

Square-densities, and volume forms

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Introduction

The Greek geometers (Heron et al.) discovered a remarkable result, namely that the area of a triangle in the plane can be expressed in terms of the lengths of the three sides. Here, length and area are seen as non-negative numbers, which involves, in modern terms, formation of *absolute value* and *square root*. To express the notions and results involved without these non-smooth constructions, one can express the Heron Theorem in terms of the *squares* of the quantities in question: if $g(A, B)$ denotes the square of the length of the line segment given by A and B , the Heron formula says that the square of area of triangle ABC may be calculated by a simple algebraic formula out the three numbers $g(A, B)$, $g(A, C)$, and $g(B, C)$. Explicitly, the formula appears in (1) below. In modern terms, the formula is (except for a combinatorial constant -16^{-1}) the determinant of a certain symmetric 4×4 matrix constructed out of three numbers; see (2) below. This determinant, called the Cayley-Menger determinant, generalizes to simplices of higher dimensions, so that e.g. the square of the volume of a tetrahedron (3-simplex) $(ABCD)$ in space is given (except for a combinatorial constant) by the determinant of a certain 5×5 matrix constructed out of the six square lengths of the edges of the tetrahedron.

The Heron method has the advantage that it is symmetric w.r.to permutations of the $k + 1$ vertices of a k -simplex. Also, Heron's formula does not refer to the vector- or affine structure of the space.

In case where the n -dimensional Euclidean space, in which the k -simplex lives, is provided with an orthonormal coordinate system, one

may identify the space with R^n , with the standard inner product $\langle -, - \rangle$. Then the square lengths, square areas, square volumes etc. of the simplices can also be calculated in coordinates by an other well known and simple expression: namely as $(1/k!)^2$ times the Gram determinant of a certain $k \times k$ matrix which one may build from the coordinates of the vertices $A_0, A_1, \dots, A_k$ of the simplex. (The Gram determinant itself expresses the square volume of the parallelepipedum spanned by the vectors $A_i - A_0$.)

An important difference between the two formulae is the $(k+1)!$ -fold symmetry in the Heron formula, where the Gram formula is only $k!$ -fold symmetric, because of the special role of A_0 .

This Gram method of calculating the square-volumes has the advantage that it is easy to describe algebraically, and in particular, it is easy to describe what happens if one changes the metric; this is needed, when dealing with Riemannian manifolds, where the metric (the metric tensor) changes from point to point.

We begin in Section 1 by recalling the classical case of Euclidean spaces, i.e. finite dimensional vector spaces over R equipped with a positive definite inner product; equivalently, with a constant metric tensor. In particular, we recall the comparison (standard, but non-trivial) between the Heron and Gram calculations. In Section 2, we study Riemannian metrics on manifolds, in synthetic, or combinatorial terms, and the resulting “infinitesimal” simplices, and their square volume. In Section 3, we relate the volume form of an n -dimensional Riemannian manifold to the volume of certain infinitesimal n -simplices. This was also attempted in my [5]; I hope that the present account will be less ad hoc.

Conventions etc.

We use “SDG” as an abbreviation of “Synthetic Differential Geometry”, as expounded in [3], say, and in particular in [4]; R denotes the number line. Elements in R^n we think of as $n \times 1$ matrices (column vectors). The word *square-density* is used, where one in the positive definite case would rather say “square length”, “square area”, “square volume” It is useful to think in terms of the quantities occurring as being quantities whose physical dimension is some power of length (measured in

meter m , say), so that length is measured in m , area in m^2 , square area in m^4 , etc. Tangent vectors are not used in the following; they would have physical dimension of $m \cdot t^{-1}$ (velocity).

1 Square volumes in Euclidean spaces

1.1 Heron's formula

The basic idea for the construction of a square k -volume function goes, for the case $k = 2$, back to Heron of Alexandria (perhaps even to Archimedes); they knew how to express the square of the area of a triangle S in terms of an expression involving only the lengths a, b, c of the three sides:

$$area^2(S) = t \cdot (t - a) \cdot (t - b) \cdot (t - c)$$

where $t = \frac{1}{2}(a + b + c)$. Substituting for t , and multiplying out, one discovers (cf. [2] §1.53) that all terms involving an odd number of any of the variables a, b, c cancel, and we are left with

$$area^2(S) = -16^{-1}(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2), \quad (1)$$

an expression that only involves the *squares* a^2, b^2 and c^2 of the lengths of the sides.

The expression in the parenthesis here may be written in terms of a determinant of a 4×4 matrix (described in (2) below, with $g(01)$ denoting a^2 etc.), which makes it easy to generalize from 2-simplices (= triangles) to k simplices, in terms of determinants of certain $(k + 2) \times (k + 2)$ matrices, “Cayley-Menger matrices/determinants”; they again only involve the square lengths of the $\binom{k+1}{2}$ edges of the simplex.

