1}(x,n)$ be the principal curvatures and $a_1(x,n), \ldots, a_{d-1}(x,n)$ the corresponding principal directions defined almost everywhere on $(x,n) \in \operatorname{nor} X$, see [18] for further details. If X is smooth, then the unit normal $n = n(x)$ is a function of $x \in \partial X$ and

$$\operatorname{nor} X = \{(x, n(x)) : x \in \partial X\}.$$

Hence, all the functions defined on the unit normal bundle $\operatorname{nor} X$ can be considered as functions on ∂X.

In [18, Theorem], it was shown for $0 \le k < j \le d$ that, under the following assumptions

(A1) $O \notin \partial X$,

(A2) for almost all $L_j \in \mathcal{L}_j^d$, there is no $(x,n) \in \operatorname{nor} X$ with $x \in L_j$ and $n \perp x$,

the rotational integral equals

$$\begin{aligned}\int_{\mathcal{L}_j^d} & V_k(X \cap L_j) \mathrm{d}L_j^d \\ &= \int_{\operatorname{nor} X} \sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = j-1-k}} w_{I,j,k}(x,n) \frac{\prod_{i \in I} \kappa_i(x,n)}{\prod_{i=1}^{d-1} \sqrt{1+\kappa_i^2(x,n)}} \mathcal{H}^{d-1}(\mathrm{d}(x,n)),\end{aligned} \tag{10}$$

provided that the integral exists. Here,

$$w_{I,j,k}(x,n) = \sigma_{j-k}^{-1} |x|^{j-d} Q_j(x,n,A_I(x,n)), \tag{11}$$

where

$$A_I(x,n) = \operatorname{span}\{a_i(x,n) : i \notin I\} \in \mathcal{L}_{d-1-|I|}^d.$$

For $A_q \in \mathcal{L}_q^d$, the function Q_j is defined as the following integral

$$Q_j(x,n,A_q) = \int_{\mathcal{L}_{j(1)}^d} \frac{\mathcal{G}(L_j, A_q)^2}{|p(n|L_j)|^{d-q}} \mathrm{d}L_{j(1)}^d, \tag{12}$$

where $\mathcal{L}_{j(1)}^d$ is the set of $j-$dimensional subspaces containing the line spanned by x and $p(\cdot|L_j)$ indicates orthogonal projection onto L_j. If $j=1$ and $x \perp n$ we set $Q_j(x,n,A_q) := 0$. In the case where X is smooth, (10) reduces to (2) with $w_{I,j,k}(x) = w_{I,j,k}(x,n(x))$ and $\kappa_i(x) = \kappa_i(x,n(x))$.

Our main result formulated in Theorem 1 below follows from an expression of Q_j as a linear combination of hypergeometric functions. We use the notation

$$\beta(x,n) := \angle(x,n) \in [0,\pi], \quad \alpha_I(x,n) := \angle(x, A_I(x,n)) \in [0,\pi/2].$$

Note that $x \neq O$ if $(x,n) \in \operatorname{nor} X$ by (A1). The proof of Theorem 1 is deferred to Section 7. The cases $j=1$ and $j=d$ are treated separately in Remark 1 below.

Theorem 1. *Let $0 \le k < j < d$, $j \ge 2$ and let I be a subset of $\{1, \ldots, d-1\}$ with $|I| = j-1-k$ elements. Let $X \subset \mathbb{R}^d$ be a set with positive reach. If (A1) and (A2) are satisfied, then* (10) *holds with*

$$w_{I,j,k}(x,n) = C_{d,k,j}|x|^{j-d}\left[f_1(\beta(x,n)) + f_2(\beta(x,n))\cos^2\alpha_I(x,n))\right]$$

and

$$C_{d,k,j} = \sigma_{j-k}^{-1} c_{d-1,j-1}\frac{(j-1)!(d+k-j-1)!}{(d-1)!k!}.$$

The functions f_1 and f_2, defined on $[0,\pi]$, are given by

$$f_1(\beta) = (d+k-j)F\left(\tfrac{j-k}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right),$$

$$f_2(\beta) = \begin{cases} 0, & \beta = 0, \pi, \\ (j-d)F\left(\tfrac{j-k}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; 1\right), & \beta = \tfrac{\pi}{2}, \\ (j-d-(d-1)\cot^2\beta)F\left(\tfrac{j-k}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) & \\ \qquad +(d-1)\cot^2\beta F\left(\tfrac{j-k}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right), & \beta \ne 0, \tfrac{\pi}{2}, \pi. \end{cases}$$

Remark 1. Note that if $j = 1$ then, necessarily, $k = 0$, $I = \emptyset$ and $Q_1(x, n, n^\perp) = |\cos\beta(x,n)|$; hence, $w_{\emptyset,1,0}(x,n) = \frac{1}{2}|x|^{1-d}|\cos\beta(x,n)|$. If $j = d$ then no integration is carried out in (1) and we have $w_{I,d,k} = \sigma_{d-k}^{-1}$. These two particular cases are not included in Theorem 1. □

Two special cases were already derived in [18]. Let $k = 0$ and $j = d-1$. Let $A_I(x,n) = \operatorname{span}\{a\}$. Assume that $\alpha_I(x,n) = \angle(x,a) > 0$ and $0 < \beta(x,n) = \angle(x,n) < \pi$. Let $\theta(x,n)$ be the angle formed by the projections $p(n|x^\perp)$ and $p(a|x^\perp)$ ($\cos\theta = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}$, see the end of Section 7). Then, we find, using (6),

$$\begin{aligned} w_{I,d-1,0}(x,n) = \frac{1}{2(d-1)}|x|^{-1}\sin^2\alpha_I(x,n)\Big[&\sin^2\theta(x,n)F\left(\tfrac{d-1}{2}, \tfrac{1}{2}; \tfrac{d+1}{2}; \sin^2\beta(x,n)\right) \\ &+ \cos^2\theta(x,n)F\left(\tfrac{d-1}{2}, \tfrac{3}{2}; \tfrac{d+1}{2}; \sin^2\beta(x,n)\right)\Big]. \end{aligned} \tag{13}$$

This agrees with the result presented in [18, Section 4.2].

When $k = j-1$, we have $I = \emptyset$, $A_I(x,n) = \operatorname{span}\{a_i(x,n) : i = 1, \ldots, d-1\} = n^\perp$ and $\angle(x, A_I(x,n)) = \frac{\pi}{2} - \angle(x,n)$; hence, $\cos\alpha_I(x,n) = \sin\beta(x,n)$. Then, by applying (5), we obtain

$$w_{I,j,j-1}(x,n) = \frac{c_{d-1,j-1}}{2}|x|^{-(d-j)}F\left(-\tfrac{1}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta(x,n)\right). \tag{14}$$

Combining (2) and (14), we find in case of C^2 smooth boundary (cf. [18, Section 4.1])

$$\begin{aligned} &\int_{\mathcal{L}_j^d} V_{j-1}(X \cap L_j)\mathrm{d}L_j^d \\ &\quad = \frac{c_{d-1,j-1}}{2}\int_{\partial X}|x|^{-(d-j)}F\left(-\tfrac{1}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta(x)\right)\mathrm{d}x^{d-1}. \end{aligned}$$

4 Further simplifications

At locally spherical boundary points, the rotational formula may be further simplified. First, we derive a simple expression for the sum of $w_{I,j,k}(x,n)$.

Lemma 1. *For* $0 \leq k < j \leq d$,

$$\sum_{\substack{I\subseteq\{1,\ldots,d-1\}\\|I|=j-1-k}} w_{I,j,k}(x,n) = \frac{c_{d-1,j-1}}{\sigma_{j-k}}\binom{j-1}{k}|x|^{-(d-j)}F\left(\tfrac{j-k-2}{2},\tfrac{d-j}{2};\tfrac{d-1}{2};\sin^2\beta(x,n)\right).$$

Proof. Recall that $A_I(x,n)=\operatorname{span}\{a_i(x,n): i\notin I\}$ and $\alpha_I(x)=\angle(x,A_I(x,n))$. We find

$$\begin{aligned}\sum_{|I|=j-1-k}\cos^2\alpha_I(x,n) &= \sum_{|I|=j-1-k}|p(x|A_I(x,n))|^2 = \sum_{|I|=j-1-k}\sum_{i\notin I}(x\cdot a_i(x,n))^2\\ &= \sum_{|I|=d-j+k}\sum_{i\in I}(x\cdot a_i(x,n))^2 = \sum_{i=1}^{d-1}\binom{d-2}{d-j+k-1}(x\cdot a_i(x,n))^2\\ &= \binom{d-2}{j-k-1}|p(x|\operatorname{span}\{a_1(x,n),\ldots,a_{d-1}(x,n)\}|^2\\ &= \binom{d-2}{j-k-1}|p(x|n^\perp)|^2 = \binom{d-2}{j-k-1}\sin^2\beta(x,n).\end{aligned}$$

Using Theorem 1 and (5), we arrive at the following formula

$$\begin{aligned}&\sigma_{j-k}|x|^{d-j}\sum_{|I|=j-1-k}w_{I,j,k}(x,n)\\ &= c_{d-1,j-1}\frac{(j-1)!(d+k-j-1)!}{(d-1)!k!}\binom{d-2}{j-k-1}\\ &\quad\times\Bigg[(j-1)\sin^2\beta(x,n)F\left(\tfrac{j-k}{2},\tfrac{d-j}{2};\tfrac{d+1}{2};\sin^2\beta(x,n)\right)\\ &\qquad+(d-1)\cos^2\beta(x,n)F\left(\tfrac{j-k}{2},\tfrac{d-j}{2};\tfrac{d-1}{2};\sin^2\beta(x,n)\right)\Bigg]\\ &= c_{d-1,j-1}\binom{j-1}{k}F\left(\tfrac{j-k-2}{2},\tfrac{d-j}{2};\tfrac{d-1}{2};\sin^2\beta(x,n)\right).\end{aligned}$$

□

In case $k=0$ and $j=d-1$, the expression above reduces to the one in [18], namely

$$\sum_{|I|=d-2}w_{I,d-1,0}(x,n)=\frac{c_{d-1,d-2}}{\sigma_{d-1}}|x|^{-1}F\left(\tfrac{d-3}{2},\tfrac{1}{2};\tfrac{d-1}{2};\sin^2\beta(x,n)\right).$$

Let us look at the simplifications of the rotational formula implied by Lemma 1 in the case where X is a compact $d-$dimensional C^2 manifold with boundary. In this case, there is a unique unit normal $n(x)$ at each $x\in\partial X$. The weight functions $w_{I,j,k}$ and the curvatures κ_i can be regarded as functions of x only.

It follows from Lemma 1 that at locally spherical boundary points $x \in \partial X$ where $\kappa_i(x) = \kappa(x)$, $i = 1, \dots, d-1$, the integrand of (2) simplifies to

$$\sum_{\substack{I \subseteq \{1,\dots,d-1\} \\ |I| = j-1-k}} w_{I,j,k}(x) \prod_{i \in I} \kappa_i(x)$$
$$= \frac{c_{d-1,j-1}}{\sigma_{j-k}} \binom{j-1}{k} |x|^{-(d-j)} F\left(\tfrac{j-k-2}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2 \beta(x)\right) \kappa(x)^{j-1-k}.$$

For $k = j-2$, this hypergeometric function is identically equal to 1. If almost all boundary points are locally spherical, we get

$$\int_{\mathcal{L}_j^d} V_{j-2}(X \cap L_j) \mathrm{d}L_j^d = \frac{(j-1)c_{d-1,j-1}}{2\pi} \int_{\partial X} |x|^{-(d-j)} \kappa(x) \mathrm{d}x^{d-1}.$$

Unfortunately, locally spherical boundary points are rare unless X is a finite union of disjoint balls.

5 The rotational average as valuation on convex bodies

In this section, we will show for convex bodies that the rotational averages of intrinsic volumes can be represented as integrals with respect to natural measures on supporting flats.

For this purpose, let X be a convex body in $\mathbb{R}^d$. For convenience, we introduce the following short notation for the rotational average

$$\Phi_{k,j}(X) := \int_{\mathcal{L}_j^d} V_k(X \cap L_j) \, \mathrm{d}L_j^d,$$

$0 \leq k < j \leq d$. According to [18, Proposition 2], $\Phi_{k,j}(X) < \infty$ whenever X is a convex body. Clearly, $\Phi_{k,j}$ is a valuation which is continuous with respect to the Hausdorff metric on convex bodies and $O(d)$-invariant (see [2]).

We shall find an expression of $\Phi_{k,j}$ as an integral with respect to a certain measure $\Gamma_{d+k-j}(X; \cdot)$ associated with the convex body X. This measure is supported by $(j-k-1)-$dimensional affine subspaces "locally colliding" with X. The measure has been introduced by Firey [9], see also Weil [29], and, independently, and in different settings, in connection with absolute curvature measures by Baddeley [5] for smooth bodies and by Rother and Zähle [23] for sets with positive reach.

Given a convex body X and $0 \leq i \leq d-1$, let

$$\mathcal{F}_i^d(X) = \{(x, n, L_i) : (x, n) \in \operatorname{nor} X, \, L_i \in \mathcal{L}_i^{d-1}(n^\perp)\}.$$

Note that for $(x, n, L_i) \in \mathcal{F}_i^d(X)$, $x + L_i$ is an $i-$dimensional affine subspace that supports X at x. The projection

$$f : (x, n, L_i) \mapsto (p(x|L_i^\perp), L_i)$$

maps $\mathcal{F}_i^d(X)$ into the set of $i-$dimensional affine subspaces in $\mathbb{R}^d$ supporting X. The image of f will be denoted by $\mathcal{A}_i^d(X)$. Consider the following natural invariant measure $\mu_i(X; \cdot)$ on $\mathcal{A}_i^d(X)$, defined by the following equation for an arbitrary nonnegative measurable function on $\mathcal{A}_i^d(X)$,

$$\int_{\mathcal{A}_i^d(X)} h(z, L_i) \, \mu_i(X; \mathrm{d}(z, L_i)) = \int_{\mathcal{L}_i^d} \int_{\{z : \, (z, L_i) \in \mathcal{A}_i^d(X)\}} h(z, L_i) \, \mathcal{H}^{d-i-1}(\mathrm{d}z) \, \mathrm{d}L_i^d.$$

Then, the measure $\Gamma_{d-1-i}(X;\cdot)$ on $\mathcal{F}_i^d(X)$ is defined as

$$\int_{\mathcal{F}_i^d(X)} g(x,n,L_i)\,\Gamma_{d-1-i}(X;\mathrm{d}(x,n,L_i)) \\ = \int_{\mathcal{A}_i^d(X)} \sum_{(x,n,L_i)\in f^{-1}(z,L_i)} g(x,n,L_i)\,\mu_i(X;\mathrm{d}(z,L_i)),$$

where g is now any nonnegative measurable function on $\mathcal{F}_i^d(X)$.

The following integral representation for $\Gamma_{d-1-i}(X;\cdot)$ was derived in [23]

$$\int_{\mathcal{F}_i^d(X)} g(x,n,L_i)\,\Gamma_{d-1-i}(X;\mathrm{d}(x,n,L_i)) \tag{15}$$
$$= \binom{d-1}{i}\sigma_{i+1}^{-1}\int_{\mathrm{nor}\,X}\sum_{|I|=i}\frac{\prod_{i\in I}\kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2(x,n)}}$$
$$\times \int_{\mathcal{L}_i^{d-1}(n^\perp)} g(x,n,L_i)\,\mathcal{G}^{(n^\perp)}(L_i,A_I(x,n))^2\,\mathrm{d}L_i^{d-1}\,\mathcal{H}^{d-1}(\mathrm{d}(x,n)).$$

The result of this section follows:

Theorem 2. *Let X be a convex body in $\mathbb{R}^d$ with $O\notin\partial X$. If $0\le k<j<d$, $j\ge 2$, then*

$$\Phi_{k,j}(X) = \int_{\mathcal{F}_{j-1-k}^d(X)} g(x,n,L_{j-1-k})\,\Gamma_{d+k-j}(X;\mathrm{d}(x,n,L_{j-1-k}))$$

with

$$g(x,n,L_{j-1-k}) = C'_{d,k,j}|x|^{j-d}\left[g_1(\beta(x,n)) + g_2(\beta(x,n))|p(\bar{x}|L_{j-1-k})|^2\right].$$

Here $\bar{x}=x/|x|$,

$$C'_{d,k,j} = C_{d,k,j}\sigma_{j-k}\binom{d-1}{j-1-k}^{-1},$$
$$g_2(\beta) = \frac{1}{M_{j-1-k}^{d-1}-N_{j-1-k}^{d-1}}f_2(\beta)$$

and

$$g_1(\beta) = \frac{\binom{d-1}{j-1-k}}{c_{d-1,d+k-j}}\left(f_1(\beta) - \frac{N_{j-1-k}^{d-1}}{M_{j-1-k}^{d-1}-N_{j-1-k}^{d-1}}\sin^2\beta f_2(\beta)\right).$$

The functions f_1 and f_2 are given in Theorem 1 while the constants M_{j-1-k}^{d-1} and N_{j-1-k}^{d-1} are given in Lemma 4 in the Appendix.

Proof. Let us decompose $\bar{x}$ as

$$\bar{x} = p(\bar{x}|A_I) + p(\bar{x}|\operatorname{span}\{n\}) + p(\bar{x}|A_I^\perp\cap n^\perp).$$

If $x\not\perp A_I$, i.e. if $\alpha\ne\pi/2$, we can write $p(\bar{x}|A_I) = (\cos\alpha)\pi(\bar{x}|A_I)$, with the spherical projection

$$y := \pi(\bar{x}|A_I) = p(\bar{x}|A_I)/|p(\bar{x}|A_I)|.$$

(Note that we can use this identity even when $\alpha=\pi/2$, choosing any unit vector for y.) Analogously, we denote $z=\pi(\bar{x}|A_I^\perp\cap n^\perp)$ and we get

$$\bar{x} = (\cos\alpha)\,y + (\cos\beta)\,n + \sqrt{1-\cos^2\alpha-\cos^2\beta}\,z.$$

For $L = L_{j-1-k} \perp n$, we have

$$|p(\bar{x}|L)|^2 = \cos^2\alpha |p(y|L)|^2 + (1 - \cos^2\alpha - \cos^2\beta)|p(z|L)|^2 \\ + 2\cos\alpha\sqrt{1-\cos^2\alpha - \cos^2\beta}\, p(y|L)\cdot p(z|L).$$

Integrating with respect to $\mathrm{d}L^{d-1}_{j-1-k}$, the last summand vanishes. Thus, we get, using Lemma 4,

$$\begin{aligned} &\int_{\mathcal{L}^{d-1}_{j-1-k}(n^\perp)} |p(\bar{x}|L_{j-1-k})|^2 \mathcal{G}^{(n^\perp)}(L_{j-1-k}, A_I(x,n))^2 \mathrm{d}L^{d-1}_{j-1-k} \qquad (16) \\ &\quad = \cos^2\alpha M^{d-1}_{j-1-k} + (1-\cos^2\alpha - \cos^2\beta) N^{d-1}_{j-1-k} \\ &\quad = \cos^2\alpha (M^{d-1}_{j-1-k} - N^{d-1}_{j-1-k}) + \sin^2\beta\, N^{d-1}_{j-1-k}. \end{aligned}$$

Omitting for brevity the indexes at M and N, we get from (15), (9) and (16)

$$\begin{aligned} &\int g\, \mathrm{d}\Gamma_{j-1-k}(X,\cdot) \\ &\quad = C_{d,k,j}|x|^{j-d}\left[c_{d-1,j-1-k}K^{d-1}_{j-k-1,d+k-j} g_1 + ((M-N)\cos^2\alpha + N\sin^2\beta)g_2\right]. \end{aligned}$$

The proof is finished by comparing the last expression with Theorem 1. Note that the assumptions (A1) and (A2) of Theorem 1 are fulfilled. In particular, (A2) is fulfilled for convex bodies, as shown in [18, Proposition 1]. □

6 An open question

An $O(d)$-invariant valuation Φ on the set of convex bodies is called polynomial if $x \mapsto \Phi(K+x)$ is a polynomial for any convex body K. Alesker [2] showed for $d \geq 3$ that any continuous polynomial $O(d)$-invariant valuation Φ can be expressed in the form

$$\Phi(X) = \sum_{i=0}^{d-1} \int_{\operatorname{nor} X} p_i(|x|^2, x\cdot n)\, \Theta_i(X, \mathrm{d}(x,n)),$$

where $p_1, \ldots, p_{d-1}$ are polynomials in two variables and $\Theta_i(X,\cdot)$ are the (extended) curvature measures of X defined as

$$\begin{aligned} &\int h(x,n)\, \Theta_i(\mathrm{d}(x,n)) \\ &\quad = \sigma^{-1}_{d-i} \int_{\operatorname{nor} X} h(x,n) \sum_{|I|=d-1-i} \frac{\prod_{i\in I}\kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2(x,n)}}\, \mathcal{H}^{d-1}(\mathrm{d}(x,n)). \end{aligned}$$

He also showed that any continuous $O(d)$-invariant valuation is a locally uniform limit of continuous polynomial $O(d)$-invariant valuations, but he later found a gap in the proof (see [3]) and the validity of this assertion remained open. It seems plausible to expect that if this conjecture was true, then every continuous $O(d)$-invariant valuation could be expressed as an integral over the unit normal bundle. The valuations $\Phi_{k,j}$ given by rotational integrals are expressed as integrals over the larger flag manifolds $\mathcal{F}^d_{j-1-k}(X)$ and we doubt that they could be given as integrals over $\operatorname{nor} X$ if $k < j-1$. This leads us to conjecture that these continuous $O(d)$-invariant valuations cannot be approximated by polynomial valuations.

7 Proof of Theorem 1

Let X be a set with positive reach fulfilling assumptions (A1) and (A2), let $(x,n) \in \operatorname{nor} X$, $0 \le k < j < d$, $j \ge 2$, and let A_q be a subspace perpendicular to n and of dimension

$$q = d - 1 - (j - 1 - k) = d - j + k.$$

(Note that $j + q \ge d$.) We will derive a closed form of $Q_j(x,n,A_q)$ from (12) that will prove Theorem 1.

We first rewrite the integrand in (12), using subspaces in $x^\perp$. We use here and in the following the notation $\mathcal{L}_r^s(M)$ for the set of r-dimensional linear subspaces contained in $M \in \mathcal{L}_s^d$. Recall that $\beta = \beta(x,n) = \angle(x,n)$ and $\alpha = \angle(x,A_q)$.

Lemma 2. *Let $A_q \in \mathcal{L}_q^{d-1}(n^\perp)$ and let $L_j = L_{j-1} \oplus \operatorname{span}\{x\}$, where $L_{j-1} \in \mathcal{L}_{j-1}^{d-1}(x^\perp)$. Then,*

$$\mathcal{G}(L_j, A_q)^2 = \sin^2\alpha\, \mathcal{G}^{(x^\perp)}(L_{j-1}, p(A_q|x^\perp))^2 + \cos^2\alpha\, \mathcal{G}^{(x^\perp)}(L_{j-1}, A_q \cap x^\perp)^2. \tag{17}$$

Proof. If $\alpha = \pi/2$ then $A_q \perp x$, $p(A_q|x^\perp) = A_q$ and (17) is obvious. If $\alpha = 0$ then (17) follows from (8). It is thus sufficient to consider the case $0 < \alpha < \pi/2$.

Consider first the case $j + q = d$. Then,

$$\dim(L_{j-1} + A_q \cap x^\perp) < d - 1$$

and the second summand of (17) vanishes because $\mathcal{G}^{(x^\perp)}(L_{j-1}, A_q \cap x^\perp) = 0$. In order to prove (17) in the case $j + q = d$, first notice that if $\dim(L_j + A_q) < d$, then left- and right-hand sides of (17) are both zero. If $\dim(L_j + A_q) = d$, we can proceed as follows. Let $\{a_1, \ldots, a_q\}$ be an orthonormal basis of A_q such that $a_1 = \pi(x|A_q)$ and $a_i \perp x$, $i = 2, \ldots, q$. Then we have

$$\begin{aligned}
\mathcal{G}(L_j, A_q) &= \nabla_q(p(a_1|L_j^\perp), p(a_2|L_j^\perp), \cdots, p(a_q|L_j^\perp)) \\
&= \nabla_q(p(p(a_1|x^\perp)|L_j^\perp), p(a_2|L_j^\perp), \cdots, p(a_q|L_j^\perp)) \\
&= |p(a_1|x^\perp)| \nabla_q(p(\pi(a_1|x^\perp)|L_j^\perp), p(a_2|L_j^\perp), \cdots, p(a_q|L_j^\perp)) \\
&= |p(a_1|x^\perp)| \nabla_q(p(\pi(a_1|x^\perp)|L_{j-1}^\perp), p(a_2|L_{j-1}^\perp), \cdots, p(a_q|L_{j-1}^\perp)) \\
&= |\sin\angle(x, A_q)| \mathcal{G}^{(x^\perp)}(L_{j-1}, p(A_q|x^\perp)).
\end{aligned}$$

Let now $j + q > d$ and choose an orthonormal basis $\{u_1, \ldots, u_{j-1}\}$ of L_{j-1}. Given an index set $I \subseteq \{1, \ldots, j-1\}$, we shall write L_I for the linear hull of $\{u_i : i \in I\}$. We have by [18, Lemma 1],

$$\mathcal{G}(L_j, A_q)^2 = \sum_{|I|=d-q} \mathcal{G}(L_I, A_q)^2 + \sum_{|I|=d-q-1} \mathcal{G}(L_I + \operatorname{span}\{x\}, A_q)^2.$$

By applying (8) to each summand in the first sum and by repeating the above procedure from the case $q + j = d$ to each summand of the second sum, we obtain

$$\begin{aligned}
&\mathcal{G}(L_j, A_q)^2 \\
&\quad= \sum_{|I|=d-q} \cos^2\alpha \mathcal{G}^{(x^\perp)}(L_I, A_q \cap x^\perp)^2 + \sum_{|I|=d-q-1} \sin^2\alpha \mathcal{G}^{(x^\perp)}(L_I, p(A_q|x^\perp))^2 \\
&\quad= \cos^2\alpha \mathcal{G}^{(x^\perp)}(L_{j-1}, A_q \cap x^\perp)^2 + \sin^2\alpha \mathcal{G}^{(x^\perp)}(L_{j-1}, p(A_{j-1}|x^\perp))^2.
\end{aligned}$$

$\square$

The case $x \| n$ will be treated separately, hence, we assume that $\beta = \beta(x, n) \in (0, \pi)$.

We introduce a function of a unit vector and a linear subspace in $\mathbb{R}^{d-1} \cong x^\perp$ which will be needed for the computation of Q_j. Let $d-j \le p \le d-1$, $B_p \in \mathcal{L}_p^{d-1}$ and $m \in S^{d-2}$. Define

$$I_{j-1}^{d-1}(m, B_p) = \int_{\mathcal{L}_{j-1}^{d-1}} \frac{\mathcal{G}(L_{j-1}, B_p)^2}{(\cos^2 \beta + |p(m|L_{j-1})|^2 \sin^2 \beta)^{\frac{d-q}{2}}} \, \mathrm{d}L_{j-1}^{d-1}. \tag{18}$$

Note that using Lemma 2, we have by (12)

$$Q_j(x, n, A_q) = \sin^2 \alpha I_{j-1}^{d-1}(m, p(A_q|x^\perp)) + \cos^2 \alpha I_{j-1}^{d-1}(m, A_q \cap x^\perp), \tag{19}$$

where $m = \pi(n|x^\perp)$. We thus need to evaluate the integral (18) which is done in the following lemma. Recall that the constants $K_{i,j}^d$ are defined after (9).

Lemma 3. *Let p, q, B_p, m, β be as above, and denote $\theta = \angle(m, B_p)$. If $\beta \neq \pi/2$ then,*

$$I_{j-1}^{d-1}(m, B_p) = \frac{1}{p} c_{d-1,j-1} K_{j-1,p}^{d-1} \Big[(p - (d-1)\cos^2 \theta) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2 \beta\right) \\ + (d-1)\cos^2 \theta F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2 \beta\right) \Big].$$

If $\beta = \theta = \pi/2$ then

$$I_{j-1}^{d-1}(m, B_p) = c_{d-1,j-1} K_{j-1,p}^{d-1} F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; 1\right).$$

Proof. We shall treat only the case $\beta \in (0, \pi/2) \cup (\pi/2, \pi)$; the case $\beta = \theta = \pi/2$ is similar but simpler.

The first step will be to transform the integral over a Grassmannian into an integral over a sphere. To achieve this, we apply the coarea formula to the mapping $g : L_{j-1} \mapsto \pi(m|L_{j-1}^\perp)$ defined on $\mathcal{L}_{j-1}^{d-1} \cap \{L : m \notin L\}$ with Jacobian $J_{d-2} g(L_{j-1}) = \tan^{d-j-1} \zeta$, where $\zeta = \angle(m, L_{j-1}^\perp) = \angle(m, u)$ and $u = \pi(m|L_{j-1}^\perp)$ (cf. [21, Lemma 4.2]; note that $L_{j-1} \mapsto L_{j-1}^\perp$ is an isometry). The range of g is the semisphere $S_+^{d-2} := \{y \in S^{d-1} : y \cdot m > 0\}$. Using that

$$g^{-1}(u) = \{L_{j-2} \oplus \mathrm{span}\{v\} : L_{j-2} \in \mathcal{L}_{j-2}^{d-3}(v^\perp \cap m^\perp)\}, \quad u \in S_+^{d-2},$$

where $v = \pi(m|L_{j-1})$, we get

$$\begin{aligned} I_{j-1}^{d-1}&(m, B_p) \\ &= \int_{S_+^{d-2}} \int_{g^{-1}(v)} \frac{\mathcal{G}(L_{j-1}, B_p)^2}{(\cos^2 \beta + \sin^2 \zeta \sin^2 \beta)^{\frac{d-q}{2}}} \frac{1}{J_{j-2} g(L_{j-1})} \, \mathrm{d}L_{j-2}^{d-3} \mathcal{H}^{d-2}(\mathrm{d}u) \\ &= \int_{S_+^{d-2}} \frac{1}{(\cos^2 \beta + \sin^2 \zeta \sin^2 \beta)^{\frac{d-q}{2}} \tan^{d-j-1} \zeta} \\ &\qquad \times \int_{\mathcal{L}_{j-2}^{d-3}(v^\perp \cap m^\perp)} \mathcal{G}(L_{j-2} \oplus \mathrm{span}\{v\}, B_p)^2 \mathrm{d}L_{j-2}^{d-3} \mathcal{H}^{d-2}(\mathrm{d}u). \end{aligned}$$

In order to evaluate the inner integral, we first apply (8):

$$\mathcal{G}(L_{j-1}, B_p)^2 = \cos^2 \angle(u, B_p) \mathcal{G}^{(u^\perp)}(L_{j-1}, B_p \cap u^\perp)^2,$$

and then, we use Lemma 2 for the decomposition of $\mathcal{G}^{(u^\perp)}(L_{j-1}, B_p \cap u^\perp)^2$:

$$\begin{aligned}\mathcal{G}^{(u^\perp)}(L_{j-1}, B_p \cap u^\perp)^2 = {} & \sin^2 \angle(v, B_p \cap u^\perp)\mathcal{G}^{(u^\perp \cap v^\perp)}(L_{j-2}, p(B_p \cap u^\perp | v^\perp))^2 \\ & + \cos^2 \angle(v, B_p \cap u^\perp)\mathcal{G}^{(u^\perp \cap v^\perp)}(L_{j-2}, B_p \cap u^\perp \cap v^\perp)^2.\end{aligned}$$

Note that the second term vanishes when $d = p+j$. Using now the identity (9), we obtain

$$\begin{aligned}&\int_{\mathcal{L}^{d-3}_{j-2}} \mathcal{G}(L_{j-2} \oplus \operatorname{span}\{v\}, B_p)^2 \mathrm{d}L^{d-3}_{j-2} = \cos^2 \angle(u, B_p) \\ &\qquad \times \left(\sin^2 \angle(v, B_p \cap u^\perp) c_{d-3,j-2} K^{d-3}_{j-2,p-1} + \cos^2 \angle(v, B_p \cap u^\perp) c_{d-3,j-2} K^{d-3}_{j-2,p-2}\right) \\ &\quad = \cos^2 \angle(u, B_p) c_{d-3,j-2} K^{d-3}_{j-2,p-1} \left(1 - \tfrac{d-j-1}{p-1} \cos^2 \angle(v, B_p \cap u^\perp)\right).\end{aligned}$$

Due to the definitions of u and v, we can write

$$m = p(m|v) + p(m|v^\perp) = v \sin\zeta + u \cos\zeta,$$

hence

$$\cos^2 \angle(v, B_p \cap u^\perp) = \frac{|p(m|B_p \cap u^\perp)|^2}{\sin^2 \zeta}.$$

Consequently, we obtain

$$I^{d-1}_{j-1}(m, B_p) = c_{d-3,j-2} K^{d-3}_{j-2,p-1} \left(G_1 - \tfrac{d-j-1}{p-1} G_2\right), \tag{20}$$

where

$$G_1 = \int_{S^{d-2}_+} \frac{|p(u|B_p)|^2}{(\cos^2 \beta + \sin^2 \zeta \sin^2 \beta)^{\frac{d-q}{2}} \tan^{d-j-1} \zeta} \mathcal{H}^{d-2}(\mathrm{d}u) \tag{21}$$

and

$$G_2 = \int_{S^{d-2}_+} \frac{|p(m|B_p \cap u^\perp)|^2 |p(u|B_p)|^2 \cos^{d-j-1} \zeta}{(\cos^2 \beta + \sin^2 \zeta \sin^2 \beta)^{\frac{d-q}{2}} \sin^{d-j+1} \zeta} \mathcal{H}^{d-2}(\mathrm{d}u). \tag{22}$$

Using the coarea formula with $\varphi : S^{d-2}_+ \setminus \operatorname{span}(m) \to S^{d-3}(m^\perp)$ defined by $\varphi(u) = \pi(u|m^\perp) =: u_0$ and with $J_{d-3}\varphi(u) = (\sin \angle(u, m))^{-(d-3)}$, we obtain (recall that $\zeta = \angle(u, m)$)

$$\begin{aligned}G_1 &= \int_{S^{d-3}(m^\perp)} \int_{\varphi^{-1}(u_0)} \frac{|p(u|B_p)|^2 J^{-1}_{d-3}\varphi(u)}{(\cos^2 \beta + \sin^2 \zeta \sin^2 \beta)^{\frac{d-q}{2}} \tan^{d-j-1} \zeta} \mathcal{H}^1(\mathrm{d}u) \mathcal{H}^{d-3}(\mathrm{d}u_0) \\ &= \int_{S^{d-3}(m^\perp)} \int_{\varphi^{-1}(u_0)} \frac{|p(u|B_p)|^2 \cos^{d-j-1} \zeta \sin^{j-2} \zeta}{(\cos^2 \beta + \sin^2 \zeta \sin^2 \beta)^{\frac{d-q}{2}}} \mathcal{H}^1(\mathrm{d}u) \mathcal{H}^{d-3}(\mathrm{d}u_0).\end{aligned}$$

Define $\xi : \mathbb{R}_+ \to \varphi^{-1}(u_0)$ by $\xi(r) = \frac{u_0 + rm}{|u_0 + rm|} = u$ with $J_1\xi(r) = \frac{1}{1+r^2}$. Note that $\cos^2 \zeta = \cos^2 \angle(\xi(r), m) = \frac{r^2}{1+r^2}$ and $\sin^2 \zeta = \frac{1}{1+r^2}$. The area formula implies

$$G_1 = \int_{S^{d-3}(m^\perp)} \int_0^\infty \frac{|p(\xi(r)|B_p)|^2 (\frac{r^2}{1+r^2})^{\frac{d-j-1}{2}} (\frac{1}{1+r^2})^{\frac{j-2}{2}}}{(\cos^2 \beta + \frac{\sin^2 \beta}{1+r^2})^{\frac{d-q}{2}}} \frac{\mathrm{d}r}{1+r^2} \mathcal{H}^{d-3}(\mathrm{d}u_0).$$

We now use that

$$|p(\xi(r)|B_p)|^2 = \frac{|p(u_0|B_p)|^2 + r^2 |p(m|B_p)|^2 + 2r p(u_0|B_p) \cdot p(m|B_p)}{1+r^2}$$

which, using the equality $\int_{S^{d-3}(m^\perp)} p(u_0|B_p)\cdot p(m|B_p)\mathcal{H}^{d-3}(du_0)=0$ and (7), lead us to the following expression

$$\begin{aligned}
G_1 &= \int_{S^{d-3}(m^\perp)}\int_0^\infty \frac{(|p(u_0|B_p)|^2+r^2|p(m|B_p)|^2)(r^2)^{\frac{d-j-1}{2}}}{(\cos^2\beta+\frac{1}{1+r^2}\sin^2\beta)^{\frac{d-q}{2}}\,(1+r^2)^{\frac{d+1}{2}}}\mathrm{d}r\mathcal{H}^{d-3}(\mathrm{d}u_0)\\
&= \tfrac12 B\left(\tfrac{d-j}{2},\tfrac{j+1}{2}\right)F\left(\tfrac{d-q}{2},\tfrac{d-j}{2};\tfrac{d+1}{2};\sin^2\beta\right)H_1 \qquad (23)\\
&\quad+\tfrac12 B\left(\tfrac{d-j+2}{2},\tfrac{j-1}{2}\right)F\left(\tfrac{d-q}{2},\tfrac{d-j+2}{2};\tfrac{d+1}{2};\sin^2\beta\right)|p(m|B_p)|^2\sigma_{d-2}
\end{aligned}$$

with

$$H_1=\int_{S^{d-3}(m^\perp)}|p(u_0|B_p)|^2\mathcal{H}^{d-3}(\mathrm{d}u_0).$$

The convergence criterion in (7) is satisfied since $1<j<d$ and $0<\beta<\frac{\pi}{2}$ by assumption.