An oriented k -simplex in a space M is a $(k + 1)$ -tuple of points (vertices) $(x_0, x_1, \dots, x_k)$ in M . If $g : M \times M \rightarrow R$ satisfies $g(x, x) = 0$ and $g(x, y) = g(y, x)$ for all x and y (like a metric $\text{dist}(x, y)$, or its square), one may construct a $(k + 2) \times (k + 2)$ matrix $C(x_0, x_1, \dots, x_k)$ by the following recipe: take the $(k + 1) \times (k + 1)$ matrix whose ij th entry is $g(x_i, x_j)$. It is symmetric, and has 0s down the diagonal, by the two assumption about g . Enlarge this matrix it to a $(k + 2) \times (k + 2)$ - matrix by bordering it with $(0, 1, \dots, 1)$ on the top and and on the left. The case $k = 2$ is

depicted here (writing $g(ij)$ for $g(x_i, x_j)$ for brevity; note $g(01) = g(10)$ etc., so that the matrix is symmetric.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & g(01) & g(02) \\ 1 & g(10) & 0 & g(12) \\ 1 & g(20) & g(21) & 0 \end{bmatrix} \quad (2)$$

(The indices of the rows and columns are most conveniently taken to be $-1, 0, 1, 2$.)

This is the Cayley-Menger matrix for the simplex, and its determinant is the Cayley-Menger determinant. Heron's formula then says that the value of this determinant is, modulo the "combinatorial" factor -16^{-1} , the square of the *area* of a triangle with vertices x_0, x_1, x_2 , as expressed in terms of square $g(x_i, x_j)$ of the *distances* between them. Similarly for (square-) volumes of higher dimensional simplices. Note that no coordinates are used in the construction of this determinant.

The general formula is that the square of the volume of a k -simplex is $-(-2)^k \cdot (k!)^{-2}$ times the determinant, e.g. for $k = 1, 2$, and 3 , the factors are 2^{-1} , -16^{-1} , and 288^{-1} , respectively.

Proposition 1.1 *The Cayley-Menger determinant for a k -simplex is invariant under the $(k + 1)!$ symmetries of the vertices of the simplex.*

Proof. Interchanging the vertices x_i and x_j has the effect of first interchanging the i th and j th column, and then interchanging the i th and j th row of the matrix. Each of these changes will change the determinant by a factor -1 .

We shall in the following denote the square volume of a k -simplex X in R^n , as calculated by the Heron-Cayley-Menger formula, by $vol^2(X)$. We shall later generalize it by replacing the standard inner product on R^n by an arbitrary symmetric bilinear form G on R^n . Such G is represented by a symmetric $n \times n$ matrix, likewise denoted G , and the generalized version is denoted $vol_G^2(X)$ or $vol_g^2(X)$, where g is the square length derived from G . The standard inner product on R^n corresponds to the identity matrix I ; and $vol^2(X)$ is therefore also denoted $vol_I^2(X)$.

1.2 Gram's formula

Given a k -simplex $X = (x_0, x_1, \dots, x_k)$ in R^n . Then we may form the matrix Y , which is an $n \times k$ matrix with columns $x_i - x_0$ ($i = 1, \dots, k$) where we think of the x_i s ($i = 0, 1, \dots, k$) as $n \times 1$ matrices (column matrices). The Gram matrix of the simplex X is then the $k \times k$ matrix $Y^T \cdot Y$. Its determinant is the Gram determinant of the simplex.

This determinant likewise has a volume theoretic significance: it gives the square of the volume of the parallelepipedum spanned by the k vectors $x_i - x_0$ ($i = 1, \dots, k$). So geometrically, it depends on the geometric construction of the parallelograms out of which the parallelepipedum is constructed.

The following Proposition is only included for a comparison with the issue of $(k+1)!$ symmetry of the formulae.

Proposition 1.2 *The Gram determinant for a k -simplex is invariant under the $(k+1)!$ symmetries of the k -simplex.*

Proof. Interchanging x_i and x_j , for i and $j \geq 1$ implies an interchange of the corresponding columns in the Y -matrix, and this interchanged matrix comes about by multiplying Y on the right by the $k \times k$ matrix S obtained from the unit matrix by interchanging its i th and j th column. This S has determinant -1 . So $S^T \cdot Y^T \cdot Y \cdot S$ has same determinant as $Y^T \cdot Y$. Interchanging x_0 and x_1 in the simplex corresponds to multiplying the Y -matrix on the right by the matrix S_j , obtained from the unit $k \times k$ matrix by replacing its first row by the row $(-1, -1, \dots, -1)$. The matrix S has determinant -1 , so $S^T \cdot Y^T \cdot Y \cdot S$ has same determinant as $Y^T \cdot Y$. Similarly for exchanging x_0 by x_j .