Note that the differences between G_1 and G_2 are the extra terms $\sin^2\zeta$ and

$$|p(m|B_p\cap u^\perp)|^2=|p(m|B_p)|^2\frac{|p(u|B_p\cap m^\perp)|^2}{|p(u|B_p)|^2}.$$

Hence, G_2 can be rewritten as

$$G_2=|p(m|B_p)|^2\int_{S_+^{d-2}}\frac{|p(u|B_p\cap m^\perp)|^2\cos^{d-j-1}\zeta}{(\cos^2\beta+\sin^2\zeta\sin^2\beta)^{\frac{d-q}{2}}\sin^{d-j+1}\zeta}\mathcal{H}^{d-2}(\mathrm{d}u).$$

By applying the area formula for the mappings $\varphi: u\mapsto \pi(u|m^\perp)$ and $\xi: r\mapsto\frac{u_0+rm}{|u_0+rm|}$, the integral above becomes

$$\begin{aligned}
&\int_{S^{d-3}(m^\perp)}\int_{\varphi^{-1}(u_0)}\frac{|p(u|B_p\cap m^\perp)|^2\cos^{d-j-1}\zeta\sin^{j-4}\zeta}{(\cos^2\beta+\sin^2\zeta\sin^2\beta)^{\frac{d-q}{2}}}\mathcal{H}^1(\mathrm{d}u)\mathcal{H}^{d-3}(\mathrm{d}u_0)\\
&=\int_{S^{d-3}(m^\perp)}|p(u_0|B_p\cap m^\perp)|^2\int_0^\infty\frac{(r^2)^{\frac{d-j-1}{2}}(1+r^2)^{-\frac{d-1}{2}}}{(\cos^2\beta+\frac{\sin^2\beta}{1+r^2})^{\frac{d-q}{2}}}\mathrm{d}r\mathcal{H}^{d-3}(\mathrm{d}u_0),
\end{aligned}$$

where we used $|p(\xi(r)|B_p\cap m^\perp)|^2=\frac{|p(u_0|B_p\cap m^\perp)|^2}{1+r^2}$ for the last equality. Using (7), we obtain

$$G_2=\frac{|p(m|B_p)|^2}{2}B\left(\tfrac{d-j}{2},\tfrac{j-1}{2}\right)F\left(\tfrac{d-q}{2},\tfrac{d-j}{2};\tfrac{d-1}{2};\sin^2\beta\right)H_2 \qquad (24)$$

with

$$H_2=\int_{S^{d-3}(m^\perp)}|p(u_0|B_p\cap m^\perp)|^2\mathcal{H}^{d-3}(\mathrm{d}u_0).$$

Note that $|p(u_0|B_p\cap m^\perp)|^2=\mathcal{G}(u_0^\perp,B_p\cap m^\perp)^2$ and, since the integration over $S^{d-3}(m^\perp)$ is, up to a factor 2, the integration over the Grassmannian $\mathcal{L}_{d-3}^{d-2}(m^\perp)$, we can apply (9) and obtain

$$H_2=\sigma_{d-3}K_{d-3,p-1}^{d-2}=\omega_{d-2}(p-1), \qquad (25)$$

where $\omega_k=\sigma_k/k$ is the volume of the unit ball in $\mathbb{R}^k$. In order to calculate H_1, we use the decomposition

$$|p(u_0|B_p)|^2=|p(u_0|B_p\cap m^\perp)|^2+(u_0\cdot m_0)^2,$$

where $m_0 = \pi(m|B_p)$. We have $u_0 \cdot m_0 = \sin\theta(u_0 \cdot m_1)$ with $m_1 = \pi(m_0|m^\perp)$. Since, again by (9),

$$\int_{S^{d-3}(m^\perp)} (u_0 \cdot m_1)^2 \, \mathcal{H}^{d-3}(\mathrm{d}u_0) = \omega_{d-2},$$

we get

$$H_1 = \omega_{d-2}(p-1) + \sin^2\theta\omega_{d-2} = \omega_{d-2}(p - \cos^2\theta). \tag{26}$$

By inserting (26) into (23) and (25) into (24) we get

$$\begin{aligned} G_1 = {} & \frac{\omega_{d-2}(p-\cos^2\theta)}{2} B\left(\tfrac{d-j}{2}, \tfrac{j+1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & + \frac{\sigma_{d-2}\cos^2\theta}{2} B\left(\tfrac{d-j+2}{2}, \tfrac{j-1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j+2}{2}; \tfrac{d+1}{2}; \sin^2\beta\right), \end{aligned}$$

and

$$G_2 = \frac{\omega_{d-2}(p-1)\cos^2\theta}{2} B\left(\tfrac{d-j}{2}, \tfrac{j-1}{2}\right) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right),$$

which, in combination with (20), implies

$$\begin{aligned} I^{d-1}_{j-1}(m, B_p) = {} & \tfrac{1}{2} c_{d-3,j-2} K^{d-3}_{j-2,p-1} \omega_{d-2} B\left(\tfrac{d-j}{2}, \tfrac{j-1}{2}\right) \\ & \times \Big[(p - \cos^2\theta) \tfrac{j-1}{d-1} F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & \quad + (d-2)\cos^2\theta \tfrac{d-j}{d-1} F\left(\tfrac{d-q}{2}, \tfrac{d-j+2}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & \quad - \tfrac{(d-j-1)}{p-1}(p-1)\cos^2\theta F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right) \Big]. \end{aligned}$$

Applying (6) to the middle hypergeometric function, the expression above can be rewritten

$$\begin{aligned} I^{d-1}_{j-1}(m, B_p) = {} & \tfrac{1}{2} c_{d-3,j-2} K^{d-3}_{j-2,p-1} \omega_{d-2} B\left(\tfrac{d-j}{2}, \tfrac{j-1}{2}\right) \\ & \times \Big[\tfrac{j-1}{d-1}(p - \cos^2\theta - (d-2)\cos^2\theta) F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & \quad + (j-1)\cos^2\theta F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right) \Big]. \end{aligned}$$

Use

$$\frac{1}{p} c_{d-1,j-1} K^{d-1}_{j-1,p} = \frac{(j-1) c_{d-3,j-2} K^{d-3}_{j-2,p-1} \omega_{d-2} B(\frac{d-j}{2}, \frac{j-1}{2})}{2(d-1)}$$

and the proof is complete. □

Proof of Theorem 1. If $\beta = 0$ or π then $x \| n$, $\alpha = \pi/2$, $p(A_q|x^\perp) = A_q$, $p(n|L_j) = n$ and Q_j can be obtained by using Lemma 2:

$$Q_j(x, n, A_q) = \int \mathcal{G}^{(x^\perp)}(L_{j-1}, A_q) \, \mathrm{d}L^{d-1}_{j-1}.$$

The result is then obtained using (9).

In the case $\beta = \pi/2$ we have $m = n$, hence, $\theta = \pi/2$, and the result follows using (19) and Lemma 3.

Assume in the following that $\beta \in (0, \pi/2) \cup (\pi/2, \pi)$. We use the form of $Q_j = Q_j(x, n, A_q)$ given in (19). In the first summand in (19) we have a factor of the form

$I^{d-1}_{j-1}(m, B_p)$ with $p = \dim p(A_q|x^\perp) = q$, unless $\alpha = 0$. Assume thus that $\alpha > 0$. Let $\theta := \angle(m, p(A_q|x^\perp))$. We shall show that

$$\cos\theta = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}. \tag{27}$$

Since

$$m = \frac{n - p(n|x)}{|p(n|x^\perp)|} = \frac{1}{\sin\beta}\left(n - \frac{\cos\beta}{|x|}x\right),$$

we have

$$p(m|p(A_q|x^\perp)) = \frac{1}{\sin\beta}p(n|p(A_q|x^\perp)).$$

If $\alpha = \pi/2$ then $p(A_q|x^\perp) = A_q \perp n$ and we get $\cos\theta = 0$, verifying (27) in this particular case. Assume now that $\alpha \in (0, \pi/2)$. By using the decomposition $A_q = \operatorname{span}\{a\} \oplus (A_q \cap x^\perp)$, where $a = \pi(x|A_q)$, and that $n \perp A_q$, we get

$$p(n|p(A_q|x^\perp)) = p(n|\pi(a|x^\perp)),$$

where

$$\pi(a|x^\perp) = \frac{a - p(a|x)}{|p(a|x^\perp)|} = \frac{1}{\sin\alpha}\left(a - \cos\alpha\frac{x}{|x|}\right).$$

Since $n \perp a$, we obtain

$$\cos\theta = |p(m|p(A_q|x^\perp))| = \frac{|\pi(a|x^\perp)\cdot n|}{\sin\beta} = \frac{1}{\sin\alpha\sin\beta}\left(\cos\alpha\frac{x\cdot n}{|x|}\right),$$

verifying (27) again.

In the second summand we have a similar factor with $p = \dim(A_q \cap x^\perp) = q - 1$ and $\theta = \frac{\pi}{2}$, i.e $\cos\theta = 0$. Lemma 3 together with the identity

$$K^{d-1}_{j-1,q-1} = \tfrac{j+q-d}{q}K^{d-1}_{j-1,q}$$

imply

$$\begin{aligned} Q_j = \frac{c_{d-1,j-1}K^{d-1}_{j-1,q}}{q}\Bigg\{ & \sin^2\alpha\Big[(q-(d-1)\cos^2\theta)F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right) \\ & \qquad + (d-1)\cos^2\theta F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d-1}{2}; \sin^2\beta\right)\Big] \\ & + \cos^2\alpha\frac{j+q-d}{q-1}(q-1)F\left(\tfrac{d-q}{2}, \tfrac{d-j}{2}; \tfrac{d+1}{2}; \sin^2\beta\right)\Bigg\}. \end{aligned}$$

The result now follows by using (27). $\square$

Appendix

The following lemma gives the values of the constants appearing in Theorem 2.

Lemma 4. *Let $A \in \mathcal{L}^d_{d-i}$, $y \perp A$, $z \in A$, $|y| = |z| = 1$. Then*

$$M^d_i := \int \mathcal{G}(A, V_i)^2 |p(y|V_i)|^2 \, dV^d_i = \frac{c_{d-1,i-1}}{2\sigma_{d-i}}\binom{d-1}{i-1}^{-1} B\left(\tfrac{i+4}{2}, \tfrac{d-i}{2}\right),$$

$$N^d_i := \int \mathcal{G}(A, V_i)^2 |p(z|V_i)|^2 \, dV^d_i = c_{d,i}\binom{d}{i}^{-1} - M_{d,d-i}.$$

Proof. First, we apply the coarea formula with

$$\varphi :\, V_i \mapsto \pi(y|V_i) =: v, \quad J_{d-1}\varphi(V_i) = \tan^{i-1}\gamma,$$

with $\gamma := \angle(y,v) = \angle(y,V_i)$ (cf. the beginning of the proof of Lemma 3). We get

$$M_i^d = \int_{S_+^{d-1}} \frac{\cos^2\gamma}{\tan^{i-1}\gamma} \int_{\varphi^{-1}\{v\}} \mathcal{G}(A,V_i)^2 \,\mathrm{d}V_i^d \,\mathcal{H}^{d-1}(\mathrm{d}v) \tag{28}$$

($S_+^{d-1} = \{v \in S^{d-1} :\, v\cdot y > 0\}$). Since $y \perp A$ and $v \perp V_i \cap y^\perp$, we obtain using twice (8):

$$\begin{aligned}\mathcal{G}(A,V_i)^2 &= \cos^2\gamma \mathcal{G}^{(y^\perp)}(A, V_i\cap y^\perp)^2 \\ &= \cos^2\gamma |p(v|A)|^2 \mathcal{G}^{(y^\perp\cap v^\perp)}(A\cap v^\perp, V_i\cap y^\perp)^2.\end{aligned}$$

Further, $\varphi^{-1}\{v\} = \{V_{i-1}' \oplus \operatorname{span}\{v\} :\, V_{i-1}' \in \mathcal{L}_{i-1}^{d-2}(y^\perp \cap v^\perp)\}$, thus

$$\begin{aligned}\int_{\varphi^{-1}\{v\}} \mathcal{G}(A,V_i)^2 \,\mathrm{d}V_i &= \cos^2\gamma|p(v|A)|^2 \int \mathcal{G}^{(y^\perp\cap v^\perp)}(A\cap v^\perp, V_i\cap y^\perp)^2 \,\mathrm{d}V_{i-1}^{d-2} \\ &= \cos^2\gamma|p(v|A)|^2 c_{d-2,i-1}\binom{d-2}{i-1}^{-1}\end{aligned}$$

by (9). Inserting this into (28), we get

$$M_i^d = c_{d-2,i-1}\binom{d-2}{i-1}^{-1} \int_{S_+^{d-1}} \frac{\cos^{i+3}\gamma}{\sin^{i-1}\gamma} |p(v|A)|^2 \,\mathcal{H}^{d-1}(\mathrm{d}v).$$

We apply now the coarea formula with $\psi :\, v \mapsto v\cdot y$, $J_1\psi(v) = \sin\gamma$, $\psi^{-1}\{r\} \cong \sqrt{1-r^2}S^{d-2}$:

$$M_i^d = c_{d-2,i-1}\binom{d-2}{i-1}^{-1} \int_0^1 \frac{r^{i+3}}{(1-r^2)^{i/2}} \int_{\psi^{-1}\{r\}} |p(v|A)|^2 \,\mathcal{H}^{d-2}(\mathrm{d}v)\,\mathrm{d}r,$$

where

$$\begin{aligned}\int_{\psi^{-1}\{r\}} |p(v|A)|^2 \,\mathcal{H}^{d-2}(\mathrm{d}v) &= (1-r^2)^{(d-2)/2} \int_{S^{d-2}} |p(v|A)|^2 \,\mathrm{d}v \\ &= (1-r^2)^{(d-2)/2} \int_{S^{d-2}} \mathcal{G}(A, v^\perp)^2 \,\mathrm{d}v \\ &= (1-r^2)^{(d-2)/2} \sigma_{d-1} K_{d-1,d-i-1}^{d-i,d-2} \\ &= \sigma_{d-1}\frac{d-i}{d-1}(1-r^2)^{(d-2)/2}\end{aligned}$$

(we used (9) in the last but one equality). Hence,

$$M_i^d = c_{d-2,i-1}\sigma_{d-1}\binom{d-2}{i-1}^{-1} \frac{d-i}{d-1}\int_0^1 r^{i+3}(1-r^2)^{(d-i-2)/2}\,\mathrm{d}r.$$

After routine calculation of the integral we finally arrive at

$$M_i^d = \frac{c_{d-1,i-1}}{\sigma_{d-i}}\binom{d-2}{i-1}^{-1} B\left(\tfrac{i+4}{2}, \tfrac{d-i}{2}\right).$$

For the second integral, we use:

$$
\begin{aligned}
N_i^d &= \int \mathcal{G}(A^\perp, V_i^\perp)(1 - |p(z|V_i^\perp)|^2)\,\mathrm{d}V_i^\perp \\
&= \int \mathcal{G}(A^\perp, V_i^\perp)^2\,\mathrm{d}V_i^\perp - \int \mathcal{G}(A^\perp, V_i^\perp)^2 |p(z|V_i^\perp)|^2\,\mathrm{d}V_i^\perp \\
&= c_{d,i}\binom{d}{i}^{-1} - M_i^d.
\end{aligned}
$$

□

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Paper B

Expressing intrinsic volumes as rotational integrals

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Expressing intrinsic volumes as rotational integrals

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Abstract

A new rotational formula of Crofton type is derived for intrinsic volumes of a compact subset $X \subset \mathbb{R}^d$ of positive reach. The formula provides a functional defined on the section of X with a $j-$dimensional linear subspace with rotational average equal to the intrinsic volumes of X. Simplified forms of the functional are derived in special cases.

*Keywords:*Geometric measure theory; Integral geometry; Rotational integral; Grassmann manifold; Intrinsic volume; Set of positive reach; Stereology

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1 Introduction

For a compact subset X of $\mathbb{R}^d$, satisfying certain regularity conditions, the classical Crofton formula relates integrals of intrinsic volumes defined on $j-$dimensional affine subspaces to intrinsic volumes of X,

$$\int_{\mathcal{F}_j^d} V_k(X \cap F_j)\mathrm{d}F_j^d = c_{d,j,k}V_{d-j+k}(X),$$

$j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$. Here, $\mathcal{F}_j^d$ is the set of $j-$dimensional affine subspaces and $\mathrm{d}F_j^d$ is the element of the motion invariant measure on $j-$dimensional affine subspaces in $\mathbb{R}^d$. Furthermore, $V_k(X)$, $k = 0, 1, \ldots, d$, are the intrinsic volumes of X. Finally, $c_{d,j,k}$ is a known constant.

Motivated by applications in local stereology, a rotational version of the Crofton formula has recently been derived, cf. [8]. This formula shows how rotational averages of intrinsic volumes measured on sections passing through a fixed point are related to the geometry of the sectioned object. More specifically, for a compact subset $X \subset \mathbb{R}^d$ of positive reach, the functionals $\beta_{j,k}$, satisfying

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\mathrm{d}L_j^d = \beta_{j,k}(X),$$

$j = 0, 1, \ldots, d$, $k = 0, 1, \ldots, j$, have been determined in [8]. For $k = j$, $\beta_{j,j}(X)$ is a simple integral while in the case $k < j$, $\beta_{j,k}(X)$ is a complicated integral over the unit normal bundle of X, involving principal curvatures and hypergeometric functions.

In the present paper, we address the 'opposite' problem of finding functionals $\alpha_{j,k}$, satisfying the following rotational integral equation

$$\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j)\mathrm{d}L_j^d = V_{d-j+k}(X), \tag{1}$$

$j = 0, 1, \ldots, d$ and $k = 0, 1, \ldots, j$. The solution of the problem is inspired by some recent work reported in [3] and [4].

2 The general solution

The main tools for deriving solutions to (1) are the classical Crofton formula and a well-known geometric measure decomposition from integral geometry.

The motion invariant measure on $j-$dimensional affine subspaces can be decomposed as follows. For $F_j = x + L_j$, where L_j is a $j-$dimensional linear subspace and $x \in L_j^\perp$, we have $\mathrm{d}F_j^d = \mathrm{d}x^{d-j}\mathrm{d}L_j^d$ where $\mathrm{d}L_j^d$ is the element of the rotation invariant measure on $\mathcal{L}_j^d$, the set of $j-$dimensional linear subspaces and, for given $L_j \in \mathcal{L}_j^d$, $\mathrm{d}x^{d-j}$ is the element of the Lebesgue measure in $L_j^\perp$. The total mass of $\mathrm{d}L_j^d$ is chosen to be

$$\int_{\mathcal{L}_j^d} \mathrm{d}L_j^d = c_{d,j},$$

where

$$c_{d,j} = \frac{\sigma_d \sigma_{d-1} \cdots \sigma_{d-j+1}}{\sigma_j \sigma_{j-1} \cdots \sigma_1} \tag{2}$$

and $\sigma_k = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of the unit sphere in $\mathbb{R}^k$. With this choice, the constant in the classical Crofton formula becomes

$$c_{d,j,k} = c_{d,j} \cdot \frac{\Gamma(\frac{j+1}{2})\Gamma(\frac{d+k-j+1}{2})}{\Gamma(\frac{k+1}{2})\Gamma(\frac{d+1}{2})}. \tag{3}$$

The geometric measure decomposition used in the derivation of solutions to (1) concerns the motion invariant measure on $r-$dimensional affine subpaces in $\mathbb{R}^d$. According to Gual-Arnau and Cruz-Orive [4], we have for $r = 0, 1, \ldots, d-1$ that

$$\mathrm{d}F_r^d = d(O, F_r)^{d-r-1}\mathrm{d}F_r^{r+1}\mathrm{d}L_{r+1}^d, \tag{4}$$

where $\mathrm{d}F_r^{r+1}$ is the element of the motion invariant measure on $r-$dimensional affine subspaces in L_{r+1} and $d(O, F_r)$ denotes the distance from F_r to the origin O. Note that for $r = 0$, (4) reduces to the standard polar decomposition of Lebesgue measure

$$\mathrm{d}x^d = |x|^{d-1}\mathrm{d}x^1\mathrm{d}L_1^d.$$

We formulate the main result of this paper in the proposition below.

Proposition 1. *Let X be a compact subset of $\mathbb{R}^d$ of positive reach. Assume that for almost all $L_j \in \mathcal{L}_j^d$*

$$(x, n) \in \operatorname{nor} X, x \in L_j \Rightarrow n \not\perp L_j, \tag{5}$$

where nor X *is the unit normal bundle of X. Then,*

$$\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j)\mathrm{d}L_j^d = V_{d-j+k}(X),$$

$j = 1, \ldots, d,\ k = 1, \ldots, j,$ *where*

$$\alpha_{j,k}(X \cap L_j) = \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j} V_{k-1}((X \cap L_j) \cap F_{j-1})\mathrm{d}F_{j-1}^j. \tag{6}$$

Proof. The condition (5) of the proposition ensures that $X \cap L_j$ is of positive reach for almost all $L_j \in \mathcal{L}_j^d$, cf. [8, p. 550]. Using the Crofton formula and the measure decomposition (4), we find that

$$\begin{aligned}
&\int_{\mathcal{L}_j^d} \alpha_{j,k}(X \cap L_j)\mathrm{d}L_j^d \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}_j^d} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j} V_{k-1}(X \cap L_j \cap F_{j-1})\mathrm{d}F_{j-1}^j\mathrm{d}L_j^d \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}_j^d} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-(j-1)-1} V_{k-1}(X \cap F_{j-1})\mathrm{d}F_{j-1}^j\mathrm{d}L_j^d \\
&= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}_{j-1}^d} V_{k-1}(X \cap F_{j-1})\mathrm{d}F_{j-1}^d \\
&= V_{d-j+k}(X).
\end{aligned}$$

$\square$

3 The case $k = j$

For $k = j$, Proposition 1 provides a functional with rotational average equal to the volume $V_d(X)$. This functional can be simplified considerably, as shown in the proposition below. We use here and in the following the notation $p(x|L_r)$ for the orthogonal projection of $x \in \mathbb{R}^d$ onto $L_r \in \mathcal{L}_r^d$.

Proposition 2. *Let the situation be as in Proposition 1 and suppose that $k = j$. Then,*

$$\alpha_{j,j}(X \cap L_j) = \frac{1}{c_{d-1,j-1}} \int_{X \cap L_j} |z|^{d-j} \mathrm{d}z^j.$$

Proof. Using that $F_{j-1} = L_{j-1} + x$, where $x \in L_{j-1}^{\perp}$, we find

$$\begin{aligned}
\alpha_{j,j}(Y) &= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{F}_{j-1}^j} d(O, F_{j-1})^{d-j} V_{j-1}(Y \cap F_{j-1}) \mathrm{d}F_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^{\perp}} |x|^{d-j} V_{j-1}(Y \cap (L_{j-1} + x)) \mathrm{d}x^1 \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_{L_{j-1}^{\perp}} \int_{Y \cap (L_{j-1}+x)} |x|^{d-j} \mathrm{d}y^{j-1} \mathrm{d}x^1 \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_{\mathcal{L}_{j-1}^j} \int_Y |p(z|L_{j-1}^{\perp})|^{d-j} \mathrm{d}z^j \mathrm{d}L_{j-1}^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_Y |z|^{d-j} \left(\int_{\mathcal{L}_{j-1}^j} \frac{|p(z|L_{j-1}^{\perp})|^{d-j}}{|z|^{d-j}} \mathrm{d}L_{j-1}^j \right) \mathrm{d}z^j \\
&= \frac{1}{c_{d,j-1,j-1}} \int_Y |z|^{d-j} \left(\frac{c_{j,j-1}}{B(\frac{1}{2}, \frac{j-1}{2})} \int_0^1 y^{\frac{d-j-1}{2}} (1-y)^{\frac{j-3}{2}} \mathrm{d}y \right) \mathrm{d}z^j.
\end{aligned}$$

At the last equality sign, we have used [7, Proposition 3.9]. The result now follows immediately, using (2) and (3). □

4 The case $k < j$

It is also possible to make the expression of the functional $\alpha_{j,k}$ more explicit for $k < j$. We will concentrate on the case where X is the compact closure of an open subset of $\mathbb{R}^d$ and ∂X is a $(d-1)-$dimensional manifold of class C^2. For $k = 0, 1, \dots, d-1$, the kth intrinsic volume has the following integral representation

$$V_k(X) = \frac{1}{\sigma_{d-k}} \int_{\partial X} \sum_{|I|=d-1-k} \prod_{i \in I} \kappa_i(x) \mathcal{H}^{d-1}(\mathrm{d}x), \tag{7}$$

where $\kappa_i(x)$, $i = 1, \dots, d-1$, are the principal curvatures of ∂X at $x \in \partial X$ and $\mathcal{H}^{d-1}$ denotes the $(d-1)-$dimensional Hausdorff measure. We will assume that for all $j = 1, \dots, d$, almost all $z \in \partial X$ and almost all $L_j \in \mathcal{L}_j^d$

$$x \in (\partial X) \cap (L_j + z) \Rightarrow n(x) \not\perp L_j. \tag{8}$$

For an affine subspace $F_j = L_j + z$, satisfying (8), we have, cf. [5, p. 59 and 60],

$$\partial(X \cap F_j) = (\partial X) \cap F_j$$

and $\partial(X \cap F_j)$ is a $(j-1)-$dimensional manifold of class C^2. The principal curvatures of $\partial(X \cap F_j)$ at $x \in \partial(X \cap F_j)$ are denoted by $\kappa_{F_j,i}(x)$, $i = 1, \dots, j-1$.

The proposition below gives a more explicit expression for $\alpha_{j,k}$ for $k < j$ than the one given in (6).

Proposition 3. *Let the situation be as in Proposition 1 and let $k < j$. Suppose that X is the compact closure of an open subset of $\mathbb{R}^d$ and ∂X is a $(d-1)-$dimensional manifold of class C^2 for which (8) is satisfied. Then,*

$$\begin{aligned}&c_{d,j-1,k-1}\ \sigma_{j-k}\ \alpha_{j,k}(X \cap L_j)\\&\quad= \int_{\partial(X\cap L_j)} \int_{\mathcal{L}^j_{j-1}} \kappa(z; L_{j-1}+z)\, |p(n(z)|L_{j-1})|\, |p(z|L^{\perp}_{j-1})|^{d-j} \mathrm{d}L^j_{j-1} \mathcal{H}^{j-1}(\mathrm{d}z),\end{aligned}$$

where $n(z)$ is the unit normal of $\partial(X \cap L_j)$ at z and

$$\kappa(z; F_{j-1}) = \begin{cases} 1 & \text{if } k = j-1 \\ \sum_{|I|=j-k-1} \prod_{i\in I} \kappa_{F_{j-1},i}(z) & \text{if } k < j-1. \end{cases}$$

Proof. Note that the condition (8) ensures that for almost all $L_j \in \mathcal{L}^d_j$, $z \in \partial(X \cap L_j)$ and $L_{j-1} \in \mathcal{L}^j_{j-1}$, $\partial(X \cap L_j) \cap (L_{j-1} + z)$ is a $(j-2)-$dimensional manifold of class C^2. This can be seen by first noting that

$$\begin{aligned}\partial(X \cap L_j) \cap (L_{j-1}+z) &= (\partial X) \cap L_j \cap (L_{j-1}+z)\\ &= (\partial X) \cap (L_{j-1}+z),\end{aligned}$$

and then combining (8) with [7, Proposition 5.4]. The function $\kappa(z; L_{j-1}+z)$ is well-defined when $\partial(X \cap L_j) \cap (L_{j-1}+z)$ is a $(j-2)-$dimensional manifold of class C^2.

Letting $Y = X \cap L_j$, we have according to (6)

$$\begin{aligned}\alpha_{j,k}(Y) &= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{F}^j_{j-1}} d(O, F_{j-1})^{d-j} V_{k-1}(Y \cap F_{j-1})\, \mathrm{d}F^j_{j-1}\\ &= \frac{1}{c_{d,j-1,k-1}} \int_{\mathcal{L}^j_{j-1}} \int_{L^{\perp}_{j-1}} |x|^{d-j} V_{k-1}(Y \cap (L_{j-1}+x))\, \mathrm{d}x^1\, \mathrm{d}L^j_{j-1}.\end{aligned}$$

Using the integral representation (7) of intrinsic volumes, the expression above becomes

$$\begin{aligned}&c_{d,j-1,k-1}\alpha_{j,k}(Y)\\&\quad= \frac{1}{\sigma_{(j-1)-(k-1)}} \int_{\mathcal{L}^j_{j-1}} \int_{L^{\perp}_{j-1}} |x|^{d-j} \int_{\partial Y \cap (L_{j-1}+x)} \kappa(y; L_{j-1}+x) \mathcal{H}^{(j-1)-1}(\mathrm{d}y)\, \mathrm{d}x^1\, \mathrm{d}L^j_{j-1}\\&\quad= \frac{1}{\sigma_{j-k}} \int_{\mathcal{L}^j_{j-1}} \int_{L^{\perp}_{j-1}} \int_{\partial Y \cap (L_{j-1}+x)} |p(y|L^{\perp}_{j-1})|^{d-j} \kappa(y; L_{j-1}+y) \mathcal{H}^{j-2}(\mathrm{d}y)\, \mathrm{d}x^1\, \mathrm{d}L^j_{j-1}.\end{aligned}$$

At the first equality sign we have used that $\partial(Y \cap F_{j-1}) = \partial Y \cap F_{j-1}$ for almost all F_{j-1}. Using [7, Propositions 2.10 and 5.2] and Fubini, we finally get

$$\begin{aligned}&c_{d,j-1,k-1}\alpha_{j,k}(Y)\\&\quad= \frac{1}{\sigma_{j-k}} \int_{\mathcal{L}^j_{j-1}} \int_{\partial Y} |p(n(z)|L_{j-1})|\ \ |p(z|L^{\perp}_{j-1})|^{d-j} \kappa(z, L_{j-1}+z) \mathcal{H}^{j-1}(\mathrm{d}z)\, \mathrm{d}L^j_{j-1}\\&\quad= \frac{1}{\sigma_{j-k}} \int_{\partial Y} \int_{\mathcal{L}^j_{j-1}} \kappa(z, L_{j-1}+z) |p(n(z)|L_{j-1})|\ \ |p(z|L^{\perp}_{j-1})|^{d-j}\, \mathrm{d}L^j_{j-1} \mathcal{H}^{j-1}(\mathrm{d}z).\end{aligned}$$

□

For $k = j-1$, the expression for $\alpha_{j,k}(Y)$ given in Proposition 3 can be further simplified, using the following proposition. The proof is deferred to the Appendix.

Proposition 4. *Let $L_j \in \mathcal{L}_j^d$, $j = 1, \ldots, d$. Let x and y be unit vectors in L_j. Then, for all $m, n \in \mathbb{N}$,*

$$\int_{\mathcal{L}_{j-1}^j} |p(x|L_{j-1})|^m \, |p(y|L_{j-1}^\perp)|^n \mathrm{d}L_{j-1}^j$$
$$= \frac{\sigma_{j-1}}{2} B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{j-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{j-1}{2}, \sin^2 \angle(x,y)\right).$$

□

Using Proposition 4 with $m = 1$ and $n = d - j$, we find

$$\alpha_{j,j-1}(Y) = \frac{1}{2c_{d-1,j-1}} \int_{\partial Y} |z|^{d-j} F\left(-\frac{1}{2}, -\frac{d-j}{2}; \frac{j-1}{2}; \sin^2 \angle(n(z), z)\right) \mathcal{H}^{j-1}(\mathrm{d}z).$$

Appendix

In this appendix, we will prove Proposition 4. Without loss of generality, we assume that $x \cdot y > 0$. For simplicity, we write $\mathrm{d}z^j$ instead of $\mathcal{H}^j(\mathrm{d}z)$.

The Gauss *hypergeometric series* or *hypergeometric function* is defined for $a, b, c \in \mathbb{R}$ and $z \in [-1, 1]$ as

$$F(a, b; c; z) = F(b, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(x)_k$ is the *rising sequential product* or *Pochhammer symbol* defined for a non-negative integer k and $x \in \mathbb{R}$ by

$$(x)_k = \begin{cases} \frac{\Gamma(x+k)}{\Gamma(x)} & \text{if } x > 0 \\ (-1)^k \frac{\Gamma(-x+1)}{\Gamma(-x-k+1)} & \text{if } x \leq 0. \end{cases}$$

Note that $(x)_k = 0$ whenever $x \in \{0, -1, -2, \ldots\}$ and $k > -x$.

An application of [7, Propositions 3.2 and 3.3] gives

$$\int_{\mathcal{L}_{j-1}^j} |p(x|L_{j-1})|^m \;\; |p\left(y|L_{j-1}^\perp\right)|^n \, \mathrm{d}L_{j-1}^j$$
$$= \int_{\mathcal{L}_1^j} |p(x|L_1^\perp)|^m \;\; |p\left(y|L_1\right)|^n \, \mathrm{d}L_1^j$$
$$= \frac{1}{2} \int_{S^{j-1}} |p(x|\mathrm{span}\{\omega\}^\perp)|^m \;\; |p\left(y|\mathrm{span}\{\omega\}\right)|^n \, \mathrm{d}\omega^{j-1}$$
$$= \frac{1}{2} \int_{S^{j-1}} \sqrt{1 - (x \cdot \omega)^2}^m \;\; |y \cdot \omega|^n \, \mathrm{d}\omega^{j-1}$$
$$= \frac{1}{2} \sum_{k=0}^{\infty} \binom{\frac{m}{2}}{k} (-1)^k \int_{S^{j-1}} |x \cdot \omega|^{2k} |y \cdot \omega|^n \, \mathrm{d}\omega^{j-1}. \tag{9}$$

Now note that

$$\int_{S^{j-1}} |x \cdot \omega|^{2k} |y \cdot \omega|^n \, \mathrm{d}\omega^{j-1}$$
$$= \int_{S^{j-1}} |p(p(\omega|x \oplus y)|x)|^{2k} |p(p(\omega|x \oplus y)|y)|^n \, \mathrm{d}\omega^{j-1}. \tag{10}$$

In order to compute (10), we will use the following lemma.

Lemma 1. *Let $B_p \in \mathcal{L}_p^d$. Then, for any non-negative measurable function $g : \mathbb{R}^d \to \mathbb{R}$,*

$$\int_{S^{d-1}} g(p(x|B_p))\,\mathrm{d}x^{d-1} = \frac{\sigma_{d-p}}{2} \int_{S^{p-1}(B_p)} \int_0^1 g\left(t^{\frac{1}{2}} x_0\right) t^{\frac{p-2}{2}} (1-t)^{\frac{d-p-2}{2}}\,\mathrm{d}t\,\mathrm{d}x_0^{p-1},$$

where $S^{p-1}(B_p)$ is the unit sphere in B_p.

Proof. First, we use the co-area formula with

$$\begin{aligned} \psi \ &: S^{d-1} \backslash B_p^\perp \to S^{p-1}(B_p) \\ &x \to \pi(x|B_p) := p(x|B_p)/|p(x|B_p)|. \end{aligned}$$

The $(p-1)$-dimensional Jacobian of ψ is given by

$$J_{p-1}\psi(x, S^{d-1}) = |p(x|B_p)|^{-(p-1)}.$$

Hence, the co-area formula yields

$$\begin{aligned} \int_{S^{d-1}} g(p(x|B_p))\,\mathrm{d}x^{d-1} &= \int_{S^{d-1}} g\left(|p(x|B_p)|\pi(x|B_p)\right)\,\mathrm{d}x^{d-1} \\ &= \int_{S^{p-1}(B_p)} \int_{\psi^{-1}\{x_0\}} g(|p(x|B_p)|x_0)|p(x|B_p)|^{p-1}\,\mathrm{d}x^{d-p}\,\mathrm{d}x_0^{p-1}. \end{aligned} \tag{11}$$

Next, let $x_0 \in S^{p-1}(B_p)$ be fixed and apply the area formula with

$$\begin{aligned} \xi \ &: B_p^\perp \to \psi^{-1}\{x_0\} \\ &\omega \mapsto \frac{\omega + x_0}{|\omega + x_0|}. \end{aligned}$$

The $(d-p)$-dimensional Jacobian of ξ is

$$J_{d-p}\xi(\omega, B_p^\perp) = \left(\frac{1}{1+|\omega|^2}\right)^{\frac{d-p+1}{2}}.$$

Hence, since ξ maps $B_p^\perp$ bijectively onto $\psi^{-1}\{x_0\}$ and $|p(\xi(\omega)|B_p)| = \frac{1}{|\omega+x_0|} = \left(\frac{1}{1+|\omega|^2}\right)^{\frac{1}{2}}$, we have

$$\begin{aligned} &\int_{\psi^{-1}\{x_0\}} g(|p(x|B_p)|x_0)|p(x|B_p)|^{p-1}\,\mathrm{d}x^{d-p} \\ &\quad = \int_{\psi^{-1}\{x_0\}} g(|p(x|B_p)|x_0)|p(x|B_p)|^{p-1}\,\mathrm{d}x^{d-p} \\ &\quad = \int_{\psi^{-1}\{x_0\}} g\left(\left(\frac{1}{1+|\xi^{-1}(x)|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|\xi^{-1}(x)|^2}\right)^{\frac{p-1}{2}}\,\mathrm{d}x^{d-p} \\ &\quad = \int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{p-1}{2}}\left(\frac{1}{1+|x|^2}\right)^{\frac{d-p+1}{2}}\,\mathrm{d}x^{d-p} \\ &\quad = \int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{d}{2}}\,\mathrm{d}x^{d-p}. \end{aligned}$$

Using [7, Proposition 2.8], we get

$$\int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{d}{2}} \mathrm{d}x^{d-p}$$

$$= \sigma_{d-p}\int_0^\infty g\left(\left(\frac{1}{1+t^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+t^2}\right)^{\frac{d}{2}} t^{d-p-1}\,\mathrm{d}t. \tag{12}$$

Substitution with $s = \frac{1}{1+t^2}$ yields

$$\int_0^\infty g\left(\left(\frac{1}{1+t^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+t^2}\right)^{\frac{d}{2}} t^{d-p-1}\,\mathrm{d}t$$

$$= \frac{1}{2}\int_0^1 g\left(s^{\frac{1}{2}}x_0\right) s^{\frac{p-2}{2}}(1-s)^{\frac{d-p-2}{2}}\,\mathrm{d}s.$$

The last equation combined with (11) and (12) implies

$$\int_{S^{d-1}} g(p(x|B_p))\,\mathrm{d}x^{d-1} = \frac{\sigma_{d-p}}{2}\int_{S^{p-1}(B_p)}\int_0^1 g\left(t^{\frac{1}{2}}x_0\right)t^{\frac{p-2}{2}}(1-t)^{\frac{d-p-2}{2}}\,\mathrm{d}t\,\mathrm{d}x_0^{p-1}.$$

$\square$

Applying Lemma 1 with $B = \mathrm{span}\{x, y\}$, we get

$$\int_{S^{j-1}} |p(p(\omega|x\oplus y)|x)|^{2k}|p(p(\omega|x\oplus y)|y)|^n\,\mathrm{d}\omega^{j-1}$$

$$= \frac{\sigma_{j-2}}{2}\int_{S^1(B)}\int_0^1 t^k|p(\omega_0|x)|^{2k}t^{n/2}|p(\omega_0|y)|^n t^{\frac{2-2}{2}}(1-t)^{\frac{j-2-2}{2}}\,\mathrm{d}t\,\mathrm{d}\omega_0^1$$

$$= \frac{\sigma_{j-2}}{2}\int_{S^1(B)} |p(\omega_0|y)|^n|p(\omega_0|x)|^{2k}\,\mathrm{d}\omega_0^1\int_0^1 t^{\frac{n+2k}{2}}(1-t)^{\frac{j-4}{2}}\,\mathrm{d}t$$

$$= \frac{\sigma_{j-2}B\left(\frac{n}{2}+k+1, \frac{j-2}{2}\right)}{2}\int_{S^1(B)} |p(\omega_0|y)|^n|p(\omega_0|x)|^{2k}\,\mathrm{d}\omega_0^1. \tag{13}$$

Successive application of [7, Proposition 3.2] and [6, Corollary 4.2] yield

$$\int_{S^1(B)} |p(\omega_0|y)|^n|p(\omega_0|x)|^{2k}\,\mathrm{d}\omega_0^1 = 2\int_{\mathcal{L}_1^2(B)} |p(x|L_1)|^{2k}|p(y|L_1)|^n\,\mathrm{d}L_1^2$$

$$= 2\int_{-1}^1\int_{S^1(B)\cap y^\perp}(1-t^2)^{\frac{2-1-2}{2}}|p(x|ty+\sqrt{1-t^2}\omega)|^{2k}|p(y|ty+\sqrt{1-t^2}\omega)|^n\,\mathrm{d}\omega\,\mathrm{d}t$$

$$= 2\int_{-1}^1\int_{S^1(B)\cap y^\perp}(1-t^2)^{\frac{2-1-2}{2}}|t|^n|t(y\cdot x)+\sqrt{1-t^2}(x\cdot\omega)|^{2k}\,\mathrm{d}\omega\,\mathrm{d}t$$

$$= 2\int_{-1}^1(1-t^2)^{\frac{2-1-2}{2}}|t|^n\Bigg(|t(y\cdot x)+\sqrt{1-t^2}\sqrt{1-(y\cdot x)^2}|^{2k}$$

$$+ |t(y\cdot x)-\sqrt{1-t^2}\sqrt{1-(y\cdot x)^2}|^{2k}\Bigg)\,\mathrm{d}t.$$

Using the binomial formula, the last expression becomes

$$
\begin{aligned}
&2\sum_{l=0}^{2k}\binom{2k}{l}\int_{-1}^{1}\Big((1-t^2)^{\frac{2-1-2}{2}}|t|^n t^l (y\cdot x)^l (1-t^2)^{\frac{2k-l}{2}}\sqrt{1-(x\cdot y)^2}^{2k-l}\\
&\qquad\qquad + (-1)^l (1-t^2)^{\frac{2-1-2}{2}}|t|^n t^l (y\cdot x)^l (1-t^2)^{\frac{2k-l}{2}}\sqrt{1-(x\cdot y)^2}^{2k-l}\Big)\,\mathrm{d}t\\
&\qquad = 4\sum_{l=0}^{k}(x\cdot y)^{2l}(1-(x\cdot y)^2)^{k-l}\binom{2k}{2l}\int_0^1 (1-t^2)^{k-l-\frac{1}{2}}t^{n+2l}\,\mathrm{d}t\\
&\qquad = 2\sum_{l=0}^{k}(x\cdot y)^{2l}(1-(x\cdot y)^2)^{k-l}\binom{2k}{2l}B\left(\frac{n}{2}+l+\frac{1}{2},k-l+\frac{1}{2}\right).
\end{aligned}
$$

Applying the duplication formula for the Gamma function,

$$
\Gamma(2z)=\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)\pi^{-\frac{1}{2}}2^{2z-1},
$$

we obtain

$$
\begin{aligned}
&2\sin^{2k}\angle(x,y)B\left(\frac{n}{2}+\frac{1}{2},k+\frac{1}{2}\right)\sum_{l=0}^{k}\frac{(-k)_l\left(\frac{n}{2}+\frac{1}{2}\right)_l}{\left(\frac{1}{2}\right)_l}\frac{\left(-\tan^{-2}\angle(x,y)\right)^l}{l!}\\
&\qquad = 2\sin^{2k}\angle(x,y)B\left(\frac{n}{2}+\frac{1}{2},k+\frac{1}{2}\right)F\left(-k,\frac{n}{2}+\frac{1}{2};\frac{1}{2};-\tan^{-2}\angle(x,y)\right).
\end{aligned}
$$

According to [1, (15.3.4)] with $z=\cos^2\angle(x,y)$,

$$
\sin^{2k}\angle(x,y)F\left(-k,\frac{n}{2}+\frac{1}{2};\frac{1}{2};-\tan^{-2}\angle(x,y)\right)=F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right).
$$

By insertion in (13), we get

$$
\begin{aligned}
&\int_{S^{j-1}}|x\cdot\omega|^{2k}|y\cdot\omega|^n\,\mathrm{d}\omega^{j-1}\\
&\qquad=\sigma_{j-2}B\left(\frac{n}{2}+k+1,\frac{j-2}{2}\right)B\left(k+\frac{1}{2},\frac{n}{2}+\frac{1}{2}\right)F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right).
\end{aligned}
$$

Hence, (9) becomes

$$
\begin{aligned}
&\int_{\mathcal{L}^j_{j-1}}|p(x|L_{j-1})|^m\ \ |p\left(y|L_{j-1}^{\perp}\right)|^n\,\mathrm{d}L^j_{j-1}\\
&\qquad=\frac{\sigma_{j-2}}{2}\sum_{k=0}^{\infty}\binom{\frac{m}{2}}{k}(-1)^kB\left(\frac{n}{2}+k+1,\frac{j-2}{2}\right)\\
&\qquad\qquad\times B\left(\frac{n+1}{2},k+\frac{1}{2}\right)F\left(-k,-\frac{n}{2};\frac{1}{2};\cos^2\angle(x,y)\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{\sigma_{j-2}}{2}\binom{\frac{m}{2}}{k}(-1)^kB\left(\frac{n}{2}+k+1,\frac{j-2}{2}\right)B\left(\frac{n+1}{2},k+\frac{1}{2}\right)\\
&\qquad=\frac{\sigma_{j-1}}{2}B\left(\frac{j-1}{2},\frac{n+1}{2}\right)\frac{\left(-\frac{m}{2}\right)_k}{k!}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{n+j}{2}\right)_k},
\end{aligned}
$$

we now have

$$\int_{\mathcal{L}_{j-1}^{j}} |p(x|L_{j-1})|^{m} \ \ |p\left(y|L_{j-1}^{\perp}\right)|^{n} \, \mathrm{d}L_{j-1}^{j}$$
$$= \frac{\sigma_{j-1}}{2} B\left(\frac{j-1}{2}, \frac{n+1}{2}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{m}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{n+j}{2}\right)_k} \frac{F\left(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2 \angle(x,y)\right)}{k!}.$$

Using the power series expansion of the hypergeometric function, then expanding the polynomial $(1-\sin^2\angle(x,y))^k$ and applying the identities

$$\frac{\binom{k+l}{l}}{(k+l)!} = \frac{1}{l!}\frac{1}{k!} \qquad \text{and} \qquad (a)_{k+l} = (a)_l (a+l)_k,$$

it is straightforward to prove that the last expression equals

$$\frac{\sigma_{j-1}}{2} B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{j-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{j-1}{2}; \sin^2 \angle(x,y)\right).$$

The proof is complete.

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Paper C

A rotational Crofton formula for flagged intrinsic volumes of sets of positive reach.

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A rotational Crofton formula for flagged intrinsic volumes of sets of positive reach

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Abstract

A rotational Crofton formula is derived relating the *flagged* intrinsic volumes of a compact set of positive reach with the flagged intrinsic volumes measured on sections passing through a fixed point. In particular cases, the flagged intrinsic volumes defined in the present paper are identical to the classical intrinsic volumes. The tight connection between our main result and other recent rotational integral formulae involving intrinsic volumes is pointed out.

Keywords: Crofton formula; Geometric measure theory; Grassmann manifold; Integral geometry; Intrinsic volume; Rotational integral; Set with positive reach; Unit normal bundle

1 Introduction

In *classical* stereology, the well-known Crofton formula relates the intrinsic volumes of a compact subset X of $\mathbb{R}^d$ with the intrinsic volumes of its affine sections

$$c_{d,r,k} V_{d-r+k}(X) = \int_{\mathcal{F}_r^d} V_k(X \cap F_r)\, \mathrm{d}F_r^d, \tag{1}$$

$r = 0, \dots, d$, $k = 0 \dots, r$. Here, $\mathcal{F}_r^d$ is the set of r-dimensional affine subspaces in $\mathbb{R}^d$ and $\mathrm{d}F_r^d$ is the element of its motion invariant measure. The kth intrinsic volume of X is denoted by $V_k(X)$, $k = 0, \dots, d$. Finally, $c_{d,r,k}$ is a known constant. In *local* stereology, the focus of interest is instead on integral geometric relations of the type

$$\beta(X) = \int_{\mathcal{L}_r^d} \alpha(X \cap L_r)\, \mathrm{d}L_r^d, \tag{2}$$

where α and β are functionals, $\mathcal{L}_r^d$ denotes the set of r-dimensional subspaces in $\mathbb{R}^d$ and $\mathrm{d}L_r^d$ is the element of its rotation invariant measure. In the special case where α is an intrinsic volume of a compact set X of positive reach, i.e. for relations of the type

$$\beta(X) = \int_{\mathcal{L}_r^d} V_k(X \cap L_r)\, \mathrm{d}L_r^d, \tag{3}$$

Jensen and Rataj proved in [8] that β can be expressed as a certain integral over the unit normal bundle of X. In the same paper, the problem was raised of finding functionals α

satisfying the integral equation (2) in the particular case where β is an intrinsic volume of X,

$$V_{d-k}(X) = \int_{\mathcal{L}_r^d} \alpha(X \cap L_r)\,\mathrm{d}L_r^d. \tag{4}$$

Recently, a solution to this problem was given independently in [3] and [6]. It was shown, for all $0 \le k < r \le d$ and for any compact set Y of positive reach contained in L_r, that the functional α given by

$$\alpha(Y) = \frac{1}{c_{d,r-1,r-k-1}} \int_{\mathcal{F}_{r-1}^r} V_{r-k-1}(Y \cap F_{r-1})\, d(F_{r-1}, O)^{d-r}\,\mathrm{d}F_{r-1}^d, \tag{5}$$

where d is the distance function, is a solution to (4). In the present paper, we shall demonstrate that solutions to (4) can be expressed as an integral over the unit normal bundle of the section $X \cap L_r$, for all $1 \le k < r$, or, as an integral over $X \cap L_r$, when $k = 0$. It appears that the functionals α and β in (3) and (4) share the same integral representation, $\alpha_{r,k}^d$, parametrized by three integers. This family of functionals *generalizes* the classical intrinsic volumes and in fact, $\alpha_{d,k}^d(X) = V_{d-k}(X)$, for any d-dimensional set X of positive reach. As a main result of the present paper, a *rotational Crofton formula* shall be derived,

$$\alpha_{j,k}^d(X) = c_{d-r,j-r} \int_{\mathcal{L}_r^d} \alpha_{j,k}^r(X \cap L_r)\,\mathrm{d}L_r^d,$$

which turns formula (3) and (4) into special cases. Here, $c_{d-r,j-r}$ is a known constant, see Section 2 below.

The paper is organized as follows. In Section 2, we present the notation and some background knowledge. Section 3 shows that the solution (5) can be expressed as an integral with respect to a q-dimensional affine subspace in L_r. In Section 4, the solution is given a more explicit expression as an integral over a unit normal bundle and the rotational Crofton formula for *flagged* intrinsic volumes is presented. Proofs are deferred to Section 5.

2 Preliminaries

In this section, we shall fix the conventions used in this paper. For any compact set $X \subseteq \mathbb{R}^d$ of positive reach, we define its kth intrinsic volume by

$$V_k(X) = \frac{1}{\sigma_{d-k}} \int_{\operatorname{nor} X} \sum_{\substack{|J|=d-k-1 \\ J\subset\{1,\dots,d-1\}}} \frac{\prod_{j\in J} \kappa_j(x,n)}{\prod_{j=1}^{d-1} \sqrt{1+\kappa_j^2(x,n)}}\,\mathcal{H}^{d-1}(\mathrm{d}(x,n)),$$

where $\kappa_j(x,n)$ denotes the jth (generalized) principal curvature at $(x,n) \in \operatorname{nor} X$ and $\sigma_{d-k} = 2\pi^{\frac{d-k}{2}}/\Gamma\left(\frac{d-k}{2}\right)$ is the surface area of the unit sphere in $\mathbb{R}^{d-k}$. An r-dimensional affine subspace $F_r \in \mathcal{F}_r^d$ can be written uniquely as $F_r = L_r + x$, where $L_r \in \mathcal{L}_r^d$ and $x \in L_r^\perp$. The corresponding measure decomposition is

$$\mathrm{d}F_r^d = \mathrm{d}x^{d-r}\,\mathrm{d}L_r^d. \tag{6}$$

Here, $\mathrm{d}x^{d-r}$ is a shortcut notation for the Hausdorff (Lebesgue) measure $\mathcal{H}^{d-r}(\mathrm{d}x)$. Moreover, since the superscript in $\mathrm{d}F_r^d$ and $\mathrm{d}L_r^d$ is often superfluous, we shall write $\mathrm{d}L_r$ and $\mathrm{d}F_r$, when the context allows it. The total mass of $\mathcal{L}_r^d$ is chosen to be

$$\int_{\mathcal{L}_r^d} \mathrm{d}L_r = c_{d,r},$$

where

$$c_{d,r} = \frac{\sigma_d \sigma_{d-1} \cdots \sigma_{d-r+1}}{\sigma_r \sigma_{r-1} \cdots \sigma_1}.$$

With this convention, the constant in the classical Crofton formula is given by

$$c_{d,r,k} = c_{d,r} \frac{\Gamma\left(\frac{r+1}{2}\right) \Gamma\left(\frac{d+k-r+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{d+1}{2}\right)}.$$

The Gauss *hypergeometric series* or *hypergeometric function* is defined for $a, b, c \in \mathbb{R}$ and $z \in [-1, 1]$ as

$$F(a, b; c; z) = F(b, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(x)_k$ is the *rising sequential product* or *Pochhammer symbol* defined for a non-negative integer k and $x \in \mathbb{R}$ by

$$(x)_k = \begin{cases} \frac{\Gamma(x+k)}{\Gamma(x)} & \text{if } x > 0 \\ (-1)^k \frac{\Gamma(-x+1)}{\Gamma(-x-k+1)} & \text{if } x \leq 0, \end{cases}$$

cf. [1, Chapter 15]. Note that $(x)_k = 0$ whenever $x \in \{0, -1, -2, \ldots\}$ and $k > -x$. The *Gamma function* is defined on $\mathbb{R}_+$ as $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$ and it has an analytic continuation on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Standard formulae for the Gamma function can be found in [1, Chapter 6]. In particular, the duplication formula,

$$\Gamma(2z) = \Gamma(z) \, \Gamma\left(z + \frac{1}{2}\right) \pi^{-\frac{1}{2}} \, 2^{2z-1}, \tag{7}$$

will be useful in the present paper.

3 A generalized solution

From the 1936 paper [9] by Petkantschin, a particular measure decomposition can be employed to slightly generalize the solution α, derived in [3] and [6], to the integral equation (4).

Proposition 1. *Let $Y \subseteq L_r$ be a compact subset of positive reach in some fixed linear subspace $L_r \in \mathcal{L}_r^d$. The functional*

$$\alpha_{d,k}^r(Y) = \frac{1}{c_{d,r,q,k}} \int_{\mathcal{F}_q^r} V_{q-k}(Y \cap F_q) \, d(F_q, O)^{d-r} \, \mathrm{d}F_q$$

solves the integral equation

$$V_{d-k}(X) = \int_{\mathcal{L}_r^d} \alpha_{d,k}^r(X \cap L_r) \, \mathrm{d}L_r,$$

for all $0 \leq k \leq q < r \leq d$. Here, $c_{d,r,q,k} = c_{d-q-1,r-q-1} c_{d,q,q-k}$.

Proof. We shall use the following integral decomposition formula

$$\int_{\mathcal{F}_q^d} f(F_q)\,\mathrm{d}F_q = \frac{1}{c_{d-q-1,r-q-1}} \int_{\mathcal{L}_r^d} \int_{\mathcal{F}_q^r} f(F_q)\, d(F_q,O)^{d-r}\,\mathrm{d}F_q\,\mathrm{d}L_r, \tag{8}$$

which holds for all $0 \le q < r \le d$ and any integrable function f, cf. [9] and [11, p. 285]. Let us assume that $X \subseteq \mathbb{R}^d$ is a compact set of positive reach. The Crofton formula for sets of positive reach, cf. [10], and an application of (8) yield, for all $0 \le k \le q < r \le d$,

$$\begin{aligned}
c_{d,q,q-k} V_{d-k}(X) &= \int_{\mathcal{F}_q^d} V_{q-k}(X \cap F_q)\,\mathrm{d}F_q \\
&= \frac{1}{c_{d-q-1,r-q-1}} \int_{\mathcal{L}_r^d} \int_{\mathcal{F}_q^r} V_{q-k}(X \cap F_q)\, d(F_q,O)^{d-r}\,\mathrm{d}F_q\,\mathrm{d}L_r \\
&= \frac{1}{c_{d-q-1,r-q-1}} \int_{\mathcal{L}_r^d} \int_{\mathcal{F}_q^r} V_{q-k}((X \cap L_r) \cap F_q)\, d(F_q,O)^{d-r}\,\mathrm{d}F_q\,\mathrm{d}L_r.
\end{aligned}$$

The proof is complete. □

Remark 1. The constant appearing on the right-hand side of formula (8) is not given explicitly in [11], but only referred to as a constant depending on d, q and r. In order to compute this constant, we set $f(F_q) = V_0(B^d \cap F_q)$, where $V_0 = \chi$ is the Euler-Poincaré characteristic and B^d is the unit ball in $\mathbb{R}^d$. By computing both side of (8) separately, we obtain

$$\int_{\mathcal{F}_q^d} f(F_q)\,\mathrm{d}F_q = \int_{\mathcal{L}_q^d} \int_{L_q^{\perp} \cap B^d} \mathrm{d}z^{d-q}\,\mathrm{d}L_q = \omega_{d-q}\, c_{d,q}, \tag{9}$$

where $\omega_{d-q} = (d-q)\,\sigma_{d-q}$ is the volume of the unit ball in $\mathbb{R}^{d-q}$, and

$$\int_{\mathcal{L}_r^d} \int_{\mathcal{F}_q^r} f(F_q)\, d(F_q,O)^{d-r}\,\mathrm{d}F_q\,\mathrm{d}L_r = \frac{1}{d-q}\,\sigma_{r-q}\, c_{d,r}\, c_{r,q}. \tag{10}$$

Division of (9) by (10) yields the constant appearing in formula (8). □

Proposition 2. *For all $0 \le k \le q < r \le d$, the functional $\alpha_{d,k}^r$, defined in Proposition 1, does not depend on q.*

The proof of Proposition 2 is deferred to the last section. In spite of Proposition 2, the uniqueness of the solution, $\alpha_{d,k}^r$, to the integral equation (4) remains open.

Remark 2. Two representations of $\alpha_{d,k}^r$ are particularly interesting for our purposes. For $q = k$, we have

$$\alpha_{d,k}^r(Y) = \frac{1}{c_{d,r,k,k}} \int_{\mathcal{F}_k^r} \chi(Y \cap F_k)\, d(F_k,O)^{d-r}\,\mathrm{d}F_k \tag{11}$$

and, for $q = r-1$,

$$\alpha_{d,k}^r(Y) = \frac{1}{c_{d,r-1,r-1-k}} \int_{\mathcal{F}_{r-1}^r} V_{r-k-1}(Y \cap F_{r-1})\, d(F_{r-1},O)^{d-r}\,\mathrm{d}F_{r-1}. \tag{12}$$

Combining (11) and the identity $\mathrm{d}F_0^r = \mathrm{d}x^r$, we obtain, for $k = 0$,

$$\alpha_{d,0}^r(Y) = \frac{1}{c_{d-1,r-1}} \int_{\mathbb{R}^r} \chi(Y \cap \{x\})|x|^{d-r}\,\mathrm{d}x^r = \frac{1}{c_{d-1,r-1}} \int_Y |x|^{d-r}\,\mathrm{d}x^r. \tag{13}$$

Furthermore, using (12), we have shown in [3] that for $k = 1$,

$$\alpha^r_{d,1}(Y) = \frac{1}{2c_{d-1,r-1}} \int_{\partial Y} |x|^{d-r} F\left(-\frac{1}{2}, -\frac{d-r}{2}; \frac{r-1}{2}; \sin^2(n(x), x)\right) \mathrm{d}x^{r-1}, \tag{14}$$

whenever Y is a C^2-manifold with boundary. Here, $n(x)$ is the vector normal to the surface at x. $\square$

Motivated by the integral representation (14), we shall now prove that the functional $\alpha^r_{d,k}(Y)$ can be expressed as an integral over the normal bundle of Y for all $k = 1, \ldots, r-1$ and $r = 2, \ldots, d$.

4 Integral representation of the generalized solution

Let $X \subseteq \mathbb{R}^d$ be a compact set of positive reach and assume for now that the section $Y = X \cap L_r$, for some fixed $L_r \in \mathcal{L}^d_r$, also has positive reach. For the necessary background in multilinear algebra, current theory and sets of positive reach, the reader is referred to [4] and [5].

Let N_Y be the $(r-1)$-dimensional current on $\mathbb{R}^r \times \mathbb{R}^r$ given by

$$N_Y = (\mathcal{H}^{r-1} \llcorner \operatorname{nor} Y) \wedge a_Y, \tag{15}$$

i.e.

$$N_Y(\phi) = \int_{\operatorname{nor} Y} \langle a_Y(x,n), \phi(x,n)\rangle \, \mathcal{H}^{r-1}(\mathrm{d}(x,n))$$

for all $(r-1)$-forms ϕ on $\mathbb{R}^r \times \mathbb{R}^r$. Here, a_Y is a unit $(r-1)$-dimensional vectorfield orienting $\operatorname{nor} Y$ given explicitly by

$$a_Y(x,n) = \bigwedge_{i=1}^{r-1} \left(\frac{1}{\sqrt{1+\kappa_i(x,n)^2}} \, a_i(x,n), \frac{\kappa_i(x,n)}{\sqrt{1+\kappa_i(x,n)^2}} \, a_i(x,n) \right), \tag{16}$$

where $\kappa_i(x,n)$ is the ith principal curvature and $a_i(x,n)$ the corresponding principal direction at $(x,n) \in \operatorname{nor} Y$ for $i = 1, \ldots, r-1$, cf. [8, (27)] and [12]. We apply the usual convention $\frac{\infty}{\sqrt{1+\infty^2}} = 1$ and $\frac{1}{\sqrt{1+\infty^2}} = 0$ at points where some of the principal curvatures are infinite. Assume that the principal directions are ordered in such a way that

$$a_1(x,n), \ldots, a_{r-1}(x,n), n$$

constitute an orthonormal basis of $\mathbb{R}^r$. The Lipschitz-Killing curvature form ϕ_k on $\mathbb{R}^r \times \mathbb{R}^r$ of order $k = 0, \ldots, r-1$ is defined by

$$\langle (u^1_0, u^1_1) \wedge \cdots \wedge (u^{r-1}_0, u^{r-1}_1), \phi_k(x,n)\rangle = \frac{1}{\sigma_{r-k}} \sum_{\substack{\epsilon_i = 0,1 \\ \epsilon_1 + \cdots + \epsilon_{r-1} = r-1-k}} \langle u^1_{\epsilon_1} \wedge \cdots \wedge u^{r-1}_{\epsilon_{r-1}} \wedge n, \Omega_r\rangle, \tag{17}$$

where Ω_r is the volume r-form in R^r. Note that the right-hand side in (17) is strictly positive whenever the number of non-zero principal curvatures at (x,n) is at least $r-1-k$ or, alternatively, when the number of infinite principal curvatures is at most $r-1-k$. For any compact set Y with positive reach, the kth intrinsic volume of Y can be expressed as

$$V_k(Y) = N_Y(\phi_k),$$

for $k = 0, \dots, r-1$, cf. [12].

Two sets Y and F with positive reach *touch*, when there exists a pair $(y, n) \in \operatorname{nor} Y$ such that $(y, -n) \in \operatorname{nor} F$, cf. [13]. In the particular case where $F = L + z$ is an affine subspace, Y and F do not touch, whenever the following condition is satisfied

$$(y, n) \in \operatorname{nor} Y \quad \wedge \quad y \in F \quad \Longrightarrow \quad n \notin L^{\perp}. \tag{18}$$

Remark 3. The subset of j-dimensional affine subspaces in $\mathbb{R}^r$ that do touch Y has finite $(r-1+j(r-1-j))$-dimensional measure, cf. [10, (1)]. Hence, in the special case where $j = r-1$, the set of $(r-1)$-dimensional affine subspaces touching Y has finite $(r-1)$-dimensional measure, i.e. Y is not touched by $\mathcal{H}^r$ almost all $F \in \mathcal{F}^r_{r-1}$. Whenever Y and F do not touch, their intersection $Y \cap F$ has *local* positive reach, cf. [13] or [4, Theorem 4.10]. By the compactness of $\operatorname{nor} Y$ and the continuity of the reach function, we conclude that

$$\text{if } Y \text{ has positive reach, then } Y \cap F \text{ has positive reach for almost all } F \in \mathcal{F}^r_{r-1}.$$

Furthermore, for a compact subset $X \subseteq \mathbb{R}^d$ of positive reach, it is shown in [8] that for $\mathcal{H}^d$-a.a. choices of origo, the sets X and L do not touch for almost all $L \in \mathcal{L}^d_r$. In other words, whenever X has positive reach, we may choose origo such that

$$X \cap L \text{ has positive reach for almost all } L \in \mathcal{L}^d_r. \tag{19}$$

For those reasons, the assumption, which was made at the beginning of this section on the positive reach of $Y = X \cap L_r$, is mild.