1.3 Comparison formula, and $\text{vol}^2(X)$

In the vector space R^n , with its standard inner product, it makes sense to compare the values of the formulas for square volume of a k -simplex $X = (x_0, x_1, \dots, x_k)$. Let C denote the $(k+2) \times (k+2)$ matrix (Cayley-Menger) formed by the square distances between the vertices, as described above, and let $Y^T \cdot Y$ be the Gram $k \times k$ matrix, likewise described above. There is a known relation between their determinants

$$\frac{1}{-(-2)^k} \det(C) = \det(Y^T \cdot Y)^2. \quad (3)$$

For a proof, see reference [10]. Both sides express $(k!)^2$ times the square $\text{vol}^2(X)$ of the volume of the simplex X (or express the square of the volume of the parallelepipedum spanned by the column vectors $x_i - x_0$ in the matrix Y). But note that the right hand side only makes sense in a coordinatized situation, so in the following, $\text{vol}^2(X)$ is used for the more general Heron formula for the square volume of a simplex, as calculated in terms of the Cayley-Menger determinant C . In R^n the relationship is thus, for a k -simplex X in R^n ,

$$\text{vol}^2(X) = \frac{1}{k!^2} \det(Y^T \cdot Y) \quad (4)$$

1.4 Change of metric

The formula (3) refers to the vector space R^n with the standard inner product. The Gram formula is easy to extend when one wants to replace this inner product on R^n by another, and square length is given in terms of this new inner product $\langle -, - \rangle_G$. We identify the new inner product by a symmetric $n \times n$ matrix G , with $\langle u, v \rangle_G$ defined as the matrix product $u^T \cdot G \cdot v$. (The standard inner product corresponds to the identity $n \times n$ matrix I , so $\langle u, v \rangle = u^T \cdot I \cdot v$.)

Given a k -simplex $X = (x_0, x_1, \dots, x_k)$ in R^n . Let Y be the $n \times k$ matrix whose k columns are $x_j - x_0$, with $j = 1, \dots, k$. Then the $k \times k$ matrix $Y^T \cdot G \cdot Y$ is the G -generalization for the Gram matrix of the simplex, and its determinant is the square of the volume, as defined by the metric G , of the parallelepipedum spanned by the column vectors in Y .

Can we calculate the square volume of this simplex using the Heron-Cayley-Menger method, when one uses the metric G rather than the standard metric? This question can be answered by yes, provided that G is positive definite. More precisely, we shall assume that G can be written as a matrix product $H^T \cdot H$, with H another $n \times n$ matrix. (Over the real numbers $\mathbb{R}$, this property characterizes positive definite quadratic forms, cf. e.g. [7] XI Prop. 6.)

Let us write $\text{vol}_G^2(X)$ or $\text{vol}_G^2(x_0, \dots, x_k)$ for the square volume of the k -simplex X spanned by the x_i s, when the metric G is used, so $\text{vol}_I^2(x_0, \dots, x_k)$ is the square volume of the simplex for the standard metric.

Proposition 1.3 *If an $n \times n$ matrix G can be written $G = H^T \cdot H$ for an $n \times n$ matrix H , then (G is symmetric and) we have for vol_G^2*

$$\text{vol}_G^2(x_0, \dots, x_k) = \text{vol}_I^2(H \cdot x_0, \dots, H \cdot x_k),$$

and as such may be calculated in terms of the standard Cayley-Menger determinant.

Proof. We have for any $y \in R^n$ that $H(y)^T = (H \cdot y)^T = y^T \cdot H^T$, so

$$y^T \cdot G \cdot y = y^T \cdot H^T \cdot H \cdot y = (H \cdot y)^T \cdot (H \cdot y),$$

so letting y range over $x_i - x_j$ ($i, j = 0, 1, \dots, k$), the left hand side calculates the left hand expression; the right hand expression may be calculated in terms of the Cayley-Menger determinant (C).

We have, for G and H as in the Proposition, that

$$\det(G) = \det(H)^2 \tag{5}$$

so that if $\det(G)$ is invertible, there are exactly two numbers which deserve the name $\sqrt{\det(G)}$.

2 Infinitesimal simplices

As in [4], say, we consider the following kind of structure on an object M in a category $\mathcal{E}$ with finite inverse limits M , namely subobjects of $M \times M$,

$$M_{(0)} \subseteq M_{(1)} \subseteq M_{(2)} \dots \subseteq M \times M,$$

each of the $M_{(r)}$ s being a reflexive and symmetric relation, with $M_{(0)}$ being the equality relation. We have in mind the “ r th neighbourhood of the diagonal” of M , as considered in algebraic geometry, or the “prolongation spaces” of manifolds as considered in e.g. [6]. Except for M_0 , these relations are not transitive.

We use the well known “synthetic” language to express constructions in categories $\mathcal{E}$ with finite limits, in “elementwise” terms¹, in particular we consider for a natural number k the object of r -infinitesimal

¹Recall that a generalized element of an object M in a category $\mathcal{E}$ is just an arbitrary map in $\mathcal{E}$ with codomain M ; see [3] II.1, [8] V.5, or [9] 1.4.

k -simplices in M , meaning the subobject of $M \times M \times \dots \times M$ ($k + 1$ times) consisting of $k + 1$ -tuples $(x_0, x_1, \dots, x_k)$ of elements of M with $(x_i, x_j) \in M_{(r)}$ for all $i, j = 0, 1, \dots, k$; such a $k + 1$ -tuple, we shall call an *r -infinitesimal k -simplex*.