Definition 1 (Flagged intrinsic volumes)**.** *Let $Y \in \mathbb{R}^r$ be a compact set of positive reach. Define for all $s = 1, \dots, r$, $r \geq 1$ and $j \geq s$,*

$$\alpha^r_{j,0}(Y) := \frac{1}{c_{j-1,r-1}} \int_Y |x|^{j-r} \, \mathrm{d}y^r$$

and

$$\alpha^r_{j,s}(Y) := K^r_{j,s} \int_{\operatorname{nor} Y} |x|^{j-r} \sum_{\substack{|I|=s-1 \\ I \subset \{1,\dots,r-1\}}} \frac{\prod_{i \in I} \kappa_i(x,n)}{\prod_{i=1}^{r-1} \sqrt{1+\kappa_i(x,n)^2}} Q^r_{j,s}(x, n, A_I) \, \mathcal{H}^{r-1}(\mathrm{d}(x,n)),$$

where

$$\begin{aligned} Q^r_{j,s}(x, n, A_I) &:= F\left(-\frac{j-r}{2}, \frac{s}{2}; \frac{r+1}{2}; \sin^2(x,n)\right) \\ &\quad + \frac{(j-r)(r-s+1)}{r+1} \frac{\cos^2(x, A_I)}{r-s} F\left(-\frac{j-r}{2}+1, \frac{s}{2}; \frac{r+3}{2}; \sin^2(x,n)\right) \end{aligned}$$

and

$$K^r_{j,s} := \frac{1}{\sigma_s \, c_{j-1,r-1}} \frac{\Gamma(r-s+1)\,\Gamma(j)}{\Gamma(r)\,\Gamma(j-s+1)}.$$

Here, $A_I = \operatorname{span}\{a_i : i \notin I\}$ and, for the special case $r = s$, we set $\frac{\cos^2(x, A_{\{1,\dots,r-1\}})}{0} := 1$. Note that $c_{j-1,r-1} := \frac{1}{c_{r-1,j-1}}$ for $j < r$. □

Remark 4. In the special case $j = r$, $K^r_{r,s} = \frac{1}{\sigma_s}$ and $Q^r_{r,s} = 1$. Consequently,

$$\alpha^r_{r,s}(Y) = \frac{1}{\sigma_s} \int_{\operatorname{nor} Y} \sum_{\substack{|J|=s-1 \\ J \subset \{1,\dots,r-1\}}} \frac{\prod_{j \in J} \kappa_j(x,n)}{\prod_{j=1}^{r-1} \sqrt{1+\kappa_j^2(x,n)}} \mathcal{H}^{r-1}(\mathrm{d}(x,n)) = V_{r-s}(Y)$$

and

$$\alpha^r_{r,0}(Y) = \int_Y \mathrm{d}\mathcal{H}^r = V_r(Y),$$

for any compact set $Y \subseteq \mathbb{R}^r$ of positive reach. $\square$

The functionals defined above are identical to those given in Proposition 1. This result is formulated in the proposition below. The proof is deferred to the next section.

Proposition 3. *The flagged intrinsic volumes presented in Definition 1 are identical to the functional $\alpha^r_{d,s}$ given in Proposition 1 for all $s = 0, \dots, r-1$, $r = 1, \dots, d$. As a consequence, when origo is chosen such that condition (19) is satisfied, the functional $\alpha^r_{d,s}$ from Definition 1 satisfies the integral equation,*

$$V_{d-s}(X) = \int_{\mathcal{L}^d_r} \alpha^r_{d,s}(X \cap L_r)\, \mathrm{d}L_r,$$

for all $0 \le s < r \le d$.

Remark 5. Since $\cos^2(x, A_\emptyset) = \sin^2(x,n)$, the hypergeometric identity (34) implies

$$Q^r_{d,1}(x,n,A_\emptyset) = F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r-1}{2}; \sin^2(x,n)\right)$$

and with

$$K^r_{d,1} = \frac{1}{\sigma_1\, c_{d-1,r-1}} \frac{\Gamma(r)\Gamma(d)}{\Gamma(r)\Gamma(d)} = \frac{1}{2c_{d-1,r-1}},$$

we conclude that

$$\alpha^r_{d,1}(Y) = \frac{1}{2c_{d-1,r-1}} \int_{\operatorname{nor} Y} |x|^{d-r} F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r-1}{2}; \sin^2(x,n)\right) \mathcal{H}^{r-1}(\mathrm{d}(x,n)),$$

i.e. a generalization of (14) to sets of positive reach. $\square$

Remark 6. Applying the hypergeometric identity (35) with $z = \sin^2(x,n) > 0$, we obtain

$$\begin{aligned}\frac{(j-d)(d-j+k+1)}{d+1} & F\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+3}{2}; \sin^2(x,n)\right) \\ &= (j-d-(d-1)\cot^2(x,n))F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d+1}{2}; \sin^2(x,n)\right) \\ &\qquad + (d-1)\cot^2(x,n)F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d-1}{2}; \sin^2(x,n)\right),\end{aligned}$$

or, in other terms,

$$(d-j+k)\, Q^d_{j,j-k}(x,n) = f_1(\angle(x,n)) + f_2(\angle(x,n)) \cos^2 \alpha_I(x,n),$$

where the right-hand side is written in the notation of [2, Theorem 3.1]. Since $\frac{1}{d-j+k} K^d_{j,j-k}$ is equal to the constant $C_{d,j,k}$ defined in [2], (recall that $c_{j-1,d-1} := \frac{1}{c_{d-1,j-1}}$), we conclude that

$$|x|^{j-d}\, K^d_{j,j-k}\, Q^d_{j,j-k}(x,n) = \omega_{I,j,k}(x,n),$$

where the functional $\omega_{I,j,k}$ given in [2, Theorem 3.1] satisfies the integral equation

$$\int_{\mathcal{L}_j^d} V_k(X \cap L_j)\,\mathrm{d}L_j = \int_{\operatorname{nor} X} \sum_{\substack{|I|=j-1-k \\ I\subset\{1,\dots,d-1\}}} \omega_{I,j,k} \frac{\prod_{i\in I}\kappa_i}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}}\,\mathrm{d}\mathcal{H}^{r-1} = \alpha^d_{j,j-k}(X), \quad (20)$$

for all $0 \le k < j \le d$, whenever $X \subseteq \mathbb{R}^d$ is a set of positive reach and the origin is chosen such that condition (19) is satisfied, cf. Remark 3. □

Having made these two important remarks, the following Theorem can be proven easily.

Theorem 1 (Rotational Crofton Formula)**.** *Let $X \subset \mathbb{R}^d$ be a compact subset of positive reach and assume origo is chosen such that condition (19) is satisfied. Then,*

$$\alpha^d_{j,k}(X) = c_{d-r,j-r} \int_{\mathcal{L}_r^d} \alpha^r_{j,k}(X \cap L_r)\,\mathrm{d}L_r,$$

for all $0 \le k < r \le j \le d$.

Remark 7. With Remark 4 in mind, we notice that [8, Theorem] and Proposition 1 are special cases of Theorem 1 for $r = j$ and $d = j$, respectively. The case $r = k$ is not covered by Theorem 1. Nevertheless, for $r = j = k$, formula (20) implies

$$\alpha^d_{r,r}(X) = \int_{\mathcal{L}_r^d} V_0(X \cap L_r)\,\mathrm{d}L_r = c_{d-r,r-r} \int_{\mathcal{L}_r^d} \alpha^r_{r,r}(X \cap L_r)\,\mathrm{d}L_r.$$

Proof of Theorem 1. On the one hand, we have, according to Remark 6 and under the mild assumption on the choice of origo,

$$\alpha^d_{j,k}(X) = \int_{\mathcal{L}_j^d} V_{j-k}(X \cap L_j)\,\mathrm{d}L_j$$

for all $0 < k \le j \le d$. The case $k = 0$ follows from the Blaschke-Petkantschin formula, cf. [7, Proposition 4.5],

$$\int_{\mathcal{L}_j^d} V_j(X \cap L_j)\,\mathrm{d}L_j = \int_{\mathcal{L}_j^d}\int_{X\cap L_j} \mathrm{d}x^j\,\mathrm{d}L_j = c_{d-1,j-1}\int_X |x|^{j-d}\,\mathrm{d}x^d = \alpha^d_{j,0}(X).$$

On the other hand, Proposition 1 and [7, (3.17)] imply

$$\begin{aligned}\int_{\mathcal{L}_j^d} V_{j-k}(X \cap L_j)\,\mathrm{d}L_j &= \int_{\mathcal{L}_j^d}\int_{\mathcal{L}_r^j} \alpha^r_{j,k}(X \cap L_j \cap L_r)\,\mathrm{d}L_r\,\mathrm{d}L_j \\ &= c_{d-r,j-r}\int_{\mathcal{L}_r} \alpha^r_{j,k}(X \cap L_r)\,\mathrm{d}L_r,\end{aligned}$$

for all $0 \le k < r \le j \le d$. □

5 Proofs

Proof of Proposition 2. If $k = r-1$, there is only one possible choice for q satisfying $r > q \geq k$ and the proof is complete. Assume that $r > q > k \geq 0$. Crofton's formula for sets of positive reach and the measure decomposition (6) imply

$$\begin{aligned}
& c_{q,q-1,q-k-1} \int_{\mathcal{F}_q^r} V_{q-k}(Y \cap F_q)\, \mathrm{d}(F_q, O)^{d-r}\, \mathrm{d}F_q \\
&= c_{q,q-1,q-k-1} \int_{\mathcal{L}_q^r} \int_{L_q^\perp} V_{q-k}(Y \cap (L_q + x))|x|^{d-r}\, \mathrm{d}x^{r-q}\, \mathrm{d}L_q \\
&= \int_{\mathcal{L}_q^r} \int_{L_q^\perp} |x|^{d-r} \int_{\mathcal{F}_{q-1}^q} V_{q-k-1}((Y-x) \cap L_q \cap F_{q-1})\, \mathrm{d}F_{q-1}\, \mathrm{d}x^{r-q}\, \mathrm{d}L_q \\
&= \int_{\mathcal{L}_q^r} \int_{L_q^\perp} |x|^{d-r} \int_{\mathcal{L}_{q-1}^q} \int_{L_{q-1}^\perp \cap L_q} V_{q-k-1}(Y \cap (L_{q-1} + x + y))\, \mathrm{d}y^1\, \mathrm{d}L_{q-1}\, \mathrm{d}x^{r-q}\, \mathrm{d}L_q.
\end{aligned}$$

An application of the measure transformation $\mathrm{d}L_{q-1}^q\, \mathrm{d}L_q^r = \mathrm{d}L_{q(q-1)}^r\, \mathrm{d}L_{q-1}^r$, cf. [7, (3.15)], and of the orthogonal decomposition $L_{q-1}^\perp = L_q^\perp \oplus (L_{q-1}^\perp \cap L_q)$, turns the last expression into

$$\begin{aligned}
& \int_{\mathcal{L}_{q-1}^r} \int_{\mathcal{L}_{q(q-1)}^r} \int_{L_q^\perp} |x|^{d-r} \int_{L_{q-1}^\perp \cap L_q} V_{q-k-1}(Y \cap (L_{q-1} + x + y))\, \mathrm{d}y^1\, \mathrm{d}x^{r-q}\, \mathrm{d}L_{q(q-1)}\, \mathrm{d}L_{q-1} \\
&= \int_{\mathcal{L}_{q-1}^r} \int_{L_{q-1}^\perp} V_{q-k-1}(Y \cap (L_{q-1} + z)) \int_{\mathcal{L}_{q(q-1)}^r} |p(z|L_q^\perp)|^{d-r}\, \mathrm{d}L_{q(q-1)}\, \mathrm{d}z^{d-q+1}\, \mathrm{d}L_{q-1}.
\end{aligned}$$

It can be shown, as we shall see in Remark 11, that

$$\int_{\mathcal{L}_{q(q-1)}^r} |p(z|L_q^\perp)|^{d-r} \mathrm{d}L_{q(q-1)} = |z|^{d-r} \frac{\sigma_{r-q}}{2} B\left(\frac{1}{2}, \frac{d-q}{2}\right), \tag{21}$$

whenever $z \in L_{q-1}^\perp$. Thus,

$$\begin{aligned}
& \int_{\mathcal{L}_{q-1}^r} \int_{L_{q-1}^\perp} V_{q-k-1}(Y \cap (L_{q-1} + z)) \int_{\mathcal{L}_{q(q-1)}^r} |p(z|L_q^\perp)|^{d-r}\, \mathrm{d}L_{q(q-1)}\, \mathrm{d}z^{d-q+1}\, \mathrm{d}L_{q-1} \\
&\qquad = \frac{\sigma_{r-q}}{2} B\left(\frac{1}{2}, \frac{d-q}{2}\right) \int_{\mathcal{F}_{q-1}^r} V_{q-k-1}(Y \cap F_{q-1})\, d(F_{q-1}, O)^{d-r}\, \mathrm{d}F_{q-1}.
\end{aligned}$$

Since $c_{q,q-1,q-k-1} = \frac{\sigma_{q+1}}{2B(\frac{q-k}{2}, \frac{1}{2})}$, we conclude that

$$\begin{aligned}
& \int_{\mathcal{F}_q^r} V_{q-k}(Y \cap F_q)\, d(F_q, O)^{d-r}\, \mathrm{d}F_q \\
&\qquad = \frac{\sigma_{r-q} B\left(\frac{1}{2}, \frac{d-q}{2}\right) B\left(\frac{q-k}{2}, \frac{1}{2}\right)}{\sigma_{q+1}} \int_{\mathcal{F}_{q-1}^r} V_{q-k-1}(Y \cap F_{q-1})\, d(F_{q-1}, O)^{d-r}\, \mathrm{d}F_{q-1},
\end{aligned}$$

for all $0 \leq k < q < r$. A routine calculation shows that

$$\frac{\sigma_{r-q}\, B\left(\frac{1}{2}, \frac{d-q}{2}\right)\, B\left(\frac{q-k}{2}, \frac{1}{2}\right)}{\sigma_{q+1}\, c_{d-q-1,r-q-1}\, c_{d,q,q-k}} = \frac{1}{c_{d-(q-1)-1,r-(q-1)-1}\, c_{d,q-1,q-1-k}},$$

therefore,

$$\alpha_{r,k}(Y) = \frac{1}{c_{d-(q-1)-1,j-(q-1)-1}\, c_{d,q-1,q-1-k}} \int_{\mathcal{F}^r_{q-1}} V_{q-1-k}(Y \cap F_{q-1})\, d(F_{q-1}, O)^{d-r}\, \mathrm{d}F_{q-1}.$$

Hence, in the definition of $\alpha^r_{d,k}$, the variable q can be replaced by $q-1$. A recursive argument implies that $\alpha^r_{d,k}$ is independent of q. □

Proof of Proposition 3. Without loss of generality, we will use the representation (12) of $\alpha^r_{d,s}$ (i.e. $q = r-1$). Let $L_r \in \mathcal{L}^d_r$ be a fixed r-dimensional subset of $\mathbb{R}^d$ and let $Y \subset L_r$ be a compact set of positive reach. The case $s = 0$ holds by definition, cf. formula (13). Assume that $0 < s < r-1$. According to (12),

$$c_{d,r,r-1,s}\, \alpha^r_{d,s}(Y) = \int_{\mathcal{F}^r_{r-1}} V_{r-s-1}(Y \cap F_{r-1})\, d(F_{r-1}, O)^{d-r}\, \mathrm{d}F_{r-1} = \int_{\mathcal{L}^r_{r-1}} \mathcal{I}(L_{r-1})\, \mathrm{d}L_{r-1},$$

where

$$\mathcal{I}(L_{r-1}) = \int_{L^\perp_{r-1}} V_{r-s-1}(Y \cap (L_{r-1} + x))\, |x|^{d-r} \mathrm{d}x^1.$$

Note that $c_{d,r,r-1,s} = c_{d-r,0}\, c_{d,r-1,r-s-1} = c_{d,r-1,r-s-1}$. Let $L_{r-1} \in \mathcal{L}^r_{r-1}$ be a fixed $(r-1)$-dimensional subspace of L_r and let ω_1 be a unit vector st. $\mathrm{span}\{\omega_1\} = L^\perp_{r-1}$. Let ω_{r-1} be a simple unit $(r-1)$-vector orienting L_{r-1} such that $\langle \omega_1 \wedge \omega_{r-1}, \Omega_r \rangle = 1$. We define the two volume forms, Ω_1 and Ω_{r-1}, to be the dual vectors of ω_1 and ω_{r-1}, respectively. Define

$$f\colon \operatorname{nor} Y \setminus \{(x,n) \,|\, n \perp L_{r-1}\} \to \mathbb{R}^r \times S^{r-2}(L_{r-1})$$
$$f(x,n) = (x, \pi(n|L_{r-1}))$$

and

$$g\colon \operatorname{nor} Y \to L^\perp_{r-1}$$
$$g(x,n) = \langle x, \omega_1 \rangle \omega_1 = p(x|L^\perp_{r-1}).$$

Since the differential of the spherical projection $\pi_L\colon n \mapsto \pi(n|L)$ is

$$D\pi_L(n)v = \frac{p(v|L \cap n^\perp)}{|p(n|L)|},$$

cf. [8, Lemma 2], we have

$$D_{(x,n)} f(u,v) = \left(u, \frac{p(v|L_{r-1} \cap n^\perp)}{|p(n|L_{r-1})|} \right)$$

and the linearity of g implies

$$D_{(x,n)} g(u,v) = g(u,v) = \langle u, \mathbf{e} \rangle \mathbf{e} = p(u|L^\perp_{r-1})$$

for all $(u,v) \in \operatorname{Tan}(\operatorname{nor} Y, (x,n))$. Next, we show that for almost all $z \in L^\perp_{r-1}$, the point $(x,n) \in \operatorname{nor} Y$ is uniquely determined by the projection $f(x,n) = (x, n_0)$ for $\mathcal{H}^{r-2}$-a.a. $(x, n_0) \in f(g^{-1}(z))$.

Lemma 1. *For almost all $L_{r-1} \in \mathcal{L}^r_{r-1}$ and $\mathcal{H}^1$-almost all $z \in L^\perp_{r-1}$,*

$$\mathcal{H}^{r-2}\left(\{(x,n_0) \in f(g^{-1}(z)) \,:\, \operatorname{card} f^{-1}\{(x,n_0)\} > 1\} \right) = 0.$$

Remark 8. Let $f^{(z)}$ be the restriction of f to $g^{-1}(z)$. Note that f is well defined on the set $g^{-1}(z)$ only when $L_{r-1}+z$ and Y do not touch, which is the case for almost all pairs (L_{r-1}, z). When Y and $L_{r-1}+z$ do not touch, there is no point $(y,n) \in \operatorname{nor} Y$ such that $y \in L_{r-1}+z$ and $n \in L_{r-1}^{\perp}$, i.e f is well-defined at all points $(y,n) \in \operatorname{nor} Y$ with $y \in L_{r-1}+z$. Since the normal bundle, $\operatorname{nor} Y$, is compact, the function f can be extended to a locally Lipschitz (differentiable!) function on an open set containing $g^{-1}(z) \setminus \{(y,n) \,|\, n \perp L_{r-1}\}$. Thus, the assumptions for the area and coarea formulae are satisfied, cf. [5].

Proof. Assume that $L_{r-1}+z$ and $\operatorname{nor} Y$ are do not touch, i.e. that (18) is satisfied. Then, $f(x,n)$ is well-defined for all $(x,n) \in \operatorname{nor} Y \cap (L_{r-1}+z) \times \mathbb{R}^d$. Let N be the subset of $\operatorname{nor} Y$ where f is well-defined but not injective. More precisely N is the set of all $(x,n) \in \operatorname{nor} Y$ with $n \notin L_{r-1}^{\perp}$ such that there exist $n' \neq n$ with $(x,n') \in \operatorname{nor} Y$ and $n' \notin L_{r-1}^{\perp}$ and $f(x,n) = f(x,n')$. It is enough to show that

$$\mathcal{H}^{r-2}(f(N \cap g^{-1}(z))) \leq \int_{N \cap g^{-1}(z)} J_{r-2} f^{(z)} \, \mathrm{d}\mathcal{H}^{r-2} = 0,$$

for almost all $z \in L_{r-1}^{\perp}$, cf. [5, (3.2.3)]. Using the coarea formula, [5, (3.2.22)], we obtain

$$\int_{L_{r-1}^{\perp}} \int_{N \cap g^{-1}(z)} J_{r-2} f^{(z)} \, \mathrm{d}\mathcal{H}^{r-2} \, \mathrm{d}z = \int_N J_{r-2} f^{(z)}(x,n) J_1 g(x,n) \, \mathcal{H}^{r-1}(\mathrm{d}(x,n)).$$

Without loss of generality, assume that $\operatorname{Tan}(Y,(x,n))$ is the $(r-1)$-dimensional subspace given by (16), cf. [5, (3.2.19)]. Note that

$$\ker D_{(x,n)} g = (L_{r-1} \times \mathbb{R}^r) \cap \operatorname{Tan}(Y,(x,n)) = \operatorname{Tan}(g^{-1}(z),(x,n)).$$

If $\dim \ker D_{(x,n)} g \geq r-1$, then $J_1 g(x,n)$ must be equal to zero. Let us assume that $\dim \ker D_{(x,n)} g \leq r-2$. Since the domain of $D_{(x,n)} f^{(z)}$ is equal to $\ker D_{(x,n)} g$, the $(r-2)$-dimensional Jacobian of $f^{(z)}$ is zero if there exists a point $(u,v) \in \operatorname{Tan}(g^{-1}(z),(x,n))$ such that $D_{(x,n)} f^{(z)}(u,v) = 0$. Note that $(x,n_t) := \left(x, \frac{\sin(1-t)\theta}{\sin\theta} n + \frac{\sin t\theta}{\sin\theta} n'\right) \in g^{-1}(z)$ for all $t \in [0,1]$, where $\theta = \angle(n,n')$. By definition of the tangent cone, we have

$$(0, \pi(n-n'|n^{\perp})) = \lim_{t \downarrow 0} \frac{(x,n_t)-(x,n)}{|(x,n_t)-(x,n)|} \in \operatorname{Tan}(g^{-1}(z),(x,n)).$$

Since, $p(n'|L_{r-1} \cap n^{\perp}) = p(p(n'|L_{r-1})|L_{r-1} \cap n^{\perp}) = p(n|L_{r-1} \cap n^{\perp}) = 0$, we deduce that $D_{(x,n)} f^{(z)}(0, \pi(n-n'|n^{\perp})) = 0$ and therefore, $J f^{(z)}(x,n) = 0$. ☐

Given a subspace $L_j \in \mathcal{L}_j^r$ and $y \in L_j^{\perp}$ such that $L_j + y$ and Y do not touch, the restriction of the normal bundle of $Y \cap (L_j + y)$ to $L_j + y$ is given by

$$\operatorname{nor}^{(j)}(Y \cap (L_j + y)) = \{(x, \pi(n, L_j)) \,|\, x \in Y \cap (L_j + y) \text{ and } (x,n) \in \operatorname{nor} Y\}, \tag{22}$$

i.e. the intersection of $\operatorname{nor} Y + \operatorname{nor}(L_j + y)$ with $(L_j + y) \times S^{r-1}$, see [4, Theorem 4.10]. The corresponding orienting unit vectorfield $a_{Y \cap (L_j + x)}$ will be computed later.

Lemma 2. *Let $Y \subseteq \mathbb{R}^r$ be a compact set with positive reach and let $L_{r-1} \in \mathcal{L}_{r-1}^r$. Then,*

$$N_{Y \cap (L_{r-1}+z)} = f_{\sharp} \langle N_Y, g, z \rangle$$

for almost all $z \in L_{r-1}^{\perp}$, whenever Y and $L_j + z$ do not touch.

Proof. Applying [5, Section 4.3.8 and 4.3.13] (with $n = 1$ and $m = r - 1$) to the integral current (15), we get

$$\langle N_Y, g, z\rangle = (\mathcal{H}^{r-1-1} \llcorner g^{-1}(z)) \wedge \zeta \tag{23}$$

for almost all $z \in L_{j-1}^{\perp}$. Here, ζ is the $(r-2)$-vectorfield such that

$$\zeta(x,n) = \frac{a_Y(x,n) \llcorner \left\langle \Omega_1, \bigwedge^1 Dg(x,n)\right\rangle}{J_1 g(x,n)} = a_Y(x,n) \llcorner \left\langle \Omega_1, \bigwedge^1 Dg(x,n)\right\rangle \tag{24}$$

(because $J_1 g(x,n) = \sqrt{\det(Dg(x,n)Dg(x,n)^t)} = 1$). Applying the area formula for currents [5, Section 4.1.30] to (23), we obtain

$$f_\sharp\langle N_Y, g, z\rangle = \left(\mathcal{H}^{r-2} \llcorner f(g^{-1}\{z\})\right) \wedge \eta,$$

with unit vector field

$$\eta(x,v) = \frac{\left(\bigwedge_{r-2} Df(f^{-1}(x,v))\right) \zeta(f^{-1}(x,v))}{J_{r-2} f(f^{-1}(x,v))}. \tag{25}$$

for $\mathcal{H}^{2r-3}$-almost all $(x,v) \in g^{-1}(z) \times S^{r-2}(L_{r-1})$. Whenever $\operatorname{nor} Y$ and $L_{r-1} + z$ do not touch, then $f(g^{-1}(z))$ is equal to the normal bundle $\operatorname{nor}^{(r-1)}(Y \cap (L_{r-1} + z))$. Hence, in order to prove the lemma, it is enough to check that the orientation of the respective orienting vectorfields, $a_{X\cap(L_{r-1}+z)}$ and η, have the same orientation. By convention, the orientation of $a_{X\cap(L_{r-1}+z)}$ is chosen such that $\left\langle a_{X\cap(L_{r-1}+z)}(x,n), \varphi^{(r-1)}_{k(x,n)}(n)\right\rangle > 0$, where $k(x,n)$ is the number of principal curvatures at (x,n) that are 0. Hence, we have to check that $\left\langle \eta(x,n), \varphi^{(r-1)}_{k(x,n)}(n)\right\rangle > 0$. By combining (25) and (24), we see that η is proportional to $\bigwedge_{r-2} Df\left(a_{Y\cap(L_{r-1}+z)} \llcorner g^{\#}\Omega_1\right)$ with some strictly positive proportionality constant λ. Thus,

$$\lambda\left\langle \eta(x,n), \varphi^{(r-1)}_{k(x,n)}(x,n)\right\rangle = \lambda\left\langle a_{Y\cap(L_{r-1}+z)}(x,n), g^{\#}\Omega_1 \wedge f^{\#}\varphi^{(r-1)}_{k(x,n)}(x,n)\right\rangle \geq 0$$

and the inequality is strict when $k(x,n)$ is the number of zero principal curvatures at (x,n), cf. (29) below. □

Using the normal current of $Y \cap (L_{r-1} + |x|)$, the integral $\mathcal{I}(L_{r-1})$ can be written as

$$\mathcal{I}(L_{j-1}) = \int_{L_{r-1}^{\perp}} V_{r-s-1}(Y \cap (L_{r-1} + x))|x|^{d-r} \mathrm{d}x^1 = \int_{L_{r-1}^{\perp}} N_{Y\cap(L_{r-1}+x)}(\phi^{(r-1)}_{r-s-1})|x|^{d-r} \mathrm{d}x^1.$$

By applying Lemma 2 and the coarea formula for currents, [5, Section 4.3.13], we obtain

$$\begin{aligned}
\mathcal{I}(L_{r-1}) &= \int_{L_{r-1}^{\perp}} |x|^{d-r} N_{Y\cap(L_{r-1}+x)}(\phi^{(r-1)}_{r-s-1})\,\mathrm{d}x^1 \\
&= \int_{L_{r-1}^{\perp}} |x|^{d-r} f_\sharp\langle N_Y, g, x\rangle(\phi^{(r-1)}_{r-s-1})\,\mathrm{d}x^1 \\
&= \int_{L_{r-1}^{\perp}} |x|^{d-r} \langle N_Y, g, x\rangle(f^\sharp \phi^{(r-1)}_{r-s-1})\,\mathrm{d}x^1 \\
&= N_Y \llcorner g^\sharp(q \cdot \Omega_1)(f^\sharp \phi^{(r-1)}_{r-s-1}) \\
&= \int_{\operatorname{nor} Y} \langle a_Y, g^\sharp(q \cdot \Omega_1) \wedge f^\sharp \phi^{(r-1)}_{r-s-1}\rangle\,\mathrm{d}\mathcal{H}^{r-1},
\end{aligned}$$

where $q(x) = |x|^{d-r}$, $x \in \mathbb{R}^r$. Using the shuffle formula, [5, Section 1.4.2], the integrand can be written

$$
\begin{aligned}
&\langle a_Y, g^\sharp(q\cdot\Omega_1)\wedge f^\sharp\phi^{(r-1)}_{r-s-1}\rangle \\
&= \sum_{i=1}^{r-1}(-1)^{i+1}\left\langle (u_i,v_i), g^\sharp(q\cdot\Omega_1)\right\rangle\cdot\left\langle (u_1,v_1)\wedge\cdots\wedge\widehat{(u_i,v_i)}\wedge\cdots\wedge(u_{r-1},v_{r-1}), f^\sharp\phi^{(r-1)}_{r-s-1}\right\rangle .
\end{aligned}
\tag{26}
$$

The second term can we expressed more explicitly (for the definition of the push-forward, the reader is referred to [5, Section 4.1.6]),

$$
\begin{aligned}
&\left\langle (u_1,v_1)\wedge\cdots\wedge\widehat{(u_i,v_i)}\wedge\cdots\wedge(u_{r-1},v_{r-1}), f^\sharp\phi^{(r-1)}_{r-s-1}(x,n)\right\rangle \\
&\quad=\left\langle\left[\textstyle\bigwedge_{r-2}Df(x,n)\right](u_1,v_1)\wedge\cdots\wedge\widehat{(u_i,v_i)}\wedge\cdots\wedge(u_{r-1},v_{r-1}), \phi^{(r-1)}_{r-s-1}\circ f(x,n)\right\rangle \\
&\quad=\left\langle \textstyle\bigwedge_{j\in\{1,\dots,r-1\}\setminus\{i\}}\left(u_j,\dfrac{p(v_j|L_{r-1}\cap n^\perp)}{|p(n|L_{r-1})|}\right), \phi^{(r-1)}_{r-s-1}(\pi(n|L_{r-1}))\right\rangle \\
&\quad=\frac{1}{\sigma_s}\sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}}(\operatorname{sgn} J)\left\langle \bigwedge_{j\in J}u_j\wedge\bigwedge_{j\in J^c}\left(\frac{p(v_j|L_{r-1}\cap n^\perp)}{|p(n|L_{r-1})|}\right)\wedge\pi(n|L_{r-1}),\Omega_{r-1}\right\rangle \\
&\quad=\frac{1}{\sigma_s}\frac{1}{|p(n|L_{r-1})|^s} \\
&\qquad\times\sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}}(\operatorname{sgn} J)\left\langle \bigwedge_{j\in J}u_j\wedge\bigwedge_{j\in J^c}p(v_j|L_{r-1}\cap n^\perp)\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle ,
\end{aligned}
\tag{27}
$$

for all $(x,n)\in\operatorname{nor}Y$. Here, $\operatorname{sgn} J$ is the number of permutations needed to map $J\cup J^c$ into $(1,\dots,r-1)\setminus\{i\}$, where J and J^c are sorted in increasing order. The first term in (26) can be expressed as

$$
\begin{aligned}
\langle (u_i,v_i), g^\sharp(q\cdot\Omega_1(x,n))\rangle &= \langle Dg(x,n)(u_i,v_i), q\circ g(x,n)\cdot\Omega_1\circ g(x,n)\rangle \\
&= |p(x|L_{r-1}^\perp)|^{d-r}\left\langle p(u_i|L_{r-1}^\perp),\Omega_1\right\rangle .
\end{aligned}
\tag{28}
$$

By inserting into (27) and (28) the explicit representation of a_Y given in (16), we can write (26) as

$$
\begin{aligned}
&\langle a_Y, g^\sharp(q\cdot\Omega_1)\wedge f^\sharp\phi^{(r-1)}_{r-s-1}\rangle \\
&\quad=\sum_{i=1}^{r-1}\sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}}\frac{(-1)^{i+1}(\operatorname{sgn} J)|p(x|L_{r-1}^\perp)|^{d-r}}{\sigma_s|p(n|L_{r-1})|^s}\frac{\prod_{l\in J^c}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}} \\
&\qquad\times\left\langle p(a_i|L_{r-1}^\perp),\Omega_1\right\rangle\left\langle\bigwedge_{j\in J}a_j\wedge\bigwedge_{j\in J^c}p(a_j|L_{r-1}\cap n^\perp)\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle .
\end{aligned}
$$

Note that $(-1)^{i+1}(\operatorname{sgn} J)$ is the sign of the permutation necessary to order $\{i\}\cup J\cup J^c$ increasingly. Moreover, orthogonal projections are orientation preserving (eigenvalues are either 0 or 1), therefore,

$$
\begin{aligned}
&(-1)^{i+1}(\operatorname{sgn} J)\left\langle p(a_i|L_{r-1}^{\perp}),\Omega_1\right\rangle\left\langle \bigwedge_{j\in J} a_j\wedge\bigwedge_{j\in J^c} p(a_j|L_{r-1}\cap n^{\perp})\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle\\
&\quad=(-1)^{i+1}(\operatorname{sgn} J)\left\langle p(a_i|L_{r-1}^{\perp}),\Omega_1\right\rangle\left\langle \bigwedge_{j\in J\cup J^c} p(a_j|L_{r-1}\cap n^{\perp})\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle\\
&\quad=\left\langle \bigwedge_{j=1}^{i-1} p(a_j|L_{r-1}\cap n^{\perp})\wedge p(a_i|L_{r-1}^{\perp})\wedge\bigwedge_{j=i+1}^{r-1} p(a_j|L_{r-1}\cap n^{\perp})\wedge p(n|L_{r-1}),\Omega_r\right\rangle>0,
\end{aligned}
$$

where we used the decomposition

$$
a_j=p(a_j|L_{r-1}\cap n^{\perp})+p(a_j|\pi(n|L_{r-1}))+p(a_j|L_{r-1}^{\perp})
$$

for the first equality. Thus,

$$
\begin{aligned}
&\langle a_Y, g^{\sharp}(q\cdot\Omega_1)\wedge f^{\sharp}\phi_{r-s-1}^{(r-1)}\rangle\\
&\quad=\frac{1}{\sigma_s}\sum_{i=1}^{r-1}\sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}}\frac{|p(x|L_{r-1}^{\perp})|^{d-r}}{|p(n|L_{r-1})|^{s}}\frac{\prod_{l\in J^c}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}}\\
&\qquad\qquad\times\left|\left\langle p(a_i|L_{r-1}^{\perp}),\Omega_1\right\rangle\right|\left|\left\langle \bigwedge_{j\in J\cup J^c} p(a_j|L_{r-1}\cap n^{\perp})\wedge p(n|L_{r-1}),\Omega_{r-1}\right\rangle\right|\\
&\quad=\frac{1}{\sigma_s}\sum_{i=1}^{r-1}\sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}}\frac{|p(x|L_{r-1}^{\perp})|^{d-r}|p(a_i|L_{r-1}^{\perp})|^2}{|p(n|L_{r-1})|^{s}}\frac{\prod_{l\in J^c}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}},
\end{aligned}
$$

where the last equality follows from [7, Proposition 5.2]. Applying the re-indexing identity

$$
\sum_{i=1}^{r-1}\sum_{\substack{|J|=r-s-1\\ J\subset\{1,\dots,r-1\}\setminus\{i\}}}\frac{\prod_{l\in J^c}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}}|p(a_i|L_{r-1}^{\perp})|^2=\sum_{\substack{|J|=s-1\\ J\subset\{1,\dots,r-1\}}}\frac{\prod_{l\in J}\kappa_l}{\prod_{l=1}^{r-1}\sqrt{1+\kappa_l^2}}\sum_{i\in J^c}|p(a_i|L_{r-1}^{\perp})|^2,
$$

we conclude that

$$
\begin{aligned}
c_{d,r-1,r-s-1}\alpha_{d,s}^{r}(Y)&=\int_{\mathcal{L}_{r-1}^{r}}\mathcal{I}(L_{r-1})\,\mathrm{d}L_{r-1}\\
&=\int_{\mathcal{L}_{r-1}^{r}}\int_{\operatorname{nor}Y}\langle a_Y, g^{\sharp}(q\cdot\Omega_1)\wedge f^{\sharp}\phi_{r-s-1}^{(r-1)}\rangle\,\mathrm{d}\mathcal{H}^{r-1}\,\mathrm{d}L_{r-1}\\
&=\int_{\operatorname{nor}Y}\int_{\mathcal{L}_{r-1}^{r}}\langle a_Y, g^{\sharp}(q\cdot\Omega_1)\wedge f^{\sharp}\phi_{r-s-1}^{(r-1)}\rangle\,\mathrm{d}L_{r-1}\,\mathrm{d}\mathcal{H}^{r-1}\\
&=\frac{1}{\sigma_s}\int_{\operatorname{nor}Y}|x|^{d-r}\sum_{\substack{|J|=s-1\\ J\subset\{1,\dots,r-1\}}}\frac{\prod_{j\in J}\kappa_j}{\prod_{j=1}^{r-1}\sqrt{1+\kappa_j^2}}\widetilde{Q}_{r,s}^{d}(x,n,A_J)\,\mathrm{d}\mathcal{H}^{r-1},
\end{aligned}
\tag{29}
$$

where

$$
\widetilde{Q}_{d,s}^{r}(x,n,A_J)=\int_{\mathcal{L}_{r-1}^{r}}\frac{|p(x/|x||L_{r-1}^{\perp})|^{d-r}}{|p(n|L_{r-1})|^{s}}\sum_{i\in J^c}|p(a_i|L_{r-1}^{\perp})|^2\,\mathrm{d}L_{r-1}.
$$

In the remainder of this section, we shall prove that the above expression for $\widetilde{Q}^r_{d,s}$ can be written in terms of two hypergeometric series.