Note that the question of whether a k -simplex is r -infinitesimal only depends on the “edges” (x_i, x_j) (face-1-simplices) of the simplex, equivalently, it depends on the 1-skeleton of the simplex. We shall consider also a further condition on k -simplices which depends on the 2-skeleton, i.e. it is a condition on the face-2-simplices of the simplex. We shall use the predicate “thin”; this kind of condition was introduced by F. B  r in his thesis [1]; see Section 4. This condition will be important in Section 3 below. We shall only use the predicate “thin” for the case $r = 2$, so a thin k -simplex is in particular 2-infinitesimal.

We shall, as in [4], write $x_i \sim_r x_j$ for $(x_i, x_j) \in M_{(r)}$.

For $r = 1$ and $r = 2$, we shall consider maps from the object of r -infinitesimal k -simplices to R which have the property that they vanish if $x_i = x_j$ for some $i \neq j$. For $r = 1$, combinatorial differential k forms ω have this property: they vanish if $x_i = x_j$. (In the context of SDG, such maps are automatically alternating with respect to the $(k + 1)!$ permutations of the x_i s, see [4] Theorem 3.1.5.)

For $r = 2$, such maps have not been considered much², except for the case where $k = 1$, where Riemannian metrics g , in the combinatorial sense, are examples of such maps; for this case, we think of $g(x_0, x_1)$ as the square of the distance between x_0 and x_1 . The g s of interest are symmetric, $g(x_0, x_1) = g(x_1, x_0)$. In the context of SDG, we have

Proposition 2.1 *Given $g : M_{(2)} \rightarrow R$ with $g(x, x) = 0$ for all x . Then g is symmetric iff it vanishes on $M_{(1)} \subseteq M_{(2)}$.*

Proof. It suffices to consider an R^n chart around x ; we consider the degree ≤ 2 part of the Taylor expansion of g around x . Then g is given as $g(x, y) = C(x) + \Omega(x; y - x) + (y - x)^T \cdot G(x) \cdot (y - x)$, where $C(x)$ is a constant, Ω is linear in the argument after the semicolon, and $G(x)$ is a symmetric $n \times n$ matrix. To say that g vanishes on the diagonal $M_{(0)}$ (i.e. $g(x, x) = 0$ for all x) is equivalent to saying that $C(x) = 0$ for all x .

²For $r = 2$ and $k = 1$, such things were in [4] 8.1 called “quadratic differential forms”.

We now compare $g(x, y)$ and $g(y, x)$; we claim

$$(y - x)^T \cdot G(x) \cdot (y - x) = (y - x)^T \cdot G(y) \cdot (y - x). \quad (6)$$

Taylor expanding from x the $G(y)$ on the right hand side gives that this the difference between the two sides is $(y - x) \cdot dG(x; y - x) \cdot (y - x)$ which is trilinear in $y - x$, and therefore vanishes, since $x \sim_2 y$. So we have that if Ω vanishes, then g is symmetric; vice versa, if g is symmetric, its restriction to $M_{(1)}$ is likewise symmetric, and (being a differential 1-form), it is alternating, so the Ω -part vanishes, which in coordinate free terms says: $g(x, y) = 0$ for $x \sim_1 y$.

For the number line R , $(x_0, x_1) \in R_{(2)}$ iff $(x_1 - x_0)^3 = 0$, and the map g given by $g(x_0, x_1) := (x_1 - x_0)^2$ is a map as described in the Proposition. In fact, it is the restriction of the standard “square-distance” function $R \times R \rightarrow R$.

So we recall, respectively pose, the following definitions, corresponding to $r = 1$ and $r = 2$. Let M be a manifold.

Definition 2.2 *A (combinatorial) differential k -form on M is function ω from the set of 1-infinitesimal k -simplices in M to R , which is alternating with respect to the $(k + 1)!$ permutations of the x_i s, and which vanishes if $x_i = x_j$ for some $i \neq j$.*

Definition 2.3 *A k -square-density on M is an R -valued function Σ from the set of 2-infinitesimal k -simplices in M which is symmetric with respect to the $(k + 1)!$ permutations of the x_i s, and which vanishes if $x_i = x_j$ for some $i \neq j$.*

Note that for $k = 1$, Proposition 2.1 gives that 1-square densities g have the property that they vanish not just on $M_{(0)}$ (the diagonal), but also on $M_{(1)}$: $g(x, y) = 0$ if $x \sim_1 y$. So the notion of 1-square density agrees with “(combinatorial) quadratic form”, as considered in [4], Section 8.1.

2.1 k -square-densities Σ_g from 1-square-densities g

We shall argue that the Cayley-Menger determinants define square-densities. We already argued above that these determinants are symmetric: the value does not change when interchanging x_i and x_j . We

have to argue for the vanishing condition required. If $x_i = x_j$, then $g(x_i, x_m) = g(x_j, x_m)$ for all m , and this implies that the i th and j th rows in the matrix are equal, which implies that the determinant is 0.

We denote the k -square-density corresponding to a 1-square-density by Σ_g (when k is understood from the context).