The hypergeometric series and the Gamma function have been introduced earlier. The *Beta function*, $B(a,b) = B(b,a)$, is defined for all $a,b > 0$ as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$$

and is often expressed in terms of the Gamma function as

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Lemma 3. *Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. Then, for all $k > 0$, $a \in \mathbb{R}$, $b > -\frac{1}{k}$ and $c > -1$,*

$$\int_0^1 \left(\alpha + \beta t^k\right)^{-a} (t^k)^b (1-t^k)^c\,\mathrm{d}t = \frac{1}{k} B\left(b + \frac{1}{k}, c+1\right) F\left(a, c+1; c+b+\frac{1}{k}+1; \beta\right). \tag{30}$$

When $\beta = 1$, the extra assumption $b + \frac{1}{k} > a$ is necessary.

Proof. According to [1, 15.3.1], the analytic continuation of the hypergeometric function is given by

$$F(a,b;c;\beta) = \frac{1}{B\left(b, c-b\right)} \int_0^1 (1-\beta s)^{-a}(1-s)^{c-b-1} s^{b-1}\,\mathrm{d}s$$

whenever $c > b > 0$. Hence, a substitution by $s = 1 - r^k$ proves the Lemma. □

Remark 9. In the special case where $a = 0$ and $k = 2$, Lemma 3 yields the identity

$$\int_{-1}^1 (t^2)^b (1-t^2)^c\,\mathrm{d}t = B\left(b + \frac{1}{2}, c+1\right).$$

Lemma 4. *Let $x, z \in S^{d-1}$, $m, n \in \mathbb{N}$. Then,*

$$\int_{S^{d-1}} |x \cdot \omega|^m |z \cdot \omega|^n \,\mathrm{d}\omega^{d-1}$$
$$= \sigma_{d-2} B\left(\frac{n+m+2}{2}, \frac{d-2}{2}\right) B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,z)\right). \tag{31}$$

Proof. This Lemma was proven in [3] for any *even* natural number n. The exact same procedure as the one in the proof of Proposition 4 below, in particular the use of Lemma 6, shows that (31) also holds when n is odd. The details are omited here. In order to prove the main results of the present paper, the special case where $n = 2$ is sufficient, though. □

Remark 10. Let L_{d-1} be a $(d-1)$-dimensional subspace of $\mathbb{R}^d$, with $z \in L_{d-1}$ and $x \notin L_{d-1}^{\perp}$. Then, for all $m, n \in \mathbb{N}$,

$$\int_{S^{d-2}(L_{d-1})} |x \cdot \omega|^m |z \cdot \omega|^n \,\mathrm{d}\omega^{d-1} = \sigma_{d-3} B\left(\frac{n+m+2}{2}, \frac{d-3}{2}\right) B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$
$$\times \cos^m(x, L_{d-1}) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{1}{2}; \cos^2(\pi(x|L_{d-1}), z)\right).$$

Recall that the binomial coefficient $\binom{a}{k}$ is defined for all $a \in \mathbb{R}$ and all $k \in \mathbb{N}$ by

$$\binom{a}{k} = \frac{(-a)_k(-1)^k}{k!} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-k+1)\Gamma(k+1)} & \text{for } a > 0, \\ \frac{\Gamma(a+k)(-1)^k}{\Gamma(a)\Gamma(k+1)} & \text{for } a < 0, \\ 0 & \text{for } a = 0. \end{cases}$$

Lemma 5. *For all $a \in \mathbb{R}$ and all $s \in \mathbb{N}$,*

$$\binom{a}{2s} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}} 2^{2s}.$$

Proof. A routine calculation yields

$$\binom{a}{2s} = \frac{\Gamma(a+1)}{\Gamma(2s+1)\Gamma(a-2s+1)} = \binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(s+\frac{1}{2})} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}} 2^{2s}.$$

For the second equality, we applied the duplication formula on each Gamma function appearing in the second term, and for the third equality, we applied the duplication formula to $\Gamma(2s+1)$. ☐

Lemma 6. *For all $a \in \mathbb{R}$ and any function f, the following identity holds,*

$$\sum_{k=0}^{\infty} \binom{\frac{a}{2}}{k} \sum_{l=0}^{k} \binom{k}{l} 2^{k-l} f(k+l) = \sum_{s=0}^{\frac{a}{2}} \binom{a}{2s} f(2s),$$

where the double sum on the l.h.s. is over k and l ***with the same parity*** *only.*

Proof. Substition of $k+l$ by $2s$ yields

$$\sum_{k=0}^{\infty} \binom{\frac{a}{2}}{k} \sum_{l=0}^{k} \binom{k}{l} 2^{k-l} f(k+l) = \sum_{s=0}^{\infty}\sum_{l=0}^{s} \binom{\frac{a}{2}}{2s-l}\binom{2s-l}{l} 2^{2s-2l} f(2s).$$

Applying the duplication formula to $\Gamma(2s-2l+1)$, we get

$$\begin{aligned} \binom{\frac{a}{2}}{2s-l}\binom{2s-l}{l} 2^{2s-2l} &= \frac{\Gamma(\frac{a}{2}+1)}{\Gamma(\frac{a}{2}-2s+l+1)} \frac{2^{2s-2l}}{\Gamma(l+1)\Gamma(2s-2l+1)} \\ &= \frac{\Gamma(\frac{a}{2}+1)}{\Gamma(\frac{a}{2}-2s+l+1)} \frac{\Gamma(\frac{1}{2})}{\Gamma(l+1)\Gamma(s-l+1)\Gamma(s-l+\frac{1}{2})} \\ &= \binom{\frac{a}{2}}{s}\binom{s-\frac{1}{2}}{l}\binom{\frac{a-1}{2}-(s-\frac{1}{2})}{s-l}\frac{\Gamma(\frac{1}{2})\Gamma(s+1)}{\Gamma(s+\frac{1}{2})}. \end{aligned}$$

Then, the well-known identity, $\sum_{l=0}^{k}\binom{m}{l}\binom{n-m}{k-l} = \binom{n}{k}$, valid for any complex numbers m and n, and the duplication formula applied to $\Gamma(2s+1)$ imply

$$\sum_{l=0}^{s} \binom{\frac{a}{2}}{2s-l}\binom{2s-l}{l} 2^{2s-2l} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}} 2^{2s}.$$

Thanks to Lemma 5, the proof is complete. ☐

Proposition 4. *Let x, y and z be unit vectors in $\mathbb{R}^d$ with $y \perp z$ and let $a, b, c \in \mathbb{Z}$. Then, if $x \neq y$ and $x \notin y^\perp$, the following identity holds,*

$$\begin{aligned}&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^{d-1}\\ &\quad= \sigma_{d-2}\, |x\cdot y|^a B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+c+d-1}{2}\right) B\left(\frac{c}{2}+\frac{1}{2}, \frac{d-2}{2}\right)\\ &\qquad\times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(\frac{b+c+d-1}{2}\right)_s}{\left(\frac{c+d-1}{2}\right)_s} \frac{(-1)^s}{s!} \tan^{2s}(x,y) F\left(-s, -\frac{c}{2}; \frac{1}{2}; \frac{\cos^2(x,z)}{\sin^2(x,y)}\right),\end{aligned}$$

whenever both sides of the equation converge. Moreover, for $x = \pm y$,

$$\begin{aligned}&\int_{S^{d-1}} |y\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^{d-1}\\ &\quad= \sigma_{d-2}\, B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+c+d-1}{2}\right) B\left(\frac{c}{2}+\frac{1}{2}, \frac{d-2}{2}\right),\end{aligned}$$

and, for $x \perp y$,

$$\begin{aligned}&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^{d-1}\\ &\quad= \sigma_{d-3}\, B\left(\frac{a+b+c+d-1}{2}, \frac{1}{2}\right) B\left(\frac{a+c+2}{2}, \frac{d-3}{2}\right)\\ &\qquad\times B\left(\frac{a+1}{2}, \frac{c+1}{2}\right) F\left(-\frac{a}{2}, -\frac{c}{2}; \frac{1}{2}; \cos^2(x,z)\right).\end{aligned}$$

Proof. The mapping

$$\begin{aligned} f\colon S^{d-1} &\to [-1,1]\\ \omega &\mapsto \omega\cdot y = t\end{aligned}$$

has 1-dimensional Jacobian $J_1 f(\omega) = \sqrt{1-(\omega\cdot y)^2}$ for all $\omega \in S^{d-1}\setminus\{y\}$. The coarea formula implies

$$\int_{S^{d-1}} h(\omega)\,\mathrm{d}\omega^{d-1} = \int_{-1}^{1}\int_{S^{d-1}\cap f^{-1}(t)} \frac{1}{\sqrt{1-t^2}} h(\omega)\,\mathrm{d}\omega^{d-2}\,\mathrm{d}t,$$

for any positive, $\mathcal{H}^{d-1}$-measurable function $h\colon S^{d-1}\to\mathbb{R}$. Then, an application of the area formula with the injective mapping (whenever $t\in(-1,1)$)

$$\begin{aligned} g\colon S^{d-1}\cap f^{-1}(t) &\to S^{d-2}(y^\perp)\\ \omega &\mapsto \pi(\omega|y^\perp),\end{aligned}$$

with $(d-2)$-dimensional Jacobian $J_{d-2}g(\omega) = \sqrt{1-(\omega\cdot y)^2}^{\,2-d}$, yields

$$\int_{S^{d-1}} h(\omega)\,\mathrm{d}\omega^{d-1} = \int_{-1}^{1}\int_{S^{d-2}(y^\perp)} \sqrt{1-t^2}^{\,d-3} h(ty+\sqrt{1-t^2}\omega)\,\mathrm{d}\omega^{d-2}\,\mathrm{d}t.$$

Therefore,

$$\begin{aligned}&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^{d-1}\\ &\quad= \int_{-1}^{1}\int_{S^{d-2}(y^\perp)} \sqrt{1-t^2}^{\,d+b+c-3} |x\cdot(ty+\sqrt{1-t^2}\omega)|^a |z\cdot\omega|^c\,\mathrm{d}\omega^{d-2}\,\mathrm{d}t,\end{aligned}$$

and by a double application of the binomial formula, the last expression becomes

$$\begin{aligned}
&\sum_{k=0}^{\infty}\binom{\frac{a}{2}}{k}\int_{-1}^{1}\sqrt{1-t^2}^{d+b+c-3} \\
&\qquad\times\int_{S^{d-2}(y^\perp)}(t^2(x\cdot y)^2)^{\frac{a}{2}-k}\left((1-t^2)(x\cdot\omega)^2+2t\sqrt{1-t^2}(x\cdot y)(x\cdot\omega)\right)^k \,\mathrm{d}\omega^{d-2}\,\mathrm{d}t \\
&=\sum_{k=0}^{\infty}\binom{\frac{a}{2}}{k}\sum_{l=0}^{k}\binom{k}{l}2^{k-l}|x\cdot y|^{a-2k}(x\cdot y)^{k-l}\int_{-1}^{1}|t|^{a-2k}t^{k-l}\sqrt{1-t^2}^{d+k+l+b+c-3}\,\mathrm{d}t \\
&\qquad\times\int_{S^{d-2}(y^\perp)}(x\cdot\omega)^{2l}(x\cdot\omega)^{k-l}|z\cdot\omega|^c\,\mathrm{d}\omega^{d-2}.
\end{aligned}$$

Notice that both integrals are non-zero only if k an l have the same parity. Then, using Remark 9 and Lemma 4, we get

$$\begin{aligned}
&\int_{S^{d-1}}|x\cdot\omega|^a\sqrt{1-(y\cdot\omega)^2}^b|z\cdot\omega|^c\,\mathrm{d}\omega^{d-1} \\
&\quad=\sigma_{d-3}|x\cdot y|^a\sum_{k=0}^{\infty}\binom{\frac{a}{2}}{k}\sum_{l=0}^{k}\binom{k}{l}2^{k-l}\tan^{k+l}(x,y) \\
&\qquad\times B\left(\frac{a}{2}-\frac{k+l}{2}+\frac{1}{2},\frac{d}{2}+\frac{k+l}{2}+\frac{b+c}{2}-\frac{1}{2}\right)B\left(\frac{c}{2}+\frac{k+l}{2}+1,\frac{d-3}{2}\right) \\
&\qquad\times B\left(\frac{k+l}{2}+\frac{1}{2},\frac{c}{2}+\frac{1}{2}\right)F\left(-\frac{k+l}{2},-\frac{c}{2};\frac{1}{2};\cos^2(x_1,z)\right),
\end{aligned}$$

where $x_1=\pi(x|y^\perp)$ and the double sum is over k and l with the same parity only. Finally, an application of Lemma 6 yields

$$\begin{aligned}
&\int_{S^{d-1}}|x\cdot\omega|^a\sqrt{1-(y\cdot\omega)^2}^b|z\cdot\omega|^c\,\mathrm{d}\omega^{d-1} \\
&=2\sigma_{d-2}|x\cdot y|^a\sum_{s=0}^{\infty}\binom{a}{2s}B\left(\frac{a}{2}-s+\frac{1}{2},\frac{d}{2}+s+\frac{b+c}{2}-\frac{1}{2}\right) \\
&\quad\times\tan^{2s}(x,y)B\left(\frac{c}{2}+s+1,\frac{d-3}{2}\right)B\left(s+\frac{1}{2},\frac{c}{2}+\frac{1}{2}\right)F\left(-s,-\frac{c}{2};\frac{1}{2};\cos^2(x_1,z)\right).
\end{aligned}$$

Using the duplication formula for the Gamma function, the following identity can be derived,

$$\begin{aligned}
&\sigma_{d-3}\binom{a}{2s}B\left(\frac{a}{2}-s+\frac{1}{2},\frac{d}{2}+s+\frac{b+c}{2}-\frac{1}{2}\right)B\left(\frac{c}{2}+s+1,\frac{d-3}{2}\right)B\left(s+\frac{1}{2},\frac{c}{2}+\frac{1}{2}\right) \\
&\qquad=\sigma_{d-2}B\left(\frac{a}{2}+\frac{1}{2},\frac{b+c+d-1}{2}\right)B\left(\frac{c}{2}+\frac{1}{2},\frac{d-2}{2}\right)\frac{\left(-\frac{a}{2}\right)_s\left(\frac{b+c+d-1}{2}\right)_s}{\left(\frac{c+d-1}{2}\right)_s}\frac{(-1)^s}{s!}.
\end{aligned}$$

The proof of the first identity is complete. The two remaining identities are easily proven by a slight modification of the above argument. $\square$

Remark 11. Alternatively, the three identities in Proposition 4 can be written as

$$\begin{aligned}
&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^{d-1} \\
&\quad = \sigma_{d-2} B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+c+d-1}{2}\right) B\left(\frac{c}{2}+\frac{1}{2}, \frac{d-2}{2}\right) \\
&\qquad \times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{c}{2}\right)_s \left(\frac{b+c+d-1}{2}\right)_s}{\left(\frac{1}{2}\right)_s \left(\frac{c+d-1}{2}\right)_s} \frac{\cos^{2s}(x,z)}{s!} F\left(-\frac{a}{2}+s, -\frac{b}{2}; \frac{c+d-1}{2}+s; \sin^2(x,y)\right).
\end{aligned}$$

Then, for $b=0$, it is easy to check that Proposition 4 is Lemma 4, exactly. For $a=0$ or $c=0$, Proposition 4 is equivalent to [3, Proposition 4], i.e.

$$\begin{aligned}
&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b \,\mathrm{d}\omega^{d-1} \\
&\qquad = \sigma_{d-1} B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+d-1}{2}\right) F\left(-\frac{a}{2}, -\frac{b}{2}; \frac{d-1}{2}; \sin^2(x,y)\right). \qquad (32)
\end{aligned}$$

Recall the identity (21) in the proof of Proposition 2. Using [7, Proposition 3.5], we notice that

$$\begin{aligned}
2\int_{\mathcal{L}^j_{q(q-1)}} |p(z|L_q^\perp)|^{d-j} \,\mathrm{d}L_{q(q-1)} &= \int_{S^{j-q}(L_{q-1}^\perp)} |p(z|(L_{q-1}\oplus\omega)^\perp)|^{d-j} \,\mathrm{d}\omega^{j-q} \\
&= |z|^{d-j} \int_{S^{j-q}(L_{q-1}^\perp)} \sqrt{1-\left(\frac{z}{|z|}\cdot\omega\right)^2}^{d-j} \mathrm{d}\omega^{j-q},
\end{aligned}$$

where we used $z\in L_{q-1}^\perp$ for the second equality. An application of (32) with $a=0$ yields (21). $\square$

In the special case where $c=2$, it is easily seen using Remark 11, that

$$\begin{aligned}
&\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-(y\cdot\omega)^2}^b |z\cdot\omega|^2 \,\mathrm{d}\omega^{d-1} \\
&\quad = \frac{\sigma_{d-1}}{d-1} B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+d+1}{2}\right) \left[F\left(-\frac{a}{2}, -\frac{b}{2}; \frac{d+1}{2}; \sin^2(x,y)\right)\right. \\
&\qquad \left. + \frac{a(b+d+1)}{d+1}\cos^2(x,z) F\left(-\frac{a}{2}+1, -\frac{b}{2}; \frac{d+3}{2}; \sin^2(x,y)\right)\right]. \qquad (33)
\end{aligned}$$

When $(x,n)\in \operatorname{nor} Y$ and $a_i(x,n)$ is the i'th principal direction at (x,n), we have, thanks to formula (33),

$$\begin{aligned}
\widetilde{Q}^r_{d,s}(x,n,A_J) &= \int_{\mathcal{L}^r_{r-1}} \frac{|p(x/|x||L_{r-1}^\perp)|^{d-r}}{|p(n|L_{r-1})|^s} \sum_{i\in J^c} |p(a_i|L_{r-1}^\perp)|^2 \mathrm{d}L_{r-1} \\
&= \frac{1}{2}\int_{S^{r-1}} |x\cdot\omega|^{d-r} \sqrt{1-(n\cdot\omega)^2}^{-s} \sum_{i\in J^c} |a_i\cdot\omega|^2 \,\mathrm{d}\omega^{r-1} \\
&= \frac{(r-s)\,\sigma_{r-1}}{2(r-1)} B\left(\frac{d-r}{2}+\frac{1}{2}, \frac{r-s+1}{2}\right) Q^r_{d,s},
\end{aligned}$$

where

$$Q^r_{d,s} := F\left(-\frac{d-r}{2}, \frac{s}{2}; \frac{r+1}{2}; \sin^2(x,n)\right) + \frac{(d-r)(r-s+1)}{r+1}\frac{\cos^2(x,A_J)}{r-s}F\left(-\frac{d-r}{2}+1, \frac{s}{2}; \frac{r+3}{2}; \sin^2(x,n)\right).$$

Combining with (29), we obtain

$$\alpha^r_{d,s}(X) = K^r_{d,s}\int_{\operatorname{nor} Y} |x|^{d-r} \sum_{\substack{|J|=s-1\\ J\subset\{1,\dots,r-1\}}} \frac{\prod_{j\in J}\kappa_j}{\prod_{j=1}^{r-1}\sqrt{1+\kappa_j^2}} Q^r_{d,s}(x,n,A_J)\mathrm{d}\mathcal{H}^{r-1},$$

with

$$\begin{aligned}
K^r_{d,s} &:= \frac{(r-s)\sigma_{r-1}B\left(\frac{d-r}{2}+\frac{1}{2}, \frac{r-s+1}{2}\right)}{2(r-1)c_{d,r-1,r-s-1}\sigma_s}\\
&= \frac{(r-s)\sigma_{r-1}\,\sigma_{d-r+1}}{2(r-1)\,\sigma_s\,\sigma_d\,c_{d-1,r-1}}\frac{\Gamma\left(\frac{d-r+1}{2}\right)\Gamma\left(\frac{r-s+1}{2}\right)\Gamma\left(\frac{r-s}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{d-s+1}{2}\right)\Gamma\left(\frac{d-s+2}{2}\right)}\\
&= \frac{r-s}{(r-1)\,\sigma_s\,c_{d-1,r-1}}\frac{\Gamma\left(\frac{r-s}{2}\right)\Gamma\left(\frac{r-s+1}{2}\right)}{\Gamma\left(\frac{r-1}{2}\right)\Gamma\left(\frac{r}{2}\right)}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d-s+1}{2}\right)\Gamma\left(\frac{d-s+2}{2}\right)}\\
&= \frac{1}{\sigma_s\,c_{d-1,r-1}}\frac{\Gamma(r-s+1)}{\Gamma(r)}\frac{\Gamma(d)}{\Gamma(d-s+1)}.
\end{aligned}$$

The proof of Proposition 3 is now complete. □

Lemma 7 (Two hypergeometric identities)**.** *For all $d>0$, $r\in\mathbb{R}$ and $z\in[-1,1]$,*

$$\begin{aligned}
&(r+1)(r-1)F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r-1}{2}; z\right)\\
&= (r+1)(r-1)F\left(-\frac{d-r}{2}, \frac{1}{2}; \frac{r+1}{2}; z\right) - r(d-r)zF\left(-\frac{d-r}{2}+1, \frac{1}{2}; \frac{r+3}{2}; z\right).
\end{aligned} \tag{34}$$

If $z\neq 0,1,-1$, then, for all $j,k\in\mathbb{R}$,

$$\begin{aligned}
&-\frac{(d-j)(d-j+k+1)}{d+1}F\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+3}{2}; z\right)\\
&\quad= (d-1)\frac{1-z}{z}F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d-1}{2}; z\right)\\
&\quad\quad\times\left[(j-d)-(d-1)\frac{1-z}{z}\right]F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d+1}{2}; z\right).
\end{aligned} \tag{35}$$

Proof. First, apply [1, (15.2.17)] with $a=-\frac{1}{2}$, $b=-\frac{d-r}{2}$ and $c=\frac{r+1}{2}$ to obtain

$$\frac{r-1}{2}F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r-1}{2}; z\right) = \frac{r}{2}F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r+1}{2}; z\right) - \frac{1}{2}F\left(-\frac{d-r}{2}, \frac{1}{2}; \frac{r-1}{2}; z\right).$$

Then, an application of [1, (15.2.15)] to the first term of the r.h.s. with $a = -\frac{d-r}{2}$, $b = \frac{1}{2}$ and $c = \frac{r+1}{2}$ yields

$$\frac{r}{2}F\left(-\frac{d-r}{2}, -\frac{1}{2}; \frac{r+1}{2}; z\right)$$
$$= \frac{d}{2}F\left(-\frac{d-r}{2}, \frac{1}{2}; \frac{r+1}{2}; z\right) - \frac{d-r}{2}(1-z)F\left(-\frac{d-r}{2}+1, \frac{1}{2}; \frac{r+1}{2}; z\right).$$

Furthermore, we can transform the second term of the r.h.s. of the last expression using [1, (15.2.20)] with $a = -\frac{d-r}{2}+1$, $b = \frac{1}{2}$ and $c = \frac{r+1}{2}$,

$$\frac{r+1}{2}(1-z)F\left(-\frac{d-r}{2}+1, \frac{1}{2}; \frac{r+1}{2}; z\right)$$
$$= \frac{r+1}{2}F\left(-\frac{d-r}{2}, \frac{1}{2}; \frac{r+1}{2}; z\right) - \frac{r}{2}zF\left(-\frac{d-r}{2}+1, \frac{1}{2}; \frac{r+3}{2}; z\right).$$

Hence, combining the three identities above, we obtain (34). Note that in (34), the three hypergeometric series are absolute convergent on the circle of convergence, $|z| = 1$, whenever $d > 0$, cf. [1, (15.1.1)].

According to [1, (15.2.20)] with $a = \frac{d-j}{2}+1$, $b = \frac{j-k}{2}$ and $c = \frac{d+1}{2}$, we have

$$-\frac{d-j+k+1}{2}zF\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+3}{2}; z\right)$$
$$= \frac{d+1}{2}(1-z)F\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+1}{2}; z\right) - \frac{d+1}{2}F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d+1}{2}; z\right).$$

The first term on the r.h.s. can be re-written using [1, (15.2.17)] with $a = \frac{d-j}{2}$, $b = \frac{j-k}{2}$ and $c = \frac{d+1}{2}$,

$$\frac{d-j}{2}F\left(\frac{d-j}{2}+1, \frac{j-k}{2}; \frac{d+1}{2}; z\right)$$
$$= \frac{d-1}{2}F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d-1}{2}; z\right) - \frac{j-1}{2}F\left(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d+1}{2}; z\right).$$

Combining the last two identities, we obtain identity (35). Note that in (35), the absolute convergence of $F(\frac{d-j}{2}, \frac{j-k}{2}; \frac{d-1}{2}; z)$ on the circle $|z| = 1$ requires that $k > 1$, cf. [1, (15.1.1)].

$\square$

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Paper D

Integral geometric formulae

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Manuscript

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1 Introduction

The aim of this paper is to collect a number of integral geometric identities that were frequently used during the redaction of [2], [3] and [4]. In the literature, those results are often applied in a specialized version, e.g. in lower dimensions or with special values of the parameters. In practice, those computations are rather time-consuming and the validity of a result involving integrals over complicated spaces is often difficult to check, especially when we are working in higher dimensions. Nonetheless, expressing integrals in explicit forms is indispensable in order to make the formulae manageable for applications. We hope that the present paper will prove helpful for future research in applied integral geometry.

2 Notation and definitions

The Gauss *hypergeometric series* is defined for $a, b, c \in \mathbb{R}$ and $z \in [-1, 1]$ as

$$F(a, b; c; z) = F(b, a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(a)_k$ is the *rising sequential product* or *Pochhammer symbol*,

$$(x)_k = x(x + 1)(x + 2) \cdots (x + k - 1)$$

for all $x \in \mathbb{R}, k \in \mathbb{N}$, cf. [1, Chapter 15] Note that $(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}$ and

$$\begin{aligned}(-x)_k &= (-x)(-(x - 1))(-(x - 2)) \cdots (-(x - k + 1)) \\ &= (-1)^k x(x - 1)(x - 2) \cdots (x - k + 1) \\ &= (-1)^k \frac{\Gamma(x + 1)}{\Gamma(x - k + 1)}\end{aligned}$$

for $x > 0$. Moreover, $(x)_k = 0$ whenever $x \in \{0, -1, -2, \ldots\}$ and $k > -x$. The *Gamma function* is defined on $\mathbb{R}_+$ as $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$ and it has an analytic continuation on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Standard formulae for the Gamma function can be found in [1, Chapter 6]. In particular, the duplication formula,

$$\Gamma(2z) = \Gamma(z)\Gamma\left(z + \frac{1}{2}\right) \pi^{-\frac{1}{2}} 2^{2z-1}, \tag{1}$$

is often useful. The *Beta function*, $B(a,b) = B(b,a)$, is defined for all $a, b > 0$ as

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\,\mathrm{d}t$$

or, alternatively,

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t.$$

The Beta function is often expressed in terms of the Gamma function as

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

In the following, we shall denote the surface area of the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ by

$$\sigma_d = \int_{S^{d-1}} \mathrm{d}x^{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

and the volume of the unit ball in $\mathbb{R}^d$ by

$$\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}.$$

Definition 1 ([5], chap. 2)**.** Let $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^n$ be differentiable manifolds of dimension p and q, respectively. Moreover, let $f\colon X \to Y$ be a differential mapping and let $Df(x)$ be the $n \times d$ matrix of partial derivatives

$$(Df(x))_{i,j} = \frac{\partial f_i}{\partial x_j}(x), \qquad i = 1, \dots, n, j = 1, \dots, d.$$

Let $\ker Df(x)$ be the kernel of $Df(x)$ and denote the tangent space of X at x by $\mathrm{Tan}(x, X)$. Then, $J_p f(x; X)$ is non-zero if and only if the dimension of

$$\mathrm{Tan}(x, X) \cap (\ker Df(x) \cap \mathrm{Tan}(x, X))^\perp \tag{2}$$

is equal to q. If (2) has dimension q, then the p-dimensional Jacobian can be calculated as

$$J_p f(x; X) = \sqrt{\det\left(Df(x; X) Df(x; X)^t\right)},$$

where $Df(x; X)$ is a $q \times n$-matrix,

$$Df(x; X) = Df(x)B,$$

and B is a matrix whose columns form an orthonormal basis for (2). Important special cases are

$$J_p f(x; X) = \sqrt{\det\left(Df(x)^t Df(x)\right)} \quad \text{when} \quad p = d < n,$$
$$J_p f(x; X) = \sqrt{\det\left(Df(x) Df(x)^t\right)} \quad \text{when} \quad p = n < d$$

and

$$J_p f(x; X) = |\det(Df(x))| \qquad \text{when} \quad p = n = d.$$

Theorem 2 (Coarea formula)**.** *Let $f\colon \mathbb{R}^d \to \mathbb{R}^n$ be Lipschitz, $d \geq n$, and let $A \subseteq \mathbb{R}^d$ be Lebesgue measurable. Then, for any non-negative λ^d-measurable function h on A,*

$$\int_A h(x) J_n f(x)\, \mathrm{d}x^d = \int_{\mathbb{R}^n} \int_{A\cap f^{-1}(z)} h(x)\, \mathrm{d}x^{d-n}\, \mathrm{d}z^n.$$

Theorem 3 (Area formula)**.** *Let $f\colon \mathbb{R}^d \to \mathbb{R}^n$ be Lipschitz, $d \leq n$, and let $A \subseteq \mathbb{R}^d$ be Lebesgue measurable. Then, for any non-negative λ^d-measurable function h on A,*

$$\int_A h(x) J_d f(x)\, \mathrm{d}x^d = \int_{\mathbb{R}^n} \sum_{x\in A\cap f^{-1}(z)} h(x)\, \mathrm{d}z^d.$$

3 Preliminary integral transformations

Lemma 4. *For all $c > b > 0$, $a \in \mathbb{R}$ and $z \in [-1, 1]$, the following integral representation of the hypergeometric function holds,*

$$F(a, b, c; z) = \frac{1}{B\,(b, c-b)} \int_0^1 (1-tz)^{-a} t^{b-1} (1-t)^{c-b-1}\, \mathrm{d}t.$$

When $z = 1$, the extra assumption $a + b - c > 0$ is necessary. In particular,

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}.$$

Proof. See [1, (15.1.20) and (15.3.1)]. □

Lemma 5. *Let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. Then, for all $k > 0$, $a \in \mathbb{R}$ and $0 < b + \frac{1}{k} < c$,*

$$\int_0^\infty \left(\alpha + \frac{\beta}{1+r^k}\right)^{-a} (r^k)^b (1+r^k)^{-c}\, \mathrm{d}r = \frac{1}{k} B\left(b + \frac{1}{k}, c - b - \frac{1}{k}\right) F\left(a, b + \frac{1}{k}; c; \beta\right)$$

When $\beta = 1$, the extra assumption $c > a + b + \frac{1}{k}$ is required.

Proof. Successive substitutions with $s = r^k$, $t = \frac{1}{1+s}$ and $u = 1 - t$ yield

$$\begin{aligned}
&\int_0^\infty \left(\alpha + \frac{\beta}{1+r^k}\right)^{-a} (r^k)^b (1+r^k)^{-c}\, \mathrm{d}r \\
&\quad= \frac{1}{k} \int_0^\infty \left(\alpha + \frac{\beta}{1+s}\right)^{-a} s^{b+\frac{1}{k}-1} (1+s)^{-c}\, \mathrm{d}s \\
&\quad= \frac{1}{k} \int_0^1 (\alpha + \beta t)^{-a} (1-t)^{b+\frac{1}{k}-1} t^{c-(b+\frac{1}{k})-1}\, \mathrm{d}t \\
&\quad= \frac{1}{k} \int_0^1 (1 - \beta u)^{-a} u^{b+\frac{1}{k}-1} (1-u)^{c-(b+\frac{1}{k})-1}\, \mathrm{d}t \\
&\quad= \frac{1}{k} B\left(b + \frac{1}{k}, c - b - \frac{1}{k}\right) F\left(a, b + \frac{1}{k}; c; \beta\right).
\end{aligned}$$

For the last equality, we used Lemma 4. □

Lemma 6. *For all $k > 0$, $a \in \mathbb{R}$, $b > -\frac{1}{k}$ and $c > a + b + \frac{1}{k}$,*

$$\int_0^\infty \left(\frac{1}{1+r^k}\right)^{-a} (r^k)^b (1+r^k)^{-c}\, \mathrm{d}r = \frac{1}{k}\, B\left(c - a - b - \frac{1}{k}, b + \frac{1}{k}\right).$$

Proof. Use Lemma 5 with $\beta = 1$ and the identity $F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$,

$$\begin{aligned}
&\int_0^\infty \left(\alpha + \frac{\beta}{1+r^k}\right)^{-a} (r^k)^b (1+r^k)^{-c}\,\mathrm{d}r \\
&\quad = \frac{1}{k}\frac{\Gamma(b+\frac{1}{k})\Gamma(c-(b+\frac{1}{k}))}{\Gamma(c)}\frac{\Gamma(c)\Gamma(c-a-(b+\frac{1}{k}))}{\Gamma(c-a)\Gamma(c-(b+\frac{1}{k}))} \\
&\quad = \frac{1}{k}\Gamma\left(b+\frac{1}{k}\right)\frac{\Gamma(c-a-(b+\frac{1}{k}))}{\Gamma(c-a)} \\
&\quad = \frac{1}{k}B\left(c-a-b-\frac{1}{k}, b+\frac{1}{k}\right).
\end{aligned}$$

□

Lemma 7. *Let $\alpha, k > 0$, $q > -\frac{1}{k}$ and $p > \frac{q+1}{k}$,*

$$\int_0^\infty \left(\frac{1}{\alpha + r^k}\right)^p r^q\,\mathrm{d}r = \frac{1}{k}\alpha^{\frac{q+1}{k}-p} B\left(\frac{q+1}{k}, p - \frac{q+1}{k}\right).$$

Proof. Substitution by $s = \frac{r^k}{\alpha}$ implies

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{\alpha + r^k}\right)^p r^q\,\mathrm{d}r &= \alpha^{-p+\frac{q}{k}} \int_0^\infty \left(\frac{1}{1+\frac{r^k}{\alpha}}\right)^p \left(\frac{r^k}{\alpha}\right)^{\frac{q}{k}}\,\mathrm{d}r \\
&= \frac{1}{k}\alpha^{-p+\frac{q}{k}+\frac{1}{k}} \int_0^\infty \left(\frac{1}{1+s}\right)^p s^{\frac{q}{k}-1+\frac{1}{k}}\,\mathrm{d}s \\
&= \frac{1}{k}\alpha^{\frac{q+1}{k}-p} B\left(\frac{q+1}{k}, p - \frac{q+1}{k}\right).
\end{aligned}$$

□

Lemma 8. *Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. Then, for all $k > 0$, $a \in \mathbb{R}$, $b > -\frac{1}{k}$ and $c > -1$,*

$$\int_0^1 \left(\alpha + \beta r^k\right)^{-a} (r^k)^b (1-r^k)^c\,\mathrm{d}r = \frac{1}{k}B\left(b+\frac{1}{k}, c+1\right) F\left(a, c+1; c+b+\frac{1}{k}+1; \beta\right).$$

When $\beta = 1$, the extra assumption $b + \frac{1}{k} > a$ is necessary.

Proof. Substitution with $s = r^k$ yields

$$\begin{aligned}
&\int_0^1 \left(\alpha + \beta r^k\right)^{-a} (r^k)^b (1-r^k)^c\,\mathrm{d}r \\
&\quad = \frac{1}{k}\int_0^1 (\alpha + \beta s)^{-a} s^{b+\frac{1}{k}-1}(1-s)^c\,\mathrm{d}s \\
&\quad = \frac{1}{k}B\left(b+\frac{1}{k}, c+1\right) F\left(a, c+1; c+b+\frac{1}{k}+1; \beta\right).
\end{aligned}$$

For the last equality, we used a substitution by $t = 1 - s$. □

Lemma 9. *Let $k > 0$, $b > -\frac{1}{k}$ and $a > -1$. Then,*

$$\int_0^1 (1-t^k)^a (t^k)^b\,\mathrm{d}t = \frac{1}{k}B\left(a+1, b+\frac{1}{k}\right).$$

Proof. Use Lemma 8 with $\alpha = 0$. □

4 Integration over subspaces

In the following, we denote by $\mathcal{L}_p^d$ the space of p-dimensional linear subspaces of $\mathbb{R}^d$ with the element of rotation invariant measure $\mathrm{d}L_p^d$.

Lemma 10. *Let $B_p \in \mathcal{L}_p^d$ and let*

$$\begin{aligned} f \colon B_p \setminus \{0\} &\to S^{p-1}(B_p) \\ x &\mapsto \frac{x}{|x|}. \end{aligned}$$

The $(p-1)$-dimensional Jacobian of f is given by

$$J_{p-1} f(x) = |x|^{-(p-1)}.$$

Proof. Use the proof of [5, Proposition 2.8] with a suitable basis. □

Proposition 11. *Let $B_p \in \mathcal{L}_p^d$ and let $g \colon B_p \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int_{B_p} g(x)|x|^{-(p-1)} \,\mathrm{d}x^p = \int_{S^{p-1}(B_p)} \int_{f^{-1}(x)} g(y) \,\mathrm{d}y^1 \,\mathrm{d}x^{p-1},$$

where f is the mapping from Proposition 10.

Proof. The statement is equivalent to [5, Proposition 2.8] with a suitable basis. Alternatively, use Lemma 10 and apply the coarea formula. □

Lemma 12. *Let $B_1 = \operatorname{span}\{b\}$ be a 1-dimensional subspace of $\mathbb{R}^d$ and let $g \colon \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int_{B_1} g(|x|) \,\mathrm{d}x^1 = 2 \int_0^\infty g(r) \,\mathrm{d}r.$$

Proof. By using the co-area formula with $f \colon \mathbb{R}^d \to \mathbb{R}$, $f(x) = |x|^2 = x \cdot x$ and

$$J_1 f(x) = \sqrt{\det(Df(x)Df(x)^t)} = \sqrt{4|x|^2} = 2|x|,$$

(because $k = n = 1$) we get

$$\int_{B_1} g(|x|) \,\mathrm{d}x^1 = \int_0^\infty \int_{B_1 \cap f^{-1}(r)} \frac{g(|x|)}{J_1 f(x)} \,\mathrm{d}x \,\mathrm{d}r = 2 \int_0^\infty \frac{g(\sqrt{r})}{J_1 f(\sqrt{r} b)} \,\mathrm{d}r = 2 \int_0^\infty \frac{g(\sqrt{r})}{2\sqrt{r}} \,\mathrm{d}r.$$

Substitution by $s = \sqrt{r}$ completes the proof. □

Proposition 13. *Let $B_p \in \mathcal{L}_p^d$ and let $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-negative measurable function. Then,*

$$\int_{B_p} g(|x|) \,\mathrm{d}x^p = \sigma_p \int_0^\infty g(t) t^{p-1} \,\mathrm{d}t.$$

Proof. Combine Lemma 12 and Proposition 11,

$$\int_{B_p} g(|x|) \,\mathrm{d}x^p = \int_{S^{p-1}(B_p)} \mathrm{d}x^{p-1} \int_0^\infty g(t) t^{p-1} \,\mathrm{d}t = \sigma_p \int_0^\infty g(t) t^{p-1} \,\mathrm{d}t.$$

□

Proposition 14. *Let $B_p \in \mathcal{L}_p^d$ and let $g\colon B_p \to \mathbb{R}_+$ be given by*

$$g(x) = \left(\alpha + \frac{\beta}{1+|x|^m}\right)^{-a} (|x|^m)^b (1+|x|^m)^{-c}$$

for some $m > 0$ and $a, b, c \in \mathbb{R}$ with $c > b + \frac{p}{m} > 0$. Then,

$$\int_{B_p} g(x)\,\mathrm{d}x^p = \frac{\sigma_p}{m} B\left(b + \frac{p}{m}, c - b - \frac{p}{m}\right) F\left(a, b + \frac{p}{m}; c; \beta\right).$$

When $\beta = 1$, the additional condition $c - a - b - \frac{1}{k} > 0$ is necessary.

Proof. By combining Lemma 5 and Proposition 13, we obtain

$$\begin{aligned}\int_{B_p} g(x)\,\mathrm{d}x^p &= \sigma_p \int_0^\infty \left(\alpha + \frac{\beta}{1+t^m}\right)^{-a} (1+t^m)^{-c}(t^m)^b (t^m)^{\frac{p}{m}-\frac{1}{m}}\,\mathrm{d}t \\ &= \frac{\sigma_p}{m} B\left(b + \frac{p}{m}, c - b - \frac{p}{m}\right) F\left(a, b + \frac{p}{m}; c; \beta\right).\end{aligned}$$

□

Proposition 15. *Let $B_p \in \mathcal{L}_p^d$. Then, for all $m > 0$ and $k > \frac{p}{m}$,*

$$\int_{B_p} \left(\frac{1}{1+|x|^m}\right)^k \mathrm{d}x^p = \frac{\sigma_p}{m} B\left(k - \frac{p}{m}, \frac{p}{m}\right).$$

Proof. Apply Proposition 13 and Lemma 7,

$$\begin{aligned}\int_{B_p} \left(\frac{1}{1+|x|^m}\right)^k \mathrm{d}x^p &= \sigma_p \int_0^\infty \left(\frac{1}{1+t^m}\right)^k t^{p-1}\,\mathrm{d}t \\ &= \frac{\sigma_p}{m} B\left(k - \frac{p}{m}, \frac{p}{m}\right).\end{aligned}$$

□

5 Integration over spheres

For any p-dimensional linear subspace B_p of $\mathbb{R}^d$, the orthogonal projection of $x \in \mathbb{R}^d$ onto B_p is denoted by $p(x|B_p)$. By $\pi(x|B_p)$ we shall denote the *spherical* projection of x onto B_p, i.e. $\pi(x|B_p) = \frac{p(x|B_p)}{|p(x|B_p)|}$. Furthermore, for any q-dimensional subspace B_q of $\mathbb{R}^d$, we define the $\mathcal{G}$-function as

$$\mathcal{G}(B_p, B_q) = \begin{cases} \sqrt{\det(A^t A)} & \text{if } q < d \text{ and } \dim(B_p \cap (B_q \cap B_p)^\perp) = d - q \\ 1 & \text{if } q = d \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Here, $A = [p(b_1|B_q^\perp), \dots, p(b_{d-q}|B_q^\perp)]$, where $b_1, \dots, b_{d-q}$ is an orthonormal basis of the linear subspace $B_p \cap (B_q \cap B_p)^\perp$. The $\mathcal{G}$-function is symmetric in its arguments and can be regarded as a generalized sinus of the angle between the subspaces B_p and B_q. For any subspace B of $\mathbb{R}^d$, $\mathcal{G}^{(B)}$ is the function $\mathcal{G}$ considered relatively in B.

Proposition 16. *Let $e \in S^{d-1}$. Then, the mapping*

$$\begin{aligned} f\colon S^{d-1} &\to [0,1] \\ x &\mapsto (e\cdot x)^2 \end{aligned}$$

has the following 1*-dimensional Jacobian*

$$J_1 f(x; S^{d-1}) = 2|e\cdot x|\sqrt{1-(e\cdot x)^2}.$$

Proof. See the proof of [5, Proposition 2.11]. □

Proposition 17. *Let $e \in S^{d-1}$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int_{S^{d-1}} g((e\cdot x)^2)\, \mathrm{d}x^{d-1} = \sigma_{d-1} \int_0^1 g(t) t^{-\frac{1}{2}}(1-t)^{\frac{d-1}{2}-1} dt.$$

Proof. See [5, Proposition 2.11]. □

Proposition 18. *Let $e \in S^{d-1}$and let $g\colon \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function. Then, for all $k > 0$,*

$$\int_{S^{d-1}} g(|e\cdot x|^k)\, \mathrm{d}x^{d-1} = \sigma_{d-1} \int_0^1 g(t^{\frac{k}{2}}) t^{-\frac{1}{2}}(1-t)^{\frac{d-1}{2}-1} dt.$$

Proof. See Proposition 17. □

Proposition 19. *Let $e \in S^{d-1}$ and let $n \in \mathbb{N}$.Then,*

$$\int_{S^{d-1}} |e\cdot x|^n\, \mathrm{d}x^{d-1} = B\left(\frac{n+1}{2}, \frac{d-1}{2}\right)\sigma_{d-1}.$$

Proof. Use Proposition 18,

$$\begin{aligned} \int_{S^{d-1}} |e\cdot x|^n\, \mathrm{d}x^{d-1} &= \sigma_{d-1}\int_0^1 t^{\frac{n+1}{2}-1}(1-t)^{\frac{d-1}{2}-1} dt \\ &= \sigma_{d-1} B\left(\frac{n+1}{2}, \frac{d-1}{2}\right). \end{aligned}$$

□

Proposition 20. *Let $e \in S^{d-1}$. Then,*

$$\int_{S^{d-1}} (e\cdot x)^2\, \mathrm{d}x^{d-1} = \omega_d.$$

Proof. Use Proposition 19 with $n = 2$. □

Proposition 21. *Let $e \in S^{d-1}$, $\beta \in [-1, 1]$ and $a \in \mathbb{R}$. Then,*

$$\int_{S^{d-1}} \left(1-\beta(e\cdot x)^2\right)^{-a}\, \mathrm{d}x^{d-1} = \sigma_{d-1} B\left(\frac{1}{2}, \frac{d-1}{2}\right) F\left(a, \frac{1}{2}; \frac{d}{2}; \beta\right).$$

Proof. Use Proposition 17 with $g(t) = (1 - \beta t)^{-a}$,

$$\begin{aligned}
\int_{S^{d-1}} g((e \cdot x)^2) \, \mathrm{d}x^{d-1} &= \sigma_{d-1} \int_0^1 g(t) t^{-\frac{1}{2}} (1-t)^{\frac{d-1}{2}-1} \, \mathrm{d}t \\
&= \sigma_{d-1} \int_0^1 (1-\beta t)^{-a} t^{\frac{1}{2}-1} (1-t)^{\frac{d}{2}-\frac{1}{2}-1} \, \mathrm{d}t \\
&= \sigma_{d-1} B\left(\frac{1}{2}, \frac{d-1}{2}\right) F\left(a, \frac{1}{2}; \frac{d}{2}; \beta\right).
\end{aligned}$$

$\square$

Lemma 22. *Let $h \colon S^{d-1} \to \mathbb{R}$ be a non-negative measurable function and let $y \in S^{d-1}$. Then,*

$$\int_{S^{d-1}} h(\omega) \, \mathrm{d}\omega^{d-1} = \int_{-1}^1 \int_{S^{d-2}(y^\perp)} \sqrt{1-t^2}^{d-3} h(ty + \sqrt{1-t^2}\omega) \, \mathrm{d}\omega^{d-2} \, \mathrm{d}t.$$

Proof. The mapping

$$\begin{aligned}
f \colon S^{d-1} &\to [-1, 1] \\
\omega &\mapsto \omega \cdot y = t
\end{aligned}$$

has 1-dimensional Jacobian $J_1 f(\omega) = \sqrt{1 - (\omega \cdot y)^2}$ for all $\omega \in S^{d-1} \setminus \{y\}$. The coarea formula implies

$$\int_{S^{d-1}} h(\omega) \, \mathrm{d}\omega^{d-1} = \int_{-1}^1 \int_{S^{d-1} \cap f^{-1}(t)} \frac{1}{\sqrt{1-t^2}} h(\omega) \, \mathrm{d}\omega^{d-2} \, \mathrm{d}t.$$

An application of the area formula with the injective mapping (whenever $t \in (-1, 1)$)

$$\begin{aligned}
g \colon S^{d-1} \cap f^{-1}(t) &\to S^{d-2}(y^\perp) \\
\omega &\mapsto \pi(\omega | y^\perp),
\end{aligned}$$

with $(d-2)$-dimensional Jacobian $J_{d-2} g(\omega) = \sqrt{1 - (\omega \cdot y)^2}^{2-d}$, yields

$$\int_{S^{d-1}} h(\omega) \, \mathrm{d}\omega^{d-1} = \int_{-1}^1 \int_{S^{d-2}(y^\perp)} \sqrt{1-t^2}^{d-3} h(ty + \sqrt{1-t^2}\omega) \, \mathrm{d}\omega^{d-2} \, \mathrm{d}t.$$

$\square$

Lemma 23. *Let $B_p \in \mathcal{L}_p^d$ and let $x_1 \in B_p^\perp$. The mapping*

$$\begin{aligned}
\xi \colon B_p &\to S^p(B_p \oplus \operatorname{span}(x_1)) \\
x &\mapsto \frac{x_1 + x}{|x_1 + x|}
\end{aligned}$$

has p-dimensional Jacobian

$$J_p \xi(x) = \frac{|x_1|}{(|x_1|^2 + |x|^2)^{\frac{p+1}{2}}}.$$

Proof. The kernel of

$$D\xi(x) = \frac{p(\,\cdot\,|\mathrm{span}(x_1+x)^\perp)}{|x_1+x|}$$

is

$$\ker D\xi(x) = \mathrm{span}(x_1+x) \nsubseteq B_p.$$

Since $\mathrm{Tan}(x,B_p) = B_p$, the orthogonal complement of $\ker D\xi(x)$ in $\mathrm{Tan}(B_p,x)$ is

$$\mathrm{Tan} \cap (\ker \cap \mathrm{Tan})^\perp = B_p \cap (\mathrm{span}(x_1+x)\cap B_p)^\perp = B_p.$$

Let $e_1,\dots,e_p$ be an orthonormal basis of B_p such that $e_1 = \pi(x_1+x|B_p) = x/|x|$ and $e_2,\dots,e_p \perp (x_1+x)$. Then, by Definition 1,

$$\begin{aligned} D\xi(x,B_p) &= \begin{pmatrix} e_1^* \\ \vdots \\ e_{d-p}^* \end{pmatrix}^* D\xi(x)^* = \frac{1}{|x_1+x|}\begin{pmatrix} p(e_1|\mathrm{span}(x_1+x)^\perp)^* \\ \vdots \\ p(e_p|\mathrm{span}(x_1+x)^\perp)^* \end{pmatrix} \\ &= \left(\frac{1}{|x_1|^2+|x|^2}\right)^{\frac12}\begin{pmatrix} |x|^{-1}p(x|\mathrm{span}(x_1+x)^\perp)^* \\ e_2^* \\ \vdots \\ e_p^* \end{pmatrix}. \end{aligned}$$

Hence,

$$J_p\xi(x)^2 = \left(\frac{1}{|x_1|^2+|x|^2}\right)^p \frac{1}{|x|^2}|p(x|\mathrm{span}(x_1+x)^\perp)|^2.$$

An easy computation yields $\frac{1}{|x|^2}|p(x|\mathrm{span}(x_0+x)^\perp)|^2 = \frac{|x_1|^2}{|x_1|^2+|x|^2}$ and, as a consequence,

$$J_p\xi(x) = \frac{|x_1|}{(|x_1|^2+|x|^2)^{\frac{p+1}{2}}}.$$

□

Lemma 24. *Let $\psi(x) = \pi(x|B_p)$ be defined on S^{d-1}. Then, for all $x_0 \in S^{p-1}(B_p)$ and all $k \geq 0$,*

$$\int_{\psi^{-1}(x_0)} |p(x|B_p)|^k \,\mathrm{d}x^{d-p} = \frac{\sigma_{d-p}}{2}\, B\left(\frac{d-p}{2}, \frac{k+1}{2}\right).$$

Proof. Note that $\psi^{-1}(x_0) = S^{d-p}(\mathrm{span}(x_0)\oplus B_p^\perp)$. Define the mapping

$$\begin{aligned} \xi\colon B_p^\perp &\to \psi^{-1}(x_0) \\ x &\mapsto \frac{x_0+x}{|x_0+x|}. \end{aligned}$$

According to Lemma 23, the $(d-p)$-dimensional Jacobian of ξ is

$$J_{d-p}\xi(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{d-p+1}{2}}.$$

Since ξ is bijective, the area formula implies

$$\begin{aligned}\int_{\psi^{-1}(x_0)} |p(x|B_p)|^k \,\mathrm{d}x^{d-p} &= \int_{\psi^{-1}(x_0)} \sum_{y\in\xi^{-1}(x)} |p(\xi(y)|B_p)|^k \,\mathrm{d}x^{d-p} \\ &= \int_{\psi^{-1}(x_0)} \sum_{y\in\xi^{-1}(x)} \frac{1}{|x_0+y|^k} \,\mathrm{d}x^{d-p} \\ &= \int_{B_p^\perp} \left(\frac{1}{1+|x|^2}\right)^{\frac{k}{2}} \left(\frac{1}{1+|x|^2}\right)^{\frac{d-p+1}{2}} \mathrm{d}x^{d-p} \\ &= \int_{B_p^\perp} \left(\frac{1}{1+|x|^2}\right)^{\frac{k+d-p+1}{2}} \mathrm{d}x^{d-p}.\end{aligned}$$

Using Proposition 15, we conclude that

$$\begin{aligned}\int_{B_p^\perp} \left(\frac{1}{1+|x|^2}\right)^{\frac{k+d-p+1}{2}} \mathrm{d}x^{d-p} &= \frac{\sigma_{d-p}}{2} B\left(\frac{d-p}{2}, \frac{k+d-p+1}{2} - \frac{d-p}{2}\right) \\ &= \frac{\sigma_{d-p}}{2} B\left(\frac{d-p}{2}, \frac{k+1}{2}\right).\end{aligned}$$

□

Lemma 25. *Let A_q and B_p be q- and p-dimensional linear subspaces of $\mathbb{R}^d$, respectively. Define the spherical projection*

$$\begin{aligned}\psi \colon S^{q-1}(A_q) &\to S^{p-1}(B_p) \\ x &\mapsto \pi(x|B_p).\end{aligned}$$

The $(p-1)$-dimensional Jacobian of ψ is given by

$$J_{p-1}\psi(x, S^{q-1}(A_q)) = \frac{\mathcal{G}(A_q \cap x^\perp, (B_p \cap x^\perp)^\perp)}{|p(x|B_p)|^{p-1}}.$$

Note that the Jacobian is non-zero at x only if

$$\dim\left(A_q \cap x^\perp \cap \left[(B_p \cap x^\perp)^\perp \cap (A_q \cap x^\perp)\right]^\perp\right) = p-1$$

or $\dim(B_p \cap x^\perp) = 0$, cf. definition (3).

Proof. The kernel of

$$D\psi(x) = \frac{p(\,\cdot\,|x^\perp \cap B_p)}{|p(x|B_p)|}$$

is

$$\ker D\psi(x) = (x^\perp \cap B_p)^\perp.$$

Since $\mathrm{Tan}(x, S^{d-1}) = x^\perp \cap A_q$, we may define an orthonormal basis

$$\mathrm{Tan} \cap (\ker\ \cap\ \mathrm{Tan})^\perp = \mathrm{span}\{e_1, \ldots, e_{p-1}\},$$

whenever $\dim^{(x^\perp)}(A_q \cap (A_q \cap B_p^\perp)^\perp) = p-1$. Note that the $(p-1)$-dimensional Jacobian of ψ is zero when this dimensional assumption is not satisfied. Applying the differential of ψ to the orthonormal basis $e_1, \ldots, e_{p-1}$, we obtain

$$D\psi(x, S^{q-1}(A_q)) = |p(x|B_p)|^{-1} \begin{pmatrix} p(e_1|x^\perp \cap B_p)^* \\ \vdots \\ p(e_{p-1}|x^\perp \cap B_p)^* \end{pmatrix}$$

and

$$J_{p-1}\psi(x, S^{q-1}(A_q)) = |p(x|B_p)|^{-(p-1)} \mathcal{G}\left(\mathrm{Tan}(x, S^{d-1}), (x^\perp \cap B_p)^\perp\right),$$

cf. (3). Note that the $\mathcal{G}$ function is non-zero only if

$$\dim\left(\mathrm{Tan}(x, S^{d-1}) \cap \left[(x^\perp \cap B_p)^\perp \cap \mathrm{Tan}(x, S^{d-1})\right]^\perp\right) = d - (d-p-1) = p-1$$

when $\dim(x^\perp \cap B_p) > 0$ and $\mathcal{G}$ is equal to 1 whenever $\dim(x^\perp \cap B_p) = 0$. □

Remark 26. In the special case $p = 1$, set $B_1 = \mathrm{span}\{b\}$ and note that the 0-dimensional Jacobian of ψ is given by $J_0\psi(x, S^{q-1}(A_q)) = 1$. A combined application of the coarea formula and Lemma 24 yields

$$\begin{aligned}
\int_{S^{q-1}(A_q)} |x \cdot b|^k \, \mathrm{d}x^{q-1} &= \int_{S^0(B_1)} \int_{\psi^{-1}\{x_0\}} |x \cdot b|^k \, \mathrm{d}x^{q-1} \, \mathrm{d}x_0^0 \\
&= 2\cos^k(b, A_q) \int_{\psi^{-1}\{b\}} |x \cdot \pi(b|A_q)|^k \, \mathrm{d}x^{q-1} \\
&= \sigma_{q-1} B\left(\frac{q-1}{2}, \frac{k+1}{2}\right) \cos^k(b, A_q).
\end{aligned}$$

This identity is also derived in Remark 32 and Proposition 38.

Lemma 27. *Let $B_p \in \mathcal{L}_p^d$ be fixed and define the mapping*

$$\begin{aligned}
\psi \colon S^{d-1} &\to S^{p-1}(B_p) \\
x &\mapsto \pi(x|B_p).
\end{aligned}$$

The $(p-1)$-dimensional Jacobian of ψ is given by

$$J_{p-1}\psi(x, S^{d-1}) = |p(x|B_p)|^{-(p-1)}.$$

Proof. Use Lemma 25 with $A_q = \mathbb{R}^d$. Then [5, Proposition 5.2] implies

$$\mathcal{G}(x^\perp, (B_p \cap x^\perp)^\perp) = |p(x|(B_p \cap x^\perp)^\perp)| = |x| = 1.$$

□

Remark 28. According to Lemma 27 and the co-area formula,

$$\begin{aligned}
\int_{S^{d-1}} |p(x|B_p)|^{k-p+1} \, \mathrm{d}x^{d-1} &= \int_{S^{d-1}} |p(x|B_p)|^k \, J_{p-1}\psi(x) \, \mathrm{d}x^{d-1} \\
&= \int_{S^{p-1}(B_p)} \int_{\psi^{-1}(x_0)} |p(x|B_p)|^k \, \mathrm{d}x^{d-p} \, \mathrm{d}x_0^{p-1}.
\end{aligned}$$

An application of Lemma 24 to the inner integral implies

$$\int_{S^{d-1}} |p(x|B_p)|^{k-p+1}\,\mathrm{d}x^{d-1} = \frac{\sigma_p\,\sigma_{d-p}}{2} B\left(\frac{d-p}{2}, \frac{k+1}{2}\right).$$

Proposition 38 yields the same result.

Lemma 29. *Let $B_p \in \mathcal{L}_p^d$ and $m \in S^{d-1}$ such that $m \notin B_p$. Define*

$$\begin{aligned} \psi\colon S^{d-2}(m^\perp) &\to S^{p-1}(B_p) \\ x &\mapsto \pi(x|B_p) = x_0. \end{aligned}$$

Then, for almost all $x \in S^{d-2}(m^\perp)$, the $(p-1)$-dimensional Jacobian of ψ is

$$J_{p-1}\psi(x) = \frac{\sqrt{\sin^2\theta + \cos^2\theta\,\cos^2(x_0, m_0)}}{|p(x|B_p)|^{p-1}},$$

where $\theta = \angle(m, B_p)$, $x_0 = \pi(u|B_p)$ and $m_0 = \pi(m|B_p)$ (set $\angle(x_0, m_0) = 0$ when $m \perp B_p$, in accordance with Lemma 27).

Proof. Use Lemma 25 with $A_q = m^\perp$. The $(p-1)$-dimensional Jacobian of ψ is given by

$$J_{p-1}\psi(x, S^{d-2}(m^\perp)) = \frac{\mathcal{G}(m^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp)}{|p(x|B_p)|^{p-1}}.$$

Applying [6, Lemma 4.1], [5, Proposition 5.2] and $x \perp m$, we obtain

$$\begin{aligned} \mathcal{G}(m^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp)^2 &= \mathcal{G}^{(x^\perp)}(m^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp \cap x^\perp)^2 \\ &= |p(\pi(m|x^\perp)|(B_p \cap x^\perp)^\perp \cap x^\perp)|^2 \\ &= |p(m|(B_p \cap x^\perp)^\perp)|^2 \\ &= 1 - |p(m|B_p \cap x^\perp)|^2 \\ &= 1 - |p(m|B_p)|^2 |p(\pi(m|B_p)|B_p \cap x^\perp)|^2 \\ &= 1 - |p(m|B_p)|^2 (1 - |p(\pi(m|B_p)|\pi(x|B_p))|^2) \\ &= (1 - |p(m|B_p)|^2) + |p(m|B_p)|^2 |p(\pi(m|B_p)|\pi(x|B_p))|^2 \\ &= \sin^2\theta + \cos^2\theta\cos^2(x_0, m_0). \end{aligned}$$

□

Lemma 30. *Let $B_p \in \mathcal{L}_p^d$ and $A_q \in \mathcal{L}_q^d$ such that $A_q \cap (B_p^\perp \cap A_q)^\perp = p$. Define*

$$\begin{aligned} \psi\colon S^{q-1}(A_q) &\to S^{p-1}(B_p) \\ x &\mapsto \pi(x|B_p) = x_0. \end{aligned}$$

Let $a_1, \ldots, a_{d-q}$ be an orthonormal basis of $A_q^\perp$. Then, for almost all $x \in S^{q-1}(A_q)$, the $(p-1)$-dimensional Jacobian of ψ is

$$J_{p-1}\psi(x) = \frac{\prod_{i=1}^{d-q} \sqrt{\sin^2\theta_i + \cos^2\theta_i \cos^2(\pi(x|B_p), \pi(a_i|B_p))}}{|p(u|B_p)|^{p-1}},$$

where $\theta_i = \angle(a_i, B_p)$ (set $\angle(\pi(x|B_p), \pi(a_i|B_p)) = 0$ when $a_i \perp B_p$, in accordance with Lemma 29.