2.2 k -square-densities Σ^ω from differential k -forms ω

Essentially this is the process of *squaring* (in R) the values, so it is tempting to denote the square-density Σ^ω which we are aiming for, by ω^2 . However, the values of a differential form are always numbers of square 0. We do get, however, a well defined k -square-density out of a differential k -form by a two step procedure: 1) to *extend* the given k form ω to a suitable function $\bar{\omega}$, to allow as inputs 2-infinitesimal k -simplices; and then 2) *squaring* $\bar{\omega}$ valuewise. “Suitable” implies in particular that the squared function $\bar{\omega}$ is a $(k+1)!$ -symmetric function when applied to 2-infinitesimal k -simplices. We shall prove that such an extension is possible; but is not unique. It depends on choosing a coordinate chart.

The question is local, so let us assume that the manifold M is an open subset of R^n . Then the k -form ω is given by a function $\Omega : M \times (R^n)^k \rightarrow R$, such that

$$\omega(x_0, x_1, \dots, x_k) = \Omega(x_0; x_1 - x_0, x_2 - x_0, \dots, x_k - x_0)$$

where for each $x_0 \in M$, the function $\Omega(x_0; -, \dots, -) : (R^n)^k \rightarrow R$ is k -linear and alternating in the k arguments; these arguments are arbitrary vectors in R^n , in particular, they may be of the form $x_i - x_0$ for $x_i \sim_2 x_0$, so the restriction of $\Omega(x_0; x_1 - x_0, x_2 - x_0, \dots, x_k - x_0)$ to the set of 2-infinitesimal 2-simplexes defines an extension $\bar{\omega}$ of ω , so

$$\bar{\omega}(x_0, x_1, \dots, x_k) := \Omega(x_0; x_1 - x_0, x_2 - x_0, \dots, x_k - x_0) \quad (7)$$

In this form, the fact that ω is alternating w.r.to the $k!$ permutations of the x_i s ($i = 1, \dots, k$) can be read of from the fact that $\Omega(x_0; \dots)$ is alternating. It is also alternating w.r.to permutations involving x_0 , as long as the x_i s are $\sim_1 x_0$; this can be seen from a Taylor expansion argument, see the proof of Theorem 3.1.5 in [SGM] (or: restrict the argument below to the case of 1-infinitesimal simplices).

Now if we use Ω to construct the extension $\bar{\omega}$, to 2-infinitesimal k -simplices, the constructed $\bar{\omega}$ will still be alternating w.r.to permutations of the x_i s for $i \geq 0$, but the Taylor expansion argument mentioned fails for the interchange of, say, x_0 and x_1 : we cannot conclude that $\Omega(x_1; x_0 - x_1, \dots) = -\Omega(x_0; x_1 - x_0, \dots)$. This failure get repaired by valuewise squaring:

Proposition 2.4 *For any 2-infinitesimal k -simplex $(x_0, x_1, \dots, x_k)$, we have*

$$\Omega(x_1; x_0 - x_1, x_2 - x_1, \dots)^2 = \Omega(x_0; x_1 - x_0, x_2 - x_0, \dots)^2.$$

Proof. We shall only do the case $k = 1$. (For the more general case, the further argument is essentially the same as in the proof of Proposition 1.2 above.) First, we have by a Taylor expansion from x_0

$$\begin{aligned} \Omega(x_1; x_0 - x_1) &= \Omega(x_0; x_0 - x_1) + d\Omega(x_0; x_1 - x_0; x_0 - x_1) \\ &\quad + \text{a } d^2\Omega(x_0; \dots) \text{ term, trilinear in } x_1 - x_0. \end{aligned}$$

The trilinear term vanishes, because $x_1 \sim_2 x_0$. Now we square, and get

$$\begin{aligned} \Omega(x_1; x_0 - x_1)^2 &= \Omega(x_0; x_0 - x_1)^2 + 2 \cdot \Omega(x_0; x_0 - x_1) \cdot d\Omega(x_0; x_1 - x_0, x_1 - x_0) \\ &\quad + \text{a term } (d\Omega(x_0; \dots))^2, \text{ quadrilinear in } x_1 - x_0. \end{aligned}$$

The quadrilinear term vanishes because $x_1 \sim_2 x_0$, but also the term $\Omega \cdot d\Omega$ vanishes, because it is trilinear in $x_1 - x_0$. So we get

$$\Omega(x_1; x_0 - x_1)^2 = \Omega(x_0; x_0 - x_1)^2 = (-\Omega(x_0; x_1 - x_0))^2 = \Omega(x_0; x_1 - x_0)^2,$$

as desired.

We conclude that a differential k -form ω can be extended to an $\bar{\omega}$ (whose input are 2-infinitesimal k -simplices), such that $\bar{\omega}^2$ is $(k+1)!$ -symmetric. (Also, the extension constructed also clearly has the property that it vanishes if $x_i = x_j$ for some $i \neq j$.)

Proposition 2.5 *If two extended k -forms extend the same differential k -form ω , the two resulting square-densities agree.*

This is proved by passing to a coordinate chart, in which case the proof comes about by a simple counting of degrees of the polynomial functions expressing the two forms and their difference. This in turn is best formulated in terms of k -fold tensor products of symmetric algebras; it is given in full in the Appendix.

Because of the Proposition, there is a well-defined “squaring” process, leading from differential k -forms to k -square-densities on a manifold M : extend the form ω , and square the result.

3 Riemannian structure in coordinates

The question of when a 1-square-density g qualifies as a Riemannian metric may be described in terms of a coordinate chart.³ So we consider a manifold M which is embedded as an open subset of R^n (elements of R^n we write as $n \times 1$ matrices). A 1-square-density g on M can in this case be given by a family by a family of symmetric $n \times n$ matrices $G(x)$ (for $x \in M$), such that for $x \sim_2 y$,

$$g(x, y) = (y - x)^T \cdot G(x) \cdot (y - x)$$

(which equals $(y - x)^T \cdot G(y) \cdot (y - x)$ by (6)). To say that such g is positive definite is then to say that each $G(x)$ can be written $H^T \cdot H$ for some $n \times n$ matrix $H = H(x)$, as in Proposition 1.3, and assumed to be invertible.

Recall the relationship between g and G : $g(y, z) = (z - y)^T \cdot G(y) \cdot (y - z)$. We shall prove that for thin simplices, the $G(y)$ here may be replaced by suitable $G(x)$:

Lemma 3.1 *Let (x, y, z) be a thin 2-simplex. Then $g(y, z)$ may be calculated by $(z - y)^T \cdot G(x) \cdot (z - y)$.*

Proof. Taylor expand G from x ; we have

$$\begin{aligned} g(y, z) &= (y - z)^T \cdot G(y) \cdot (y - z) \\ &= [(y - z)^T \cdot G(x) \cdot (y - z)] + [(y - z)^T \cdot dG(x; y - x) \cdot (y - z)] + \\ &\quad + \text{higher terms involving } d^2G(x; \dots) \cdot (y - z) \end{aligned}$$

³A classical description, which is coordinate free, is possible in terms of the tangent bundle $T(M)$, namely in terms of bilinear maps $T_x \times T_x \rightarrow R$. See [4] 8.1.3 for a comparison.

but the term $(y - z)^T \cdot dG(x; y - x) \cdot (y - z)$ (as well as the “higher” terms), vanish by the assumed thin-ness because they contain $(y - x), (y - z), (y - z)$ in a trilinear way.

A Riemannian metric on a manifold M allows us to describe the square volume of any 2-infinitesimal k -simplex in M : use the (generalized) Heron formula. If the k -simplex is $(x_0, \dots, x_k)$, the Heron (Cayley-Menger-) formula is an expression in the $g(x_i, x_j)$ s. In a coordinatized situation, it can therefore be calculated in terms of the $(x_j - x_i)^T \cdot G(x_i) \cdot (x_j - x_i)$. For $k \geq 2$, this expression is in general rather complicated, because there are several $G(x_i)$ s involved. However, if the simplex is thin, any one of these will do for all; for instance, they may all be replaced by $G(x_0)$: this is an immediate consequence of Lemma 3.1.

We consider a Riemannian manifold M, g , with M an open subset of R^n ; the Riemannian metric g on M is not assumed to be the one induced by the standard metric on R^n . It is given by positive definite $n \times n$ matrices $G(x)$, for $x \in M$.

Proposition 3.2 *Let $X = (x_0, \dots, x_k)$ be a thin k -simplex in $M \subseteq R^n$. Then*

$$\text{vol}_g^2(x_0, \dots, x_k) = \text{vol}_I^2(H \cdot x_0, \dots, H \cdot x_k)$$

where H satisfies $H^T \cdot H = G(x_0)$.

Proof. The left hand side is an expression in the square distances $g(x_i, x_j)$, and they are given by expressions $(x_i - x_j)^T \cdot G(x_i) \cdot (x_i - x_j)$. For thin 2-infinitesimal simplices, we may by 3.1 replace all the $G(x_i)$ occurring here by $G(x_0)$. Thus, for a thin k -simplex, we are back at the situation with a constant G , and can therefore use Proposition 1.3.

Assume that g is positive definite (in the above sense) at every $x \in M$, so $G(x) = H(x)^T \cdot H(x)$ for some $n \times n$ matrix. Also $\det(G(x)) = (\det(H(x)))^2$ so $\det(G(x))$ (as x varies over M) defines a function $W(g) : M \rightarrow R$ that pointwise admits square roots. The assumed square roots of them are only “half unique”; they are unique modulo sign. If M is connected and oriented, it is possible to choose a (smooth) function whose square is W . This is possibly in exactly two ways, by assumption. Choose one of these functions and denote it $\sqrt{\det(G)}$.