Proof. Using Lemma 25 with $A_q = a_1^\perp \cap \dots \cap a_{d-q}^\perp$, the Jacobian of ψ can be calculated as

$$J\psi(x, S^{q-1}(A_q)) = \frac{\mathcal{G}(\cap_{i=1}^{d-q} a_i^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp)}{|p(x|B_p)|^{p-1}} = \frac{\mathcal{G}^{(x^\perp)}(\cap_{i=1}^{d-q} a_i^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp \cap x^\perp)}{|p(x|B_p)|^{p-1}}.$$

Without loss of generality, assume that the a_is are orthogonal to neither $(B_p \cap x^\perp)^\perp$ nor to its complement. According to [6, Lemma 4.1], we have

$$\begin{aligned}
&\mathcal{G}^{(x^\perp)}(\cap_{i=1}^{d-q} a_i^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp \cap x^\perp) \\
&\quad = \mathcal{G}^{(x^\perp \cap a_{d-q}^\perp)}\left(\cap_{i=1}^{d-q} a_i^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp \cap a_{d-q}^\perp \cap x^\perp\right) |p(a_{d-q}|(B_p \cap x^\perp)^\perp \cap x^\perp)| \\
&\quad = \mathcal{G}^{(x^\perp \cap a_{d-q}^\perp)}\left(\cap_{i=1}^{d-q} a_i^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp \cap a_{d-q}^\perp \cap x^\perp\right) |p(a_{d-q}|(B_p \cap x^\perp)^\perp)|.
\end{aligned}$$

Using that $x \perp a_i$, the same procedure as in the proof of Lemma 29 yields

$$\begin{aligned}
&\mathcal{G}^{(x^\perp)}(\cap_{i=1}^{d-q} a_i^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp \cap x^\perp) \\
&\quad = \mathcal{G}^{(\cap_{i=2}^{d-q} a_i^\perp \cap x^\perp)}(\cap_{i=1}^{d-q} a_i^\perp \cap x^\perp, (B_p \cap x^\perp)^\perp \cap x^\perp \cap_{i=2}^{d-1} a_i^\perp) \\
&\qquad \times \prod_{i=2}^{d-q} \sqrt{\sin^2\theta_i + \cos^2\theta_i \cos^2(x_0, a_i^0)} \\
&\quad = \prod_{i=1}^{d-q} \sqrt{\sin^2\theta_i + \cos^2\theta_i \cos^2(x_0, a_i^0)}.
\end{aligned}$$

$\square$

Proposition 31. *Let B_p be a p-dimensional subspace of $\mathbb{R}^d$ and let A_q be a q-dimensional subspace of $\mathbb{R}^d$ such that $A_q \cap (A_q \cap B_p^\perp)^\perp = p$. Moreover, let $a_1, \dots, a_{d-q}$ be an orthonormal basis for $A_q^\perp$ and define the $(d-q) \times (d-q)$-matrix A as*

$$A_{i,j} = \langle \pi(a_j|B_p^\perp), a_i \rangle$$

Then,

$$\begin{aligned}
\int_{S^{q-1}(A_q)} |p(x|B_p)|^k \,\mathrm{d}x^{q-1} &= \frac{\sigma_{q-p}}{2} B\left(\frac{q-p}{2}, \frac{p+k}{2}\right) \\
&\quad \times \int_{S^{p-1}(B_p)} \frac{\left(\frac{1}{1+\beta^2}\right)^{(p+k)/2}}{\prod_{i=1}^{d-q} \sqrt{\sin^2\theta_i + \cos^2\theta_i \cos^2(x_0, \pi(a_i|B_p))}} \,\mathrm{d}x_0^{p-1}.
\end{aligned}$$

Here,

$$\beta = |\alpha_1 \pi(a_1|B_p^\perp) + \ldots + \alpha_{d-q} \pi(a_{d-q}|B_p^\perp)|,$$

where $\alpha = (\alpha_1, \dots, \alpha_{d-q}) \in \mathbb{R}^{d-q}$ solves the system of equations

$$A(\alpha_1, \dots, \alpha_{d-q}) = (-x_0 \cdot a_1, -x_0 \cdot a_2, \dots, -x_0 \cdot a_{d-q}).$$

Proof. Define the mapping

$$\begin{aligned}
\psi \colon S^{q-1}(A_q) &\to S^{p-1}(B_p) \\
x &\mapsto \pi(x|B_p) = x_0
\end{aligned}$$

and apply the coarea formula together with Lemma 30,

$$\begin{aligned}
&\int_{S^{q-1}(A_q)} |p(x|B_p)|^k \,\mathrm{d}x^{q-1} \\
&\quad = \int_{S^{p-1}(B_p)} \int_{\psi^{-1}(x_0)} |p(x|B_p)|^k J_{p-1}\psi(x, S^{q-1}(A_q))^{-1} \,\mathrm{d}x^{q-p} \,\mathrm{d}x_0^{p-1} \\
&\quad = \int_{S^{p-1}(B_p)} \int_{\psi^{-1}(x_0)} |p(x|B_p)|^{k+p-1} \,\mathrm{d}x^{q-p} \frac{\mathrm{d}x_0^{p-1}}{\prod_{i=1}^{d-q} \sqrt{\sin^2\theta_i + \cos^2\theta_i \cos^2(x_0, a_i^0)}}, \qquad (4)
\end{aligned}$$

where $a_1, \dots, a_{d-q}$ is an orthonormal basis of $A_q^\perp$ and $a_i^0 = \pi(a_i|B_p)$, $i = 1, \dots, d-q$. Note that $A_q \cap (A_q \cap B_p^\perp)^\perp = p$ implies $A_q \cap B_p^\perp = q - p$ and define the bijective mapping

$$\begin{aligned}
\xi\colon A_q \cap B_p^\perp &\to \psi^{-1}(x_0) \\
\omega &\mapsto \frac{x_0 + \beta x_1 + \omega}{|x_0 + \beta x_1 + \omega|},
\end{aligned}$$

where $\beta \in \mathbb{R}$ and $x_1 = x_1(x_0) \in S^{d-p-q}\left(p(A_q^\perp|B_p^\perp)\right)$ are uniquely chosen s.t. $x_0 + \beta x_1 \in A_q$ (see below for a more explicit expression). According to Lemma 23, the $(q-p)$-dimensional Jacobian of ξ is given by

$$J_{q-p}\xi(\omega) = \frac{\sqrt{1+\beta^2}}{(1+\beta^2+|\omega|^2)^{\frac{q-p+1}{2}}}.$$

Hence, the area formula yields

$$\begin{aligned}
&\int_{\psi^{-1}(x_0)} |p(x|B_p)|^{k+p-1} \,\mathrm{d}x^{q-p} \\
&\quad = \int_{\psi^{-1}(x_0)} \sum_{y \in \xi^{-1}(x)} |p(\xi(y)|B_p)|^{k+p-1} \,\mathrm{d}x^{q-p} \\
&\quad = \int_{\psi^{-1}(x_0)} \sum_{y \in \xi^{-1}(x)} \frac{1}{|x_0 + \beta x_1 + y|^{k+p-1}} \,\mathrm{d}x^{q-p} \\
&\quad = \int_{A_q \cap B_p^\perp} \frac{1}{(1+\beta^2+|x|^2)^{\frac{k+p-1}{2}}} \frac{\sqrt{1+\beta^2}}{(1+\beta^2+|x|^2)^{\frac{q-p+1}{2}}} \,\mathrm{d}x^{q-p} \\
&\quad = \int_{A_q \cap B_p^\perp} \frac{\sqrt{1+\beta^2}}{(1+\beta^2+|x|^2)^{\frac{q+k}{2}}} \,\mathrm{d}x^{q-p}. \qquad (5)
\end{aligned}$$

According to Proposition 13 and Lemma 7,

$$\begin{aligned}
\int_{A_q \cap B_p^\perp} \frac{\sqrt{1+\beta^2}}{(1+\beta^2+|x|^2)^{\frac{q+k}{2}}} \,\mathrm{d}x^{q-p} &= \sigma_{q-p} \int_0^\infty \frac{\sqrt{1+\beta^2}}{(1+\beta^2+r^2)^{\frac{q+k}{2}}} r^{q-p-1} \,\mathrm{d}r \\
&= \frac{\sigma_{q-p}}{2} \left(\frac{1}{1+\beta^2}\right)^{(p+k-1)/2} B\left(\frac{q-p}{2}, \frac{p+k}{2}\right). \qquad (6)
\end{aligned}$$

Then, by combining (4), (5) and (6),

$$\int_{S^{q-1}(A_q)} |p(x|B_p)|^k \, \mathrm{d}x^{q-1}$$
$$= \int_{S^{p-1}(B_p)} \int_{\psi^{-1}(x_0)} |p(x|B_p)|^{k+p-1} \, \mathrm{d}x^{q-p} \frac{\mathrm{d}x_0^{p-1}}{\prod_{i=1}^{d-q} \sqrt{\sin^2\theta_i + \cos^2\theta_i \cos^2(x_0, a_i^0)}}$$
$$= \frac{\sigma_{q-p}}{2} B\left(\frac{q-p}{2}, \frac{p+k}{2}\right) \int_{S^{p-1}(B_p)} \frac{\left(\frac{1}{1+\beta^2}\right)^{(p+k-1)/2}}{\prod_{i=1}^{d-q} \sqrt{\sin^2\theta_i + \cos^2\theta_i \cos^2(x_0, a_i^0)}} \, \mathrm{d}x_0^{p-1}.$$

We need to express β and x_1 in a more explicit way. Define the $(d-q) \times (d-q)$ matrix A by

$$A_{i,j} = \pi(a_j|B_p^\perp) \cdot a_i.$$

Let $a_1, \ldots, a_{d-1}$ be an orthonormal basis for $A_q^\perp$. The orthogonal decomposition of $\mathbb{R}^d$ into $B_p \oplus B_p^\perp$ ensures the existence of scalars $\alpha_1, \ldots, \alpha_{d-q}$ such that

$$x_0 + \sum_{j=1}^{d-q} \alpha_j \pi(a_j|B_p^\perp) \in A_q. \tag{7}$$

In other words, $\alpha_1, \ldots, \alpha_{d-q}$ solve the following system of equations

$$\begin{aligned}
0 &= x_0 \cdot a_1 + \sum_{j=1}^{d-q} \alpha_j \pi(a_j|B_p^\perp) \cdot a_1 \\
0 &= x_0 \cdot a_2 + \sum_{j=1}^{d-q} \alpha_j \pi(a_j|B_p^\perp) \cdot a_2 \\
&\vdots \\
0 &= x_0 \cdot a_{d-q} + \sum_{j=1}^{d-q} \alpha_j \pi(a_j|B_p^\perp) \cdot a_{d-q}.
\end{aligned}$$

Whenever A is invertible, the solution to this system is given by

$$\alpha = A^{-1}(-x_0 \cdot a_1, -x_0 \cdot a_2, \ldots, -x_0 \cdot a_{d-q}).$$

Now, set

$$\beta = |\alpha_1 \pi(a_1|B_p^\perp) + \ldots + \alpha_{d-q}\pi(a_{d-q}|B_p^\perp)|$$

and

$$x_1 = \frac{\alpha_1 \pi(a_1|B_p^\perp) + \ldots + \alpha_{d-q}\pi(a_{d-q}|B_p^\perp)}{|\alpha_1 \pi(a_1|B_p^\perp) + \ldots + \alpha_{d-q}\pi(a_{d-q}|B_p^\perp)|}.$$

Hence, (7) implies $x_0 + \beta x_1 \in A_q$ and the mapping ξ is well-defined. □

Remark 32. Let us consider Proposition 31 in the special case $p = 1$. Let $B_1 = \mathrm{span}(b)$. Then,

$$A_{ij} = \pi(a_j|b^\perp) \cdot a_i = \frac{a_i \cdot a_j - (a_j \cdot b)(a_i \cdot b)}{\sqrt{1 - (a_j \cdot b)}} = \begin{cases} \frac{1-(a_i \cdot b)^2}{\sqrt{1-(a_i \cdot b)^2}}, & j = i \\ \frac{-(a_i \cdot b)^2}{\sqrt{1-(a_i \cdot b)^2}}, & j \neq i. \end{cases}$$

Since $x_0 = \pi(x|b) = b$, the vector $\alpha = (\alpha_1, \dots, \alpha_{d-q})$ is a solution to the system of $d-q$ linear equations

$$A_1 \cdot \alpha_1 = \frac{\alpha_1}{\sqrt{1-(a_1\cdot b)^2}} - (a_1 \cdot b) \sum_{j=1}^{d-q} \frac{\alpha_j (a_j \cdot b)}{\sqrt{1-(a_j \cdot b)^2}} = -a_1 \cdot b$$

$$\vdots$$

$$A_{d-q} \cdot \alpha_{d-q} = \frac{\alpha_{d-q}}{\sqrt{1-(a_{d-q}\cdot b)^2}} - (a_{d-q} \cdot b) \sum_{j=1}^{d-q} \frac{\alpha_j (a_j \cdot b)}{\sqrt{1-(a_j \cdot b)^2}} = -a_{d-q} \cdot b,$$

where A_i is the ith row of the matrix A. One can easily check that

$$\alpha = \left(- \frac{(a_j \cdot b)\sqrt{1-(a_j\cdot b)^2}}{1 - \sum_i (a_i \cdot b)^2} \right)_{j=1,\dots,d-q}$$

is a solution to that system. Hence, for all $j = 1, \dots, d-q$,

$$\alpha_j \, \pi(a_j | b^\perp) = \alpha_j \frac{a_j - (a_j \cdot b) b}{\sqrt{1-(a_j\cdot b)^2}}$$

and

$$\beta^2 = \Big| \sum_{j=1}^{d-q} \alpha_j \, \pi(a_j|b) \Big|^2 = \left(\frac{1}{1 - \sum_i (a_i \cdot b)^2} \right)^2 \Big| \sum_j (a_j \cdot b) a_j - b \sum_j (a_j \cdot b)^2 \Big|^2.$$

Noting that $\sum_i (a_i \cdot b)^2 = \cos^2(b, A_q^\perp)$, we deduce that

$$\beta^2 = \frac{1}{\sin^4(b, A_q^\perp)} \Big(\cos^2(b, A_q^\perp) + \cos^4(b, A_q^\perp) - 2\cos^2(b, A_q^\perp)\cos^2(b, A_q^\perp) \Big) = \frac{\cos^2(b, A_q^\perp)}{\sin^2(b, A_q^\perp)},$$

or in other terms,

$$\frac{1}{1+\beta^2} = \sin^2(b, A_q^\perp).$$

Having computed the nominator on the right-hand side of Proposition 31, we note that the denominator is equal to 1 since $\cos^2(x_0, \pi(a_i|B_1)) = \cos^2(b,b) = 1$. Thus,

$$\int_{S^{q-1}(A_q)} |x \cdot b|^k \, \mathrm{d}x^{q-1} = \sigma_{q-1} B\left(\frac{q-1}{2}, \frac{1+k}{2} \right) \sin^k(b, A_q^\perp). \tag{8}$$

Notice that this identity can also be derived using Proposition 38 below, with $d = q$ and $p = 1$,

$$\begin{aligned} \int_{S^{q-1}(A_q)} |x \cdot b|^k \, \mathrm{d}x^{q-1} &= \cos^k(b, A_q) \int_{S^{q-1}(A_q)} |x \cdot \pi(b|A_q)|^k \, \mathrm{d}x^{q-1} \\ &= \sigma_{q-1} B\left(\frac{q-1}{2}, \frac{1+k}{2} \right) \cos^k(b, A_q). \end{aligned}$$

Proposition 33. *Let $B_p \in \mathcal{L}_p^d$ be fixed. Let m be a unit vector in $\mathbb{R}^{\mathrm{d}}$ and assume that $m \notin B_p$. Then,*

$$\begin{aligned}&\int_{S^{d-2}(m^\perp)} |p(x|B_p)|^k \,\mathrm{d}x^{d-2}\\ &\quad = \frac{\sigma_{d-p-1}\sigma_p B\left(\frac{d-p-1}{2}, \frac{p+k}{2}\right)}{2} \sin^{p+k-1}(m, B_p) F\left(\frac{p+k}{2}, \frac{p-1}{2}; \frac{p}{2}; \cos^2(m, B_p)\right).\end{aligned}$$

Proof of Proposition 33. Using Proposition 31 with $A_q = m^\perp$, $A = \pi(m|B_p^\perp) \cdot m$ and $a_1 = m$, we obtain

$$\alpha_1 = -\frac{x_0 \cdot m}{\pi(m|B_p^\perp) \cdot m} = -\frac{\cos(m, B_p)(x_0 \cdot \pi(m|B_p))}{\sin(m, B_p)}$$

and

$$\beta = |\alpha_1 \pi(m|B_p^\perp)| = \alpha_1.$$

Hence,

$$\begin{aligned}\frac{1}{1+\beta^2} &= \frac{1}{1 + \frac{\cos^2(m,B_p)(x_0\cdot\pi(m|B_p))^2}{\sin^2(m,B_p)}}\\ &= \frac{\sin^2(m, B_p)}{\sin^2(m, B_p) + \cos^2(m, B_p)(x_0 \cdot \pi(m|B_p))^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\sigma_{d-p-1} B\left(\frac{d-p-1}{2}, \frac{p+k}{2}\right)}{2} \int_{S^{d-2}(m^\perp)} |p(x|B_p)|^k \,\mathrm{d}x^{d-2}\\ &= \int_{S^{p-1}(B_p)} \frac{\left(\frac{\sin^2(m,B_p)}{\sin^2(m,B_p)+\cos^2(m,B_p)(x_0\cdot\pi(m|B_p))^2}\right)^{(p+k-1)/2}}{\sqrt{\sin(m, B_p) + \cos^2(m, B_p)\cos^2(x_0, \pi(m|B_p))}} \,\mathrm{d}x_0^{p-1}\\ &= \sin^{p+k-1}(m, B_p) \int_{S^{p-1}(B_p)} \frac{1}{(\sin^2(m, B_p) + \cos^2(m, B_p)\cos^2(x_0, \pi(m|B_p))^{(p+k)/2}} \,\mathrm{d}x_0^{p-1}.\end{aligned}$$

The last integral is of the form

$$\int_{S^{p-1}(B_p)} g((x_0 \cdot \pi(m|B_p))^2) \,\mathrm{d}x_0^{p-1},$$

with $g(t) = \frac{1}{(\sin^2(m,B_p)+\cos^2(m,B_p)t)^{(p+k)/2}}$. Hence, using Proposition 17 and Lemma 8,

$$\begin{aligned}\int_{S^{p-1}(B_p)} g((x_0 \cdot \pi(m|B_p))^2) \,\mathrm{d}x_0^{p-1} &= \sigma_{p-1} \int_0^1 g(t) t^{-\frac{1}{2}} (1-t)^{\frac{p-1}{2}-1} \,\mathrm{d}t\\ &= \sigma_{p-1} \int_0^1 \frac{1}{(\sin^2(m, B_p) + \cos^2(m, B_p)t)^{(p+k)/2}} t^{-\frac{1}{2}} (1-t)^{\frac{p-1}{2}-1} \,\mathrm{d}t\\ &= \sigma_{p-1}\, B\left(\frac{1}{2}, \frac{p-1}{2}\right) F\left(\frac{p+k}{2}, \frac{p-1}{2}; \frac{p}{2}; \cos^2(m, B_p)\right)\\ &= \sigma_p\, F\left(\frac{p+k}{2}, \frac{p-1}{2}; \frac{p}{2}; \cos^2(m, B_p)\right).\end{aligned}$$

□

Remark 34. Note that for $p = 1$, $\text{span}(b) = B_1$, Proposition 33 yields

$$\int_{S^{d-2}(m^\perp)} |x \cdot b|^k \, \mathrm{d}x^{d-2} = \sigma_{d-2} B\left(\frac{d-2}{2}, \frac{k}{2}\right) \sin^k(b, m),$$

i.e. the same expression as the one obtained in Remark 32 with $q = d - 1$. Moreover, in the special case $k = 0$, Proposition 33 yields, for all p,

$$\begin{aligned}\int_{S^{d-2}(m^\perp)} \mathrm{d}x^{d-2} &= \frac{\sigma_{d-p-1}\sigma_p B\left(\frac{d-p-1}{2}, \frac{p}{2}\right)}{2} \sin^{p-1}(m, B_p) F\left(\frac{p}{2}, \frac{p-1}{2}; \frac{p}{2}; \cos^2(m, B_p)\right) \\ &= \sigma_{d-1},\end{aligned}$$

as was expected. The second equality follows from [1, (15.1.8)].

Proposition 35. *Let $B_p \in \mathcal{L}_p^d$ be fixed. Then, for any non-negative measurable function $g \colon \mathbb{R}^d \to \mathbb{R}$,*

$$\int_{S^{d-1}} g(p(x|B_p)) \, \mathrm{d}x^{d-1} = \frac{\sigma_{d-p}}{2} \int_{S^{p-1}(B_p)} \int_0^1 g\left(t^{\frac{1}{2}} x_0\right) t^{\frac{p-2}{2}} (1-t)^{\frac{d-p-2}{2}} \, \mathrm{d}t \, \mathrm{d}x_0^{p-1}.$$

Proof. First, we use the co-area formula with

$$\begin{aligned}\psi \colon S^{d-1} &\to S^{p-1}(B_p) \\ x &\mapsto \pi(x|B_p).\end{aligned}$$

According to Lemma 27, the $(p-1)$-dimensional Jacobian of ψ is given by

$$J_{p-1}\psi(x, S^{d-1}) = |p(x|B_p)|^{-(p-1)}.$$

Hence, the co-area formula yields

$$\begin{aligned}\int_{S^{d-1}} g(p(x|B_p)) \, \mathrm{d}x^{d-1} &= \int_{S^{d-1}} g\left(|p(x|B_p)|\pi(x|B_p)\right) \, \mathrm{d}x^{d-1} \\ &= \int_{S^{p-1}(B_p)} \int_{\psi^{-1}(x_0)} g(|p(x|B_p)|x_0)|p(x|B_p)|^{p-1} \, \mathrm{d}x^{d-p} \, \mathrm{d}x_0^{p-1}. \qquad (9)\end{aligned}$$

Let $x_0 \in S^{p-1}(B_p)$ be fixed and apply the area formula with

$$\begin{aligned}\xi \colon B_p^\perp &\to \psi^{-1}(x_0) \\ \omega &\mapsto \frac{\omega + x_0}{|\omega + x_0|}.\end{aligned}$$

According to Lemma 23, the $(p-1)$-dimensional Jacobian of ξ is

$$J_{d-p}\xi(\omega, S^{d-1}) = \left(\frac{1}{1 + |\omega|^2}\right)^{\frac{d-p+1}{2}}.$$

Hence, since ξ maps $B_p^\perp$ bijectively onto $\psi^{-1}(x_0)$ and $|p(\xi(\omega)|B_p)| = \frac{1}{|\omega+x_0|} = \left(\frac{1}{1+|\omega|^2}\right)^{\frac{1}{2}}$,

$$\begin{aligned}
&\int_{\psi^{-1}(x_0)} g(|p(x|B_p)|x_0)|p(x|B_p)|^{p-1}\,\mathrm{d}x^{d-p} \\
&\quad= \int_{\psi^{-1}(x_0)} \sum_{\omega\in\xi^{-1}(x)} g(|p(\xi(\omega)|B_p)|x_0)|p(\xi(\omega)|B_p)|^{p-1}\,\mathrm{d}x^{d-p} \\
&\quad= \int_{\psi^{-1}(x_0)} \sum_{\omega\in\xi^{-1}(x)} g\left(\left(\frac{1}{1+|\omega|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|\omega|^2}\right)^{\frac{p-1}{2}}\,\mathrm{d}x^{d-p} \\
&\quad= \int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{p-1}{2}}\left(\frac{1}{1+|x|^2}\right)^{\frac{d-p+1}{2}}\,\mathrm{d}x^{d-p} \\
&\quad= \int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{d}{2}}\,\mathrm{d}x^{d-p}.
\end{aligned}$$

Using Proposition 13, we get

$$\begin{aligned}
&\int_{B_p^\perp} g\left(\left(\frac{1}{1+|x|^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+|x|^2}\right)^{\frac{d}{2}}\,\mathrm{d}x^{d-p} \\
&\qquad= \sigma_{d-p}\int_0^\infty g\left(\left(\frac{1}{1+t^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+t^2}\right)^{\frac{d}{2}} t^{d-p-1}\,\mathrm{d}t. \qquad (10)
\end{aligned}$$

Substitution with $s = \frac{1}{1+t^2}$ yield

$$\int_0^\infty g\left(\left(\frac{1}{1+t^2}\right)^{\frac{1}{2}} x_0\right)\left(\frac{1}{1+t^2}\right)^{\frac{d}{2}} t^{d-p-1}\,\mathrm{d}t = \frac{1}{2}\int_0^1 g\left(s^{\frac{1}{2}}x_0\right)s^{\frac{p-2}{2}}(1-s)^{\frac{d-p-2}{2}}\,\mathrm{d}s.$$

The last equation combined with (9) and (10) implies

$$\int_{S^{d-1}} g(p(x|B_p))\,\mathrm{d}x^{d-1} = \frac{\sigma_{d-p}}{2}\int_{S^{p-1}(B_p)}\int_0^1 g\left(t^{\frac{1}{2}}x_0\right)t^{\frac{p-2}{2}}(1-t)^{\frac{d-p-2}{2}}\,\mathrm{d}t\,\mathrm{d}x_0^{p-1}$$

□

Remark 36. Note that Proposition 35 generalizes Proposition 17 ($p = 1$).

Proposition 37. *Let $B_p \in \mathcal{L}_p^d$. Then,*

$$\int_{S^{d-1}} g(|p(x|B_p)|)\,\mathrm{d}x^{d-1} = \frac{\sigma_{d-p}\,\sigma_p}{2}\int_0^1 g\left(t^{\frac{1}{2}}\right)t^{\frac{p-2}{2}}(1-t)^{\frac{d-p-2}{2}}\,\mathrm{d}t.$$

Proof. Use Proposition 35. □

Proposition 38. *Let $B_p \in \mathcal{L}_p^d$. Then,*

$$\int_{S^{d-1}} |p(x|B_p)|^k\,\mathrm{d}x^{d-1} = \frac{\sigma_p\,\sigma_{d-p}}{2}B\left(\frac{d-p}{2},\frac{k+p}{2}\right).$$

Proof. Use Proposition 37 with $g(t) = t^k$. □

Proposition 39. *Let $B_p \in \mathcal{L}_p^d$ be fixed. Then,*

$$\int_{S^{d-1}} g(\pi(x|B_p))\, \mathrm{d}x^{d-1} = \frac{\sigma_{d-p}\, B\left(\frac{p}{2}, \frac{d-p}{2}\right)}{2} \int_{S^{p-1}(B_p)} g\left(x_0\right)\, \mathrm{d}x_0^{p-1}.$$

Proof. Use Proposition 35. $\square$

For the remainder of this section, recall that the binomial coefficient $\binom{a}{k}$ is defined for all $a \in \mathbb{R}$ and all $k \in \mathbb{N}$ by

$$\binom{a}{k} = \frac{(-a)_k(-1)^k}{k!} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-k+1)\Gamma(k+1)} & \text{for } a > 0, \\ \frac{\Gamma(a+k)(-1)^k}{\Gamma(a)\Gamma(k+1)} & \text{for } a < 0, \\ 0 & \text{for } a = 0. \end{cases}$$

Lemma 40. *For all $a, n \in \mathbb{R}$ and $k \in \mathbb{N}$,*

$$\binom{n}{2k} B\left(a + k + \frac{1}{2}, \frac{n}{2} - k + \frac{1}{2}\right) = B\left(\frac{n}{2} + \frac{1}{2}, a + \frac{1}{2}\right) \frac{\left(-\frac{n}{2}\right)_k \left(a + \frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_k} \frac{(-1)^k}{k!}.$$

Proof. The duplication formula (1) implies

$$\frac{\Gamma\left(\frac{n}{2} - k + \frac{1}{2}\right)}{\Gamma(n - 2k + 1)} = \frac{\pi^{\frac{1}{2}} 2^{-(n-2k)}}{\Gamma(\frac{n}{2} - k + 1)} = \frac{(-1)^k (-\frac{n}{2})_k \pi^{\frac{1}{2}} 2^{-(n-2k)}}{\Gamma(\frac{n}{2} + 1)},$$
$$\frac{1}{\Gamma(2k+1)} = \frac{\pi^{\frac{1}{2}} 2^{-2k}}{\Gamma\left(k + \frac{1}{2}\right) k!} = \frac{2^{-2k}}{\left(\frac{1}{2}\right)_k k!}$$

and

$$\Gamma\left(n+1\right) = \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n}{2} + 1\right) \pi^{-\frac{1}{2}} 2^n.$$

The result now follows by insertion. $\square$

Lemma 41. *For all $a, n \in \mathbb{R}$ and $k \in \mathbb{N}$,*

$$\binom{n}{2k} B\left(k + \frac{1}{2}, a + \frac{n}{2} - k + \frac{1}{2}\right) = B\left(a + \frac{n}{2} + \frac{1}{2}, \frac{1}{2}\right) \frac{\left(-\frac{n}{2} + \frac{1}{2}\right)_k (-\frac{n}{2})_k}{\left(-a - \frac{n}{2} + \frac{1}{2}\right)_k} \frac{(-1)^k}{k!}.$$

Proof. Left to the reader. $\square$

Lemma 42. *For all $a, b, c \in \mathbb{R}$ and $z \in [0, 1]$,*

$$F(a, b; c; z) = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)} F(a, b; a + b - c + 1; 1 - z).$$

Proof.

$$\begin{aligned}
F(a,b;c;z) &= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{(1-(1-z))^k}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{(a)_k(b)_k}{(c)_k} \frac{(1-z)^l}{k!} \\
&= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty} \binom{k}{l} (-1)^l \frac{(a)_k(b)_k}{(c)_k} \frac{(1-z)^l}{k!} \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \binom{k+l}{l} (-1)^l \frac{(a)_{k+l}(b)_{k+l}}{(c)_{k+l}} \frac{(1-z)^l}{(k+l)!}.
\end{aligned}$$

Note that

$$\frac{\binom{k+l}{l}}{(k+l)!} = \frac{1}{l!}\frac{1}{k!}$$

and

$$(a)_{k+l} = \frac{\Gamma(a+k+l)}{\Gamma(a)}\frac{\Gamma(a+l)}{\Gamma(a+l)} = (a)_l(a+l)_k$$

for all $k,l \in \mathbb{N}$ and all $a \in \mathbb{R}$ (when a is negative, use the corresponding definition of the rising sequential product). Thus,

$$\begin{aligned}
F(a,b;c;z) &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \binom{k+l}{l} (-1)^l \frac{(a)_{k+l}(b)_{k+l}}{(c)_{k+l}} \frac{(1-z)^l}{(k+l)!} \\
&= \sum_{l=0}^{\infty} (-1)^l \frac{(a)_l(b)_l}{(c)_l} \frac{(1-z)^l}{l!} \sum_{k=0}^{\infty} \frac{(a+l)_k(b+l)_k}{(c+l)_k} \frac{1}{k!}.
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty} \frac{(a+l)_k(b+l)_k}{(c+l)_k} \frac{1}{k!} = F(a+l,b+l;c+l;1) = \frac{\Gamma(c+l)\Gamma(c-a-b-l)}{\Gamma(c-a)\Gamma(c-b)}, \tag{11}$$

we get

$$\begin{aligned}
F(a,b;c;z) &= \sum_{l=0}^{\infty} (-1)^l \frac{(a)_l(b)_l}{(c)_l} \frac{(1-z)^l}{l!} \frac{\Gamma(c+l)\Gamma(c-a-b-l)}{\Gamma(c-a)\Gamma(c-b)} \\
&= \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \sum_{l=0}^{\infty} (-1)^l (a)_l(b)_l \frac{(1-z)^l}{l!} \Gamma(c-a-b-l) \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{l=0}^{\infty} \frac{(a)_l(b)_l}{(a+b-c+1)_l} \frac{(1-z)^l}{l!},
\end{aligned}$$

where we used that

$$(-1)^l\Gamma(c-a-b-l) = (-1)^l \frac{\Gamma(c-a-b-l)}{\Gamma(c-a-b)}\Gamma(c-a-b) = (a+b-c+1)_l\Gamma(c-a-b). \tag{12}$$

Hence,

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z).$$

□

Lemma 43. *Let $a, b, c \in \mathbb{R}$ and $z \in [0,1]$. Then,*

$$\sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{F(-k,d;b;z)}{k!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,d;a+b-c+1;z)$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{F(-k,d;b;z)}{k!} &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(a)_k\,(b)_k}{(c)_k} \frac{1}{k!} \frac{(-k)_l (d)_l}{(b)_l} \frac{z^l}{l!} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(a)_k\,(b)_k}{(c)_k} \frac{1}{k!} (-1)^l \binom{k}{l} \frac{(d)_l}{(b)_l} z^l \\
&= \sum_{l=0}^{\infty} (-1)^l \frac{(d)_l}{(b)_l} z^l \sum_{k=0}^{\infty} \frac{(a)_{k+l}\,(b)_{k+l}}{(c)_{k+l}} \frac{1}{(k+l)!} \binom{k+l}{l} \\
&= \sum_{l=0}^{\infty} (-1)^l \frac{(d)_l}{(b)_l} \frac{(a)_l\,(b)_l}{(c)_l} \frac{z^l}{l!} \sum_{k=0}^{\infty} \frac{(a+l)_k\,(b+l)_k}{(c+l)_k} \frac{1}{k!} \\
&= \sum_{l=0}^{\infty} (-1)^l \frac{(d)_l\,(a)_l}{(c)_l} \frac{z^l}{l!} \sum_{k=0}^{\infty} \frac{(a+l)_k\,(b+l)_k}{(c+l)_k} \frac{1}{k!}.
\end{aligned}$$

Applying (11) and (12), we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{F(-k,d;b;z)}{k!} &= \frac{1}{\Gamma(c-a)\Gamma(c-b)} \sum_{l=0}^{\infty} (-1)^l \frac{(d)_l\,(a)_l}{(c)_l} \frac{z^l}{l!} \Gamma(c+l)\Gamma(c-a-b-l) \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{l=0}^{\infty} \frac{(d)_l\,(a)_l}{(a+b-c+1)_l} \frac{z^l}{l!} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,d;a+b-c+1;z).
\end{aligned}$$

□

Lemma 44. *Let $a, b, c, d, e \in \mathbb{R}$ and $z \in [-1,1]$. Then,*

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z_1^k}{k!} F(-k,d,e,z_2) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (d)_k}{(c)_k (e)_k} \frac{(-1)^k z_1^k z_2^k}{k!} F(a+k, b+k; c+k; z_1).$$

Proof. First, note that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z_1^k}{k!} F(-k,d,e,z_2) &= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z_1^k}{k!} \sum_{j=0}^{k} \frac{(-k)_j (d)_j}{(e)_j} \frac{z_2^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{(d)_j}{(e)_j} \frac{z_2^j}{j!} \sum_{k=j}^{\infty} \frac{(a)_k (b)_k (-k)_j}{(c)_k} \frac{z_1^k}{k!}.
\end{aligned}$$

Then, using the identities $(-k)_j = (-1)^j \frac{k!}{(k-j)!}$ and $(a)_{k+j} = (a)_j (a+j)_k$, the last expression becomes

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{(d)_j}{(e)_j} \frac{(-1)^j z_2^j}{j!} \sum_{k=j}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z_1^k}{(k-j)!} &= \sum_{j=0}^{\infty} \frac{(d)_j}{(e)_j} \frac{(-1)^j z_2^j}{j!} \sum_{k=0}^{\infty} \frac{(a)_{k+j} (b)_{k+j}}{(c)_{k+j}} \frac{z_1^{j+k}}{k!} \\
&= \sum_{j=0}^{\infty} \frac{(a)_j (b)_j (d)_j}{(c)_j (e)_j} \frac{(-1)^j z_1^j z_2^j}{j!} \sum_{k=0}^{\infty} \frac{(a+j)_k (b+j)_k}{(c+j)_k} \frac{z_1^k}{k!}.
\end{aligned}$$

□

Proposition 45. *For all $m, n \in \mathbb{N}$, it holds that*

$$B\left(\frac{n+1}{2}, \frac{d-1}{2}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{m}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{n+d}{2}\right)_k} \frac{F\left(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,y)\right)}{k!}$$
$$= B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{d-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{d-1}{2}; \sin^2(x,y)\right).$$

Remark 46. In the particular case when $(x, y) = 0$, two successive applications of [1, (15.1.20)] yield

$$B\left(\frac{n+1}{2}, \frac{d-1}{2}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{m}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{n+d}{2}\right)_k} \frac{F\left(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,y)\right)}{k!} = B\left(\frac{n+1}{2}, \frac{d+m-1}{2}\right).$$

This is the statement in Proposition 45, since $F(a, b; c; 0) = 1$.

Proof of Proposition 45. According to Lemma 42, we have

$$F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{d-1}{2}; \sin^2(x,y)\right)$$
$$= \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d-1}{2} + \frac{m}{2} + \frac{n}{2}\right)}{\Gamma\left(\frac{d-1}{2} + \frac{m}{2}\right)\Gamma\left(\frac{d-1}{2} + \frac{n}{2}\right)} F\left(-\frac{m}{2}, -\frac{n}{2}; -\frac{m}{2} - \frac{n}{2} - \frac{d-1}{2} + 1; \cos^2(x,y)\right). \quad (13)$$

Moreover, by applying Lemma 43, we get

$$\sum_{k=0}^{\infty} \frac{\left(-\frac{m}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{n+d}{2}\right)_k} \frac{F\left(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,y)\right)}{k!}$$
$$= \frac{\Gamma\left(\frac{n+d}{2}\right)\Gamma\left(\frac{n+d}{2} + \frac{m}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{n+d}{2} + \frac{m}{2}\right)\Gamma\left(\frac{n+d}{2} - \frac{1}{2}\right)} F\left(-\frac{m}{2}, -\frac{n}{2}, -\frac{m}{2} + \frac{1}{2} - \frac{n+d}{2} + 1; \cos^2(x,y)\right) \quad (14)$$

Comparing (13) and (14), Proposition 45 follows from the identity

$$B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{d-1}{2}\right) \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d+m-1}{2}\right)} = B\left(\frac{n+1}{2}, \frac{d-1}{2}\right) \frac{\Gamma\left(\frac{n+d}{2}\right)}{\Gamma\left(\frac{n+d+m}{2}\right)}.$$

□

Lemma 47. *For all $a \in \mathbb{R}$ and all $s \in \mathbb{N}$,*

$$\binom{a}{2s} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}} 2^{2s}.$$

Proof. A routine calculation yields

$$\binom{a}{2s} = \frac{\Gamma(a+1)}{\Gamma(2s+1)\Gamma(a-2s+1)} = \binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(s+\frac{1}{2})} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}} 2^{2s}.$$

For the second equality, we applied the duplication formula on each Gamma function appearing in the second term, and for the third equality, we applied the duplication formula to $\Gamma(2s+1)$. □

Lemma 48. *For all $a \in \mathbb{R}$ and any function f, the following identity holds,*

$$\sum_{k=0}^{\infty} \binom{\frac{a}{2}}{k} \sum_{l=0}^{k} \binom{k}{l} 2^{k-l} f(k+l) = \sum_{s=0}^{\frac{a}{2}} \binom{a}{2s} f(2s),$$

where the double sum on the left-hand side is over k and l ***with the same parity*** *only.*

Proof. Substition of $k+l$ by $2s$ yields

$$\sum_{k=0}^{\infty} \binom{\frac{a}{2}}{k} \sum_{l=0}^{k} \binom{k}{l} 2^{k-l} f(k+l) = \sum_{s=0}^{\infty} \sum_{l=0}^{s} \binom{\frac{a}{2}}{2s-l} \binom{2s-l}{l} 2^{2s-2l} f(2s).$$

Applying the duplication formula to $\Gamma(2s-2l+1)$, we get

$$\begin{aligned}
\binom{\frac{a}{2}}{2s-l} \binom{2s-l}{l} 2^{2s-2l} &= \frac{\Gamma(\frac{a}{2}+1)}{\Gamma(\frac{a}{2}-2s+l+1)} \frac{2^{2s-2l}}{\Gamma(l+1)\Gamma(2s-2l+1)} \\
&= \frac{\Gamma(\frac{a}{2}+1)}{\Gamma(\frac{a}{2}-2s+l+1)} \frac{\Gamma(\frac{1}{2})}{\Gamma(l+1)\Gamma(s-l+1)\Gamma(s-l+\frac{1}{2})} \\
&= \binom{\frac{a}{2}}{s} \binom{s-\frac{1}{2}}{l} \binom{\frac{a-1}{2}-(s-\frac{1}{2})}{s-l} \frac{\Gamma(\frac{1}{2})\Gamma(s+1)}{\Gamma(s+\frac{1}{2})}.
\end{aligned}$$

Then, the well-known identity, $\sum_{l=0}^{k} \binom{m}{l}\binom{n-m}{k-l} = \binom{n}{k}$, valid for any complex numbers m and n, and the duplication formula applied to $\Gamma(2s+1)$ imply

$$\sum_{l=0}^{s} \binom{\frac{a}{2}}{2s-l} \binom{2s-l}{l} 2^{2s-2l} = \frac{\binom{\frac{a}{2}}{s}\binom{\frac{a-1}{2}}{s}}{\binom{2s}{s}} 2^{2s}.$$

Thanks to Lemma 47, the proof is complete. ☐

Lemma 49. *Let $m, n \in \mathbb{N}$. Then,*

$$\int_{S^1(x \oplus y)} |\omega \cdot x|^m |\omega \cdot y|^n \, \mathrm{d}\omega^1 = 2B\left(\frac{m}{2}+\frac{1}{2}, \frac{n}{2}+\frac{1}{2}\right) F\left(-\frac{n}{2}, -\frac{m}{2}; \frac{1}{2}; \cos^2(x,y)\right)$$

Proof. According to Lemma 22, we have

$$\int_{S^1(x \oplus y)} |\omega \cdot x|^m |\omega \cdot y|^n \, \mathrm{d}\omega^1 = \int_{-1}^{1} \int_{S^1 \cap x^\perp} (1-t^2)^{\frac{-1}{2}} |t|^m |t(y\cdot x) + \sqrt{1-t^2}(y\cdot\omega)|^n \, \mathrm{d}\omega^0 \, \mathrm{d}t.$$

Applying twice a binomial expansion, we obtain

$$\begin{aligned}
&\int_{-1}^{1} \int_{S^1 \cap x^\perp} (1-t^2)^{-\frac{1}{2}} |t|^m |t^2(y\cdot x)^2 + (1-t^2)(y\cdot\omega)^2 + 2t\sqrt{1-t^2}(y\cdot x)(y\cdot\omega)|^{\frac{n}{2}} \, \mathrm{d}\omega^0 \, \mathrm{d}t \\
&= \sum_{k=0}^{\infty} \binom{\frac{n}{2}}{k} \int_{-1}^{1} \int_{S^1 \cap x^\perp} (1-t^2)^{-\frac{1}{2}} |t|^m (t^2(y\cdot x)^2)^{\frac{n}{2}-k} \\
&\qquad \times ((1-t^2)(y\cdot\omega)^2 + 2t\sqrt{1-t^2}(y\cdot x)(y\cdot\omega))^k \, \mathrm{d}\omega^0 \, \mathrm{d}t \\
&= \sum_{k=0}^{\infty} \binom{\frac{n}{2}}{k} \sum_{l=0}^{k} \binom{k}{l} 2^{k-l} |x\cdot y|^{n-2k} (x\cdot y)^{k-l} \\
&\qquad \times \int_{-1}^{1} |t|^{m+n-2k} t^{k-l} \sqrt{1-t^2}^{k+l-1} \, \mathrm{d}t \int_{S^1 \cap x^\perp} |y\cdot\omega|^{k+l} \, \mathrm{d}\omega^0 \\
&= 2\sum_{k=0}^{\infty} \binom{\frac{n}{2}}{k} \sum_{l=0}^{k} \binom{k}{l} 2^{k-l} |x\cdot y|^{n-2k} (x\cdot y)^{k-l} \sin^{k+l}(x,y) \int_{-1}^{1} |t|^{m+n-2k} t^{k-l} \sqrt{1-t^2}^{k+l-1} \, \mathrm{d}t.
\end{aligned}$$

Notice that the terms in the double sum are non-zero only if k and l have the same parity. In that case, according to Lemma 9 and Lemma 48, we have

$$\begin{aligned} 2|x\cdot y|^n \sum_{k=0}^{\infty}\binom{\frac{n}{2}}{k}\sum_{l=0}^{k}\binom{k}{l}2^{k-l}\tan^{k+l}(x,y)B\left(\frac{m+n}{2}-\frac{k+l}{2}+\frac{1}{2},\frac{k+l}{2}+\frac{1}{2}\right) \\ = 2|x\cdot y|^n\sum_{k=0}^{\infty}\binom{n}{2k}\tan^{2k}(x,y)B\left(\frac{m+n}{2}-k+\frac{1}{2},k+\frac{1}{2}\right), \end{aligned}$$

where the double sum on the left-hand side is over k and l with the same parity. By applying Lemma 41 with $a=\frac{m}{2}$, we get

$$\binom{n}{2k}B\left(\frac{m+n}{2}-k+\frac{1}{2},k+\frac{1}{2}\right)=\frac{\left(-\frac{n}{2}\right)_k\left(-\frac{n-1}{2}\right)_k}{\left(-\frac{m+n}{2}+\frac{1}{2}\right)_k}\frac{(-1)^k}{k!}B\left(\frac{m+n}{2}+\frac{1}{2},\frac{1}{2}\right).$$

Hence, we have attained the following expression

$$\begin{aligned} 2|x\cdot y|^n B\left(\frac{m+n}{2}+\frac{1}{2},\frac{1}{2}\right)\frac{\left(-\frac{n}{2}\right)_k\left(-\frac{n-1}{2}\right)_k}{\left(-\frac{m+n}{2}+\frac{1}{2}\right)_k}\sum_{k=0}^{\frac{n}{2}}\frac{\left(-\tan^{-2}(x,y)\right)^k}{k!} \\ = 2|x\cdot y|^n B\left(\frac{m+n}{2}+\frac{1}{2},\frac{1}{2}\right)F\left(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};-\frac{m+n}{2}+\frac{1}{2};-\tan^2(x,y)\right). \end{aligned}$$

According to [1, (15.3.4)] with $z=\cos^2(x,y)$,

$$\begin{aligned} 2|x\cdot y|^n B\left(\frac{m+n}{2}+\frac{1}{2},\frac{1}{2}\right)F\left(-\frac{n}{2},-\frac{n}{2}+\frac{1}{2};-\frac{m+n}{2}+\frac{1}{2};-\tan^2(x,y)\right) \\ = 2B\left(\frac{m+n}{2}+\frac{1}{2},\frac{1}{2}\right)F\left(-\frac{n}{2},-\frac{m}{2};-\frac{m+n}{2}+\frac{1}{2};\sin^2(x,y)\right). \end{aligned}$$

An application of Lemma 42 with $c=\frac{1}{2}$ completes the proof. □

Remark 50. By using an alternative binomial expansion (or Lemma 42), the identity in Lemma 49 becomes

$$\int_{S^1(x\oplus y)}|\omega\cdot x|^m|\omega\cdot y|^n\,\mathrm{d}\omega^1=2B\left(\frac{m+n+1}{2},\frac{1}{2}\right)F\left(-\frac{n}{2},-\frac{m}{2};-\frac{m+n-1}{2};\sin^2(x,y)\right). \tag{15}$$

(The hypergeometric function is well-defined but can only be expressed as a finite series for non-negative odd integers m, cf. [1, (15.4.2)]). Moreover, when $m=0$, (15) becomes, cf. [1, (15.1.8)],

$$\int_{S^1(x\oplus y)}|\omega\cdot y|^n\,\mathrm{d}\omega^1=2B\left(\frac{n+1}{2},\frac{1}{2}\right),$$

which is also the result in Proposition 19, since $\sigma_1=2$. For $m=1$, [1, (15.1.8)] also offers a reduction,

$$\int_{S^1(x\oplus y)}|\omega\cdot x||\omega\cdot y|^n\,\mathrm{d}\omega^1=2B\left(\frac{n+1}{2},\frac{1}{2}\right)\cos(x,y).$$

Comparing with the identity in Lemma 49, we now have

$$B\left(1,\frac{n+1}{2}\right)F\left(-\frac{n}{2},-\frac{1}{2};\frac{1}{2};\cos^2(x,y)\right)=B\left(\frac{n}{2}+1,\frac{1}{2}\right)\cos(x,y).$$

Proposition 51. *Let $x, y \in S^{d-1}$, $m, n \in \mathbb{N}$. Then,*

$$\int_{S^{d-1}} |x \cdot \omega|^m |y \cdot \omega|^n \,\mathrm{d}\omega$$
$$= \sigma_{d-2} B\left(\frac{n+m+2}{2}, \frac{d-2}{2}\right) B\left(\frac{m}{2}+\frac{1}{2}, \frac{n}{2}+\frac{1}{2}\right) F\left(-\frac{n}{2}, -\frac{m}{2}; \frac{1}{2}; \cos^2(x,y)\right).$$

Proof. First, note that

$$\int_{S^{d-1}} |x \cdot \omega|^m |y \cdot \omega|^n \,\mathrm{d}\omega^{d-1}$$
$$= \int_{S^{d-1}} |p(p(\omega|x \oplus y)|x)|^m |p(p(\omega|x \oplus y)|y)|^n \,\mathrm{d}\omega^{d-1}.$$

Then, we apply Proposition 35 with $B_p = \operatorname{span}\{x, y\}$,

$$\int_{S^{d-1}} |p(p(\omega|x \oplus y)|x)|^m |p(p(\omega|x \oplus y)|y)|^n \,\mathrm{d}\omega^{d-1}$$
$$= \frac{\sigma_{d-2}}{2} \int_{S^{p-1}(B_p)} \int_0^1 t^{m/2} |p(\omega_0|x)|^m t^{n/2} |p(\omega_0|y)|^n t^{\frac{2-2}{2}} (1-t)^{\frac{d-2-2}{2}} \,\mathrm{d}t \,\mathrm{d}\omega_0^{p-1}$$
$$= \frac{\sigma_{d-2}}{2} \int_{S^{p-1}(B_p)} |p(\omega_0|y)|^n |p(\omega_0|x)|^m \,\mathrm{d}\omega_0^{p-1} \int_0^1 t^{\frac{n+m}{2}} (1-t)^{\frac{d-4}{2}} \,\mathrm{d}t$$
$$= \frac{\sigma_{d-2} B\left(\frac{n+m+2}{2}, \frac{d-2}{2}\right)}{2} \int_{S^{p-1}(B_p)} |p(\omega_0|y)|^n |p(\omega_0|x)|^m \,\mathrm{d}\omega_0^{p-1}.$$

Using Lemma 49 we get

$$\int_{S^{d-1}} |x \cdot \omega|^m |y \cdot \omega|^n \,\mathrm{d}\omega^{d-1}$$
$$= \sigma_{d-2} B\left(\frac{n+m+2}{2}, \frac{d-2}{2}\right) B\left(\frac{m}{2}+\frac{1}{2}, \frac{n}{2}+\frac{1}{2}\right) F\left(-\frac{n}{2}, -\frac{m}{2}; \frac{1}{2}; \cos^2(x,y)\right).$$

□

Lemma 52. *Let $m, n \in \mathbb{N}$ and let $x, y \in S^{d-1}$. Then,*

$$\int_{S^1(x \oplus y)} \sqrt{1 - |\omega \cdot x|^2}^m |\omega \cdot y|^n \,\mathrm{d}\omega^1 = 2B\left(\frac{n}{2}+\frac{1}{2}, \frac{m}{2}+\frac{1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2(x,y)\right).$$

Proof. A binomial expansion of the square root and an application of Lemma 49 yield

$$\int_{S^1(x \oplus y)} \sqrt{1 - |\omega \cdot x|^2}^m |\omega \cdot y|^n \,\mathrm{d}\omega^1$$
$$= \sum_{k=0}^{\infty} \binom{\frac{m}{2}}{k} (-1)^k \int_{S^1(x \oplus y)} |\omega \cdot x|^{2k} |\omega \cdot y|^n \,\mathrm{d}\omega^1$$
$$= 2 \sum_{k=0}^{\infty} \frac{\left(-\frac{m}{2}\right)}{k!} B\left(\frac{n+1}{2}, k+\frac{1}{2}\right) F\left(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,y)\right)$$
$$= 2B\left(\frac{n+1}{2}, \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{m}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{n}{2}+1\right)_k} \frac{F\left(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,y)\right)}{k!}.$$

According to Lemma 43, the last expression is equal to

$$= 2B\left(\frac{n+1}{2}, \frac{1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; -\frac{m+n}{2} + \frac{1}{2}; \cos^2(x,y)\right).$$

Finally, using Lemma 42 with $c = \frac{1}{2}$, we obtain the alternative representation

$$\int_{S^1(x \oplus y)} \sqrt{1 - |\omega \cdot x|^2}^m |\omega \cdot y|^n \, d\omega^1 = 2B\left(\frac{n}{2} + \frac{1}{2}, \frac{m}{2} + \frac{1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2(x,y)\right).$$

□

Proposition 53. *Let $x, y \in S^{d-1}$ and $m, n \in \mathbb{N}$. Then,*

$$\begin{aligned}
&\int_{S^{d-1}} \sqrt{1 - (x \cdot \omega)^2}^m |y \cdot \omega|^n \, d\omega \\
&\quad = \sigma_{d-1} B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{d-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{d-1}{2}; \sin^2(x,y)\right).
\end{aligned}$$

Proof. A binomial expansion followed by an application of Proposition 51 implies

$$\begin{aligned}
&\int_{S^{d-1}} \sqrt{1 - (x \cdot \omega)^2}^m |y \cdot \omega|^n \, d\omega \\
&\quad = \sum_{k=0}^{\infty} \binom{\frac{m}{2}}{k} (-1)^k \int_{S^{d-1}} |\omega \cdot x|^{2k} |\omega \cdot y|^n \, d\omega \\
&\quad = \sigma_{d-2} \sum_{k=0}^{\infty} \binom{\frac{m}{2}}{k} (-1)^k B\left(k + \frac{n}{2} + 1, \frac{d-2}{2}\right) B\left(\frac{n+1}{2}, k + \frac{1}{2}\right) \\
&\qquad\quad \times F\left(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,y)\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\sigma_{d-2} \binom{\frac{m}{2}}{k} (-1)^k B\left(k + \frac{n}{2} + 1, \frac{d-2}{2}\right) B\left(\frac{n+1}{2}, k + \frac{1}{2}\right) \\
&\quad = \sigma_{d-2} \frac{\Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+d}{2}\right)} \frac{\left(-\frac{m}{2}\right)_k}{k!} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{n+d}{2}\right)_k} \\
&\quad = \sigma_{d-1} B\left(\frac{d-1}{2}, \frac{n+1}{2}\right) \frac{\left(-\frac{m}{2}\right)_k}{k!} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{n+d}{2}\right)_k},
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{S^{d-1}} \sqrt{1 - (x \cdot \omega)^2}^m |y \cdot \omega|^n \, d\omega \\
&\quad = \sigma_{d-1} B\left(\frac{d-1}{2}, \frac{n+1}{2}\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{m}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{n+d}{2}\right)_k} \frac{F\left(-k, -\frac{n}{2}; \frac{1}{2}; \cos^2(x,y)\right)}{k!}.
\end{aligned}$$

According to Proposition 45, the last expression is equal to

$$\sigma_{d-1} B\left(\frac{n+1}{2}, \frac{m}{2} + \frac{d-1}{2}\right) F\left(-\frac{m}{2}, -\frac{n}{2}; \frac{d-1}{2}; \sin^2(x,y)\right)$$

and the proof is complete. □

Remark 54. In the special case where $n = 0$, Proposition 53 states

$$\int_{S^{d-1}} \sqrt{1-(x\cdot\omega)^2}^{m}\,\mathrm{d}\omega^{d-1} = \sigma_{d-1} B\left(\frac{1}{2},\frac{d-1}{2}\right) F\left(-\frac{m}{2},\frac{1}{2};\frac{d}{2};1\right).$$

Hence, when $n = 0$, Proposition 53 is equivalent to Proposition 21.

Proposition 55. *Let a, b and c be natural numbers. Let x, y and z be unit vectors with $y \perp z$, $x \neq y$ and $x \notin y^{\perp}$. Then,*

$$\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c\,\mathrm{d}\omega^2 = 2|x\cdot y|^a B\left(\frac{a+b}{2}+1,\frac{c}{2}+\frac{1}{2}\right) B\left(\frac{a+b}{2}+\frac{1}{2},\frac{1}{2}\right)$$
$$\times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s\left(-\frac{a-1}{2}\right)_s}{\left(-\frac{a+b}{2}+\frac{1}{2}\right)_s} \frac{(-1)^s}{s!} \tan^{2s}(x,y) F\left(-s,-\frac{c}{2};\frac{1}{2};\frac{\cos^2(x,z)}{\sin^2(x,y)}\right).$$

Proof. Lemma 22 with $d = 3$ implies

$$\begin{aligned}
&\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c\,\mathrm{d}\omega^2 \\
&= \int_{-1}^{1}\int_{S^2(y^\perp)} |x\cdot(ty+\sqrt{1-t^2}\omega)|^a |y\cdot(ty+\sqrt{1-t^2}\omega)|^b |z\cdot(ty+\sqrt{1-t^2}\omega)|^c\,\mathrm{d}\omega^1\,\mathrm{d}t \\
&= \int_{-1}^{1}\int_{S^2(y^\perp)} \left|\left(t^2(x\cdot y)^2+(1-t^2)(x\cdot\omega)^2+2t\sqrt{1-t^2}(x\cdot y)(x\cdot\omega)\right)^2\right|^{\frac{a}{2}} \\
&\qquad\qquad\times |t|^b\sqrt{1-t^2}^{c}|z\cdot\omega|^c\,\mathrm{d}\omega^1\,\mathrm{d}t.
\end{aligned}$$

Applying the binomial formula twice, we obtain

$$\begin{aligned}
&\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c\,\mathrm{d}\omega^2 \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{\frac{a}{2}}{k}\binom{k}{l} \int_{-1}^{1}\int_{S^2(y^\perp)} (t^2(x\cdot y)^2)^{\frac{a}{2}-k}(1-t^2)^l (x\cdot\omega)^{2l} \\
&\qquad\qquad\times \left(2t\sqrt{1-t^2}(x\cdot y)(x\cdot\omega)\right)^{k-l}|t|^b\sqrt{1-t^2}^{c}|z\cdot\omega|^c\,\mathrm{d}\omega^1\,\mathrm{d}t \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{\frac{a}{2}}{k}\binom{k}{l} |x\cdot y|^{a-(k+l)} 2^{k-l} \int_{-1}^{1} |t|^{a+b-2k} t^{k-l}\sqrt{1-t^2}^{c+k+l}\,\mathrm{d}t \\
&\qquad\qquad\times \int_{S^2(y^\perp)} |x\cdot\omega|^{2l}(x\cdot\omega)^{k-l}|z\cdot\omega|^c\,\mathrm{d}\omega^1.
\end{aligned}$$

Note that both the first and second integral are non zero only if k and l have the same parity. The first integral can be computed using Lemma 9,

$$2\int_0^1 (t^2)^{\frac{a+b}{2}-\frac{k+l}{2}}(1-t^2)^{\frac{c}{2}+\frac{k+l}{2}}\,\mathrm{d}t = B\left(\frac{a+b}{2}-\frac{k+l}{2}+\frac{1}{2},\frac{c}{2}+\frac{k+l}{2}+1\right).$$

Using Lemma 49 together with the decomposition $x = \sin(x,y)x_1 + \cos(x,y)y$, where $x_1 = \pi(x|y^\perp)$, the second integral can be written in terms of a hypergeometric function

$$\begin{aligned}
&\int_{S^2(y^\perp)} |x\cdot\omega|^{a-(k+l)}|z\cdot\omega|^c\,\mathrm{d}\omega^1 = \sin^{k+l}(x,y)\int_{S^2(y^\perp)} |x_1\cdot\omega|^{a-(k+l)}|z\cdot\omega|^c\,\mathrm{d}\omega^1 \\
&= 2\sin^{k+l}(x,y) B\left(\frac{k+l}{2}+\frac{1}{2},\frac{c}{2}+\frac{1}{2}\right) F\left(-\frac{k+l}{2},-\frac{c}{2};\frac{1}{2};\cos^2(x_1,z)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c \,\mathrm{d}\omega^2 \\
&\qquad = 2|x\cdot y|^a \sum_{k=0}^{\infty}\sum_{l=0}^{k} \binom{\frac{a}{2}}{k}\binom{k}{l} 2^{k-l} B\left(\frac{a+b}{2} - \frac{k+l}{2} + \frac{1}{2}, \frac{c}{2} + \frac{k+l}{2} + 1\right) \\
&\qquad\qquad \times \tan^{k+l}(x,y) B\left(\frac{k+l}{2} + \frac{1}{2}, \frac{c}{2} + \frac{1}{2}\right) F\left(-\frac{k+l}{2}, -\frac{c}{2}; \frac{1}{2}; \cos^2(x_1,z)\right),
\end{aligned}$$

where the double sum on the right-hand side is over k and l with $(-1)^{k+l} = 1$ only. We apply Lemma 48 and conclude

$$\begin{aligned}
&\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c \,\mathrm{d}\omega^2 \\
&\qquad = 2|x\cdot y|^a \sum_{s=0}^{\infty} \binom{a}{2s} B\left(\frac{a+b}{2} - s + \frac{1}{2}, \frac{c}{2} + s + 1\right) B\left(s + \frac{1}{2}, \frac{c}{2} + \frac{1}{2}\right) \\
&\qquad\qquad \times \tan^{2s}(x,y) F\left(-s, -\frac{c}{2}; \frac{1}{2}; \cos^2(x_1,z)\right).
\end{aligned}$$

Note that $\cos^2(x_1,z) = \frac{p(x|y^\perp)\cdot z}{|p(x|y^\perp)|} = \frac{(x-p(x|y))\cdot z}{\sin(x,y)} = \frac{\cos(x,z)}{\sin(x,y)}$, because $y \perp z$. Using the identity

$$\begin{aligned}
&B\left(\frac{a+b}{2} - s + \frac{1}{2}, \frac{c}{2} + s + 1\right) B\left(s + \frac{1}{2}, \frac{c}{2} + \frac{1}{2}\right) \\
&\qquad = B\left(\frac{a+b}{2} + 1, \frac{c}{2} + \frac{1}{2}\right) B\left(\frac{a+b}{2} - s + \frac{1}{2}, s + \frac{1}{2}\right)
\end{aligned}$$

together with Lemma 41, we obtain

$$\begin{aligned}
\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c \,\mathrm{d}\omega^2 &= 2|x\cdot y|^a B\left(\frac{a+b}{2} + 1, \frac{c}{2} + \frac{1}{2}\right) B\left(\frac{a+b}{2} + \frac{1}{2}, \frac{1}{2}\right) \\
&\quad \times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{a-1}{2}\right)_s}{\left(-\frac{a+b}{2} + \frac{1}{2}\right)_s} \frac{(-1)^s}{s!} \tan^{2s}(x,y) F\left(-s, -\frac{c}{2}; \frac{1}{2}; \frac{\cos(x,z)}{\sin(x,y)}\right).
\end{aligned}$$

□

Remark 56. In the particular case where $c = 0$, Proposition 55 combined with [1, (15.3.4)] and Lemma 42 yield

$$\begin{aligned}
&\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b \,\mathrm{d}\omega^2 \\
&\qquad = 2|x\cdot y|^a B\left(\frac{a+b}{2} + 1, +\frac{1}{2}\right) B\left(\frac{a+b}{2} + \frac{1}{2}, \frac{1}{2}\right) \\
&\qquad\qquad \times \sum_{s=0}^{\infty} F\left(-\frac{a}{2}, -\frac{a-1}{2}; -\frac{a+b}{2} + \frac{1}{2}; -\tan^2(x,y)\right) \\
&\qquad = 2B\left(\frac{a}{2} + \frac{1}{2}, \frac{b}{2} + \frac{1}{2}\right) B\left(\frac{a+b}{2} + 1, \frac{1}{2}\right) F\left(-\frac{a}{2}, -\frac{b}{2}; \frac{1}{2}; \cos^2(x,y)\right).
\end{aligned}$$

The same result is obtain, using Proposition 51 with $d = 3$. Also in the case $a = 0$ and $c = 0$, Proposition 55 and Proposition 51 are equivalent.

Proposition 57. *Let a, b and c be natural numbers. Let x, y and z be unit vectors with $y \perp z$, $x \neq y$ and $x \notin y^\perp$. Then,*

$$\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c \,\mathrm{d}\omega^2$$
$$= 2\frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{a+b+c}{2}+\frac{3}{2})} \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{c}{2}\right)_s}{\left(\frac{1}{2}\right)_s} \frac{\cos^{2s}(x,z)}{s!} F\left(-\frac{a}{2}+s, -\frac{b}{2}; \frac{1}{2}; \cos^2(x,y)\right).$$

Proof. Combining Proposition 55 and Lemma 44, we get

$$\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c \,\mathrm{d}\omega^2$$
$$= 2|x\cdot y|^a B\left(\frac{a+b}{2}+1, \frac{c}{2}+\frac{1}{2}\right) B\left(\frac{a+b}{2}+\frac{1}{2}, \frac{1}{2}\right) \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{a-1}{2}\right)_s \left(-\frac{c}{2}\right)_s}{\left(-\frac{a+b}{2}+\frac{1}{2}\right)_s \left(\frac{1}{2}\right)_s}$$
$$\times \frac{\cos^{2s}(x,z)}{\cos^{2s}(x,y)} \frac{F\left(-\frac{a}{2}+s, -\frac{a-1}{2}+s; -\frac{a+b}{2}+\frac{1}{2}+s; -\tan^2(x,y)\right)}{s!}.$$

Successive application of [1, (15.3.4)] and Lemma 42 yield

$$F\left(-\frac{a}{2}+s, -\frac{a-1}{2}+s; -\frac{a+b}{2}+\frac{1}{2}+s; -\tan^2(x,y)\right)$$
$$= |x\cdot y|^{-a+2s} F\left(-\frac{a}{2}+s, -\frac{b}{2}; -\frac{a+b}{2}+\frac{1}{2}+s; \sin^2(x,y)\right)$$
$$= |x\cdot y|^{-a+2s} \frac{\Gamma\left(\frac{a+1}{2}-s\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+b+1}{2}-s\right)\Gamma\left(\frac{1}{2}\right)} F\left(-\frac{a}{2}+s, -\frac{b}{2}; \frac{1}{2}; \cos^2(x,y)\right).$$

Thus,

$$\int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^b |z\cdot\omega|^c \,\mathrm{d}\omega^2 = 2B\left(\frac{a+b}{2}+1, \frac{c}{2}+\frac{1}{2}\right) B\left(\frac{a+b}{2}+\frac{1}{2}, \frac{1}{2}\right)$$
$$\times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{a-1}{2}\right)_s \left(-\frac{c}{2}\right)_s}{\left(-\frac{a+b}{2}+\frac{1}{2}\right)_s \left(\frac{1}{2}\right)_s} \frac{\Gamma\left(\frac{a+1}{2}-s\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+b+1}{2}-s\right)\Gamma\left(\frac{1}{2}\right)}$$
$$\times \frac{\cos^{2s}(x,z)}{s} F\left(-\frac{a}{2}+s, -\frac{b}{2}; \frac{1}{2}; \cos^2(x,y)\right).$$

The final reduction is straightforward. ∎

Remark 58. It is easily checked, whenever $a=0$, $b=0$ or $c=0$, that Proposition 57 is equivalent to Proposition 51.

Proposition 59. *Let a, b and c be natural numbers. Let x, y and z be unit vectors with $y \perp z$, $x \neq y$ and $x \notin y^\perp$. Then,*

$$\int_{S^2} |x\cdot\omega|^a \sqrt{1-|y\cdot\omega|^2}^{\,b} |z\cdot\omega|^c \,\mathrm{d}\omega^2 = 2B\left(\frac{a}{2}+\frac{1}{2}, \frac{b+c}{2}+1\right) B\left(\frac{c}{2}+\frac{1}{2}, \frac{1}{2}\right)$$
$$\times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{c}{2}\right)_s \left(\frac{b+c}{2}+1\right)_s}{\left(\frac{1}{2}\right)_s \left(\frac{c}{2}+1\right)_s} \frac{\cos^{2s}(x,z)}{s!} F\left(-\frac{a}{2}+s, -\frac{b}{2}; \frac{c}{2}+s+1; \sin^2(x,y)\right).$$

Proof. A binomial expansion of the square root yields

$$\int_{S^2} |x\cdot\omega|^a \sqrt{1-|y\cdot\omega|^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^2 = \sum_{k=0}^{\infty} \frac{(-\frac{b}{2})_k}{k!} \int_{S^2} |x\cdot\omega|^a |y\cdot\omega|^{2k} |z\cdot\omega|^c \,\mathrm{d}\omega^2. \tag{16}$$

Then, applying Proposition 57 and the identity

$$\frac{\Gamma(\frac{a+1}{2})\Gamma(k+\frac{1}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{a+c}{2}+k+\frac{3}{2})} = \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{c+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a+c}{2}+\frac{3}{2})} \frac{(\frac{1}{2})_k}{(\frac{a+c}{2}+\frac{3}{2})_k},$$

we can write (16) as

$$\begin{aligned} &2\frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{c+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a+c}{2}+\frac{3}{2})} \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{c}{2}\right)_s}{\left(\frac{1}{2}\right)_s} \frac{\cos^{2s}(x,z)}{s!} \\ &\qquad\times \sum_{k=0}^{\infty} \frac{(-\frac{b}{2})_k(\frac{1}{2})_k}{(\frac{a+c}{2}+\frac{3}{2})_k} \frac{1}{k!} F\left(-k,-\frac{a}{2}+s;\frac{1}{2};\cos^2(x,y)\right). \end{aligned} \tag{17}$$

Finally, an application of Lemma 43 followed by an application of Lemma 42 yield

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(-\frac{b}{2})_k(\frac{1}{2})_k}{(\frac{a+c}{2}+\frac{3}{2})_k} \frac{1}{k!} F\left(-k,-\frac{a}{2}+s;\frac{1}{2};\cos^2(x,y)\right) \\ &\quad= \frac{\Gamma(\frac{a+c}{2}+\frac{3}{2})\Gamma(\frac{a+b+c}{2}+1)}{\Gamma(\frac{a+b+c}{2}+\frac{3}{2})\Gamma(\frac{a+c}{2}+1)} F\left(-\frac{a}{2}+s,-\frac{b}{2};-\frac{a+b+c}{2};\cos^2(x,y)\right) \\ &\quad= \frac{\Gamma(\frac{a+c}{2}+\frac{3}{2})}{\Gamma(\frac{a+b+c}{2}+\frac{3}{2})} \frac{\Gamma(\frac{b+c}{2}+s+1)}{\Gamma(\frac{c}{2}+s+1)} F\left(-\frac{a}{2}+s,-\frac{b}{2};\frac{c}{2}+s+1;\sin^2(x,y)\right). \end{aligned} \tag{18}$$

Hence, by inserting (18) in (17), we obtain

$$\begin{aligned} &\int_{S^2} |x\cdot\omega|^a \sqrt{1-|y\cdot\omega|^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^2 = 2B\left(\frac{a}{2}+\frac{1}{2},\frac{b+c}{2}+1\right) B\left(\frac{c}{2}+\frac{1}{2},\frac{1}{2}\right) \\ &\quad\times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{c}{2}\right)_s \left(\frac{b+c}{2}+1\right)_s}{\left(\frac{1}{2}\right)_s \left(\frac{c}{2}+1\right)_s} \frac{\cos^{2s}(x,z)}{s!} F\left(-\frac{a}{2}+s,-\frac{b}{2};\frac{c}{2}+s+1;\sin^2(x,y)\right). \end{aligned}$$

□

Proposition 60. *Let a, b and c be natural numbers. Let x, y and z be unit vectors with $y\perp z$, $x\neq y$ and $x\notin y^\perp$. Then,*

$$\begin{aligned} &\int_{S^{d-1}} |x\cdot\omega|^a \sqrt{1-|y\cdot\omega|^2}^b |z\cdot\omega|^c \,\mathrm{d}\omega^2 \\ &= \sigma_{d-2} B\left(\frac{a}{2}+\frac{1}{2},\frac{b+c+d-1}{2}\right) B\left(\frac{c}{2}+\frac{d-2}{2},\frac{1}{2}\right) \\ &\quad\times \sum_{s=0}^{\infty} \frac{\left(-\frac{a}{2}\right)_s \left(-\frac{c}{2}\right)_s \left(\frac{b+c+d-1}{2}\right)_s}{\left(\frac{1}{2}\right)_s \left(\frac{c+d-1}{2}\right)_s} \frac{\cos^{2s}(x,z)}{s!} F\left(-\frac{a}{2}+s,-\frac{b}{2};\frac{c+d-1}{2}+s;\sin^2(x,y)\right). \end{aligned}$$

Proof. Use Proposition 59 and follow the same method as the one leading from Lemma 49 to Proposition 51. □

Remark 61. In the special case where $x=\pm y$ or $x\perp y$, the identities obtained in Proposition 57, 59 and 60 can be reduced further, cf. [4].

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