Theorem 3.3 *Assume that g is a Riemannian metric on an n -dimensional manifold M . Then there exists locally on M a differential n -form ω such that Σ_g and Σ^ω agree on all thin n -simplices; ω is (locally) unique up to sign, and deserves the name volume form for g .*

Proof. Since the data and assertions in the statement do not depend on the choice of a coordinate system, it suffices to prove the assertion in a coordinate chart. So assume that $M \subseteq \mathbb{R}^n$ is an open subset. We use this coordinate chart to construct the extension $\bar{\omega}$ as described in Subsection 2.2. Then g is given in terms of the symmetric positive definite $n \times n$ matrices $G(x)$ (for $x \in M$). Consider the differential n -form ω , as well as the extended n -form $\bar{\omega}$, given by the formula

$$\bar{\omega}(x_0, x_1, \dots, x_n) := \frac{\sqrt{\det G(x_0)}}{n!} \cdot \det(x_1 - x_0, \dots, x_n - x_0)$$

for any 2-infinitesimal n -simplex X . Let Y denote the $n \times n$ matrix with $x_j - x_0$ as its j th column. Then squaring the defining equality for $\bar{\omega}$ gives

$$\bar{\omega}^2(X) = \frac{\det G(x_0)}{n!^2} \cdot (\det Y)^2 = \frac{1}{n!^2} \det(Y^T \cdot G(x_0) \cdot Y)$$

using the product rule for determinants and $\det(Y^T) = \det(Y)$. By the lemma, if $X = (x_0, x_1, \dots, x_n)$ is a thin n -simplex, the $G(x_i)$ that enter in the expressions for the square distances between vertices in X may all be replaced by $G(x_0)$. We have by positive definiteness $G(x_0) = H^T \cdot H$ for some square matrix H , and then the equation continues

$$= \frac{1}{n!^2} \det(Y^T \cdot H^T \cdot H \cdot Y) = \frac{1}{n!^2} \det((H \cdot Y)^T \cdot (H \cdot Y)) = \text{vol}_I^2(H \cdot Y) = \text{vol}_g^2(Y),$$

the two last equality signs by (3) and Proposition 3.2, respectively. Restricting the $\bar{\omega}$ to the set of 1-infinitesimal n -simplices gives therefore an n -form ω such that

$$\Sigma^\omega = \Sigma_g (= \text{vol}_g^2);$$

such ω deserves the name of *volume form* of g .

(An argument for the local uniqueness, modulo sign, of such ω is not presented here, but can presumably be given.)

4 Second order i-structure

In his Cambridge thesis [1], F. Bár introduced what he calls “higher order infinitesimal structures” (i-structure) in a manifold M , with particular emphasis on second-order such structures. The crux (for this case) is that there is a refinement of the notion of 2-infinitesimal n -simplex $(x_0, \dots, x_n)$, where one not only requires that $x_i \sim_2 x_j$ for $i, j = 0, 1, \dots, n$, but furthermore requires, for all $i, j, k = 0, 1, \dots, n$, that the 2-simplex (x_i, x_j, x_k) is a *thin* 2-simplex, in a sense which we shall describe now (Bár does not use the word “thin” though). He first describes the thinness notion for the case where M is a (finite dimensional) KL vector space V . He defines $DN_2(V) \subseteq D_2(V) \times D_2(V) \times D_2(V)$ as the set of (v_1, v_2, v_3) such that any trilinear $V \times V \times V \rightarrow R$ annihilates (v_1, v_2, v_3) . Then the 2-simplex (x_0, x_1, x_2) in V is *thin* if $(x_1 - x_0, x_2 - x_0, x_2 - x_1) \in DN_2(V)$. He proves that the notion of thin 2-infinitesimal 2-simplex is preserved by (smooth) maps, and hence gives a well defined notion of thinness of 2-infinitesimal simplices in an arbitrary manifold M , (preserved by maps between manifolds): a 2-infinitesimal k -simplex in M is then called thin if all its face 2-simplices are thin.

To illustrate the notion of thinness and to show its naturality:

Consider 2-simplices on the line R , with first vertex 0, so consider $(0, x, y)$. It is 2-infinitesimal if $x^3 = 0$, $y^3 = 0$ and $(x - y)^3 = 0$. Binomial expansion of the last condition gives $x^3 - 3x^2y + 3xy^2 - y^3 = 0$, and using $x^3 = y^3 = 0$, this is equivalent to $x^2y = xy^2$. But it does not imply that $x^2y = 0$ or $xy^2 = 0$. This will hold if the simplex is thin.

Using the standard Riemannian metric on the line, $g(x, y) = (y - x)^2$, the square area of the 2-infinitesimal 2-simplex $(0, x, y)$ as calculated by Heron’s formula, is $8^{-1}x^2y^2$, which need not be 0.

But if the simplex $(0, x, y)$ is thin, x^2y is 0, implying $x^2y^2 = 0$.

So square areas of 2-infinitesimal 2-simplices on 1-dimensional manifolds with a Riemannian metric need not be 0; it is 0 if the simplex is thin. (Gram’s method gives 0 in any case.)

Given a simplex (x, y, z) with $x \sim_1 y$ and $x \sim_1 z$. Then $y \sim_2 z$, so the simplex is 2-infinitesimal. In fact, such simplex is thin. For, in any coordinate patch from a vector space U , we may form the differences $u := (x - y) \in D_1(V)$, $v := (x - z) \in D_1(V)$ and $w := (y - z) \in D_2(V)$; so $w = v - u$. Let $\phi : V^3 \rightarrow R$ be trilinear. Then $\phi(u, v, w) = \phi(u, v, v) - \phi(u, v, u)$,

and both terms here are 0, because they contain v , respectively u , in a bilinear position. 2-simplices of this kind were studied in [4] and [5] under the name “2-whisker” - an (un-symmetric) weakening of the notion of 1-infinitesimal 2-simplex. The notion of thin (2-infinitesimal) 2-simplex is a weakening of the whisker notion, and is symmetric.

Appendix: The symmetric algebra

Let k be a commutative ring; in the following, “algebra”, “module” etc. refer to (commutative) k -algebras, $-$ -modules etc. Every algebra has an underlying module. The forgetful functor from the category of algebras to modules has a left adjoint, formation of symmetric algebras. Recall that if U is a module, the symmetric algebra $S^\bullet(U)$ is the value of this left adjoint, i.e. the free commutative algebra generated by U .

It is a graded algebra: in degree 0, it has k , in degree 1, it has U (and the inclusion of U as degree 1-part has the universal property making it the unit for the adjunction); in degree 2, it has $(U \otimes U)/2!$, etc., so elements in degree 2 are represented by linear combinations of expressions $e \otimes e'$ with e and e' in U . And $e \otimes e'$ is identified with $e' \otimes e$; $e \cdot e' := e \otimes e'$, etc. In degree r , it has $(U \otimes \dots \otimes U)/r!$, the r th symmetric tensor power.

We shall mainly be interested in the case where $U = V^*$, where V is a k -module isomorphic to k^n . So the elements of degree 1 form the k -module of linear maps $V \rightarrow k$. If $e_1, \dots, e_n$ is a basis for V with dual basis $x_1, \dots, x_n$, $S^\bullet(V^*)$ may be identified with the algebra of polynomial k -valued functions in n variables $x_1, \dots, x_n$. However, it is preferable not to coordinatize (i.e. putting $U = k^n$, to avoid double indices later. The “semantics” of polynomials as polynomial *functions* $V \rightarrow k$ is then built in from the outset, since the module of degree 1 elements is taken to be V^* , thus its elements are (linear) *functions* $V \rightarrow k$.

So we consider the graded algebra $S^\bullet(U)$ for a general k -module U , with $S^1(U) = U$, $S^2(U) = (U \otimes U)/2!$, $\dots$

$$S^\bullet(U) = k \oplus S^1(U) \oplus S^2(U) \oplus \dots$$

It has some particular ideals, $I \supseteq I_2 \supseteq I_3 \supseteq \dots$ where I is the ideal generated by the degree 1 part $S^1(U)$, $I_2 (= I^2)$ is the ideal generated by the degree 2 part $S^2(U)$, etc.; equivalently, $I_r = \bigoplus_{j=r}^{\infty} S^j(U)$.

Next, we consider the k -fold tensor product (k denotes here a natural number, not the ground ring k) $(S^\bullet)^k(U)$, meaning

$$S^\bullet(U) \otimes \dots \otimes S^\bullet(U) \cong S^\bullet(U \oplus \dots \oplus U)$$

(k -fold $\otimes$ and $\oplus$). The left hand side here is multi-graded (k -graded); The elements of multidegree $(r_1, \dots, r_k)$ form the direct summand $S^{r_1} \otimes \dots \otimes S^{r_k}$. Also the symmetric group $k!$ evidently acts on $(S^\bullet)^k(U)$, so we may consider the subalgebra of $k!$ -symmetric elements; but we may also consider the submodule of alternating elements. (In particular, alternating elements of multidegree $(1, \dots, 1)$ in $S^\bullet(V^*)$ correspond to k -linear alternating functions $V^k \rightarrow k$.)

The following is now a simple “counting degrees” argument: if $\omega \in (S^\bullet)^k(U)$ is of multidegree $(1, \dots, 1)$, and if $\theta \in (S^\bullet)^k(U)$ is of multidegree $\geq (1, \dots, 1)$ and of total degree $\geq k + 1$, then in the algebra $(S^\bullet)^k(U)$ we have that $(\omega + \theta)^2 - \omega^2$ is of total degree $\geq 2k + 1$. For, the difference is $\theta^2 - 2\theta \cdot \omega$. Here, the first term is of total degree $\geq 2(k + 1) \geq 2k + 1$. In the second term, θ is a linear combination of terms of multidegree of the form $(1, 1, \dots, 1 + p, \dots, 1)$ for $p \geq 1$, so $\theta \cdot \omega$ is a linear combination of terms of multidegree

$$(1, 1, \dots, 1 + p, \dots, 1) + (1, 1, \dots, 1, \dots, 1) = (2, 2, \dots, 2 + p, \dots, 2)$$

which is of total degree $2k + p \geq 2k + 1$.

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