Torsion Geometry, Superconformal Symmetry and T-duality

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Outline

1. Torsion Geometry
   - Metric geometry with torsion
   - KT Geometry
   - HKT Geometry

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1 **Torsion Geometry**
- Metric geometry with torsion
- KT Geometry
- HKT Geometry

2 **Superconformal Symmetry**
- Superconformal Quantum Mechanics
- The Superalgebras $D(2,1;\alpha)$
- Geometric Structure

Outline

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Metric geometry with torsion

- metric g, connection ∇, torsion $T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
- $\nabla g = 0$

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- any $c \in \Omega^3(M)$ will do
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- *strong* if $dc = 0$

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Study *compact* simply-connected torsion geometries with

- compatible complex structures and
- small symmetry group

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KT Geometry

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KT geometry

additionally

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Two form $F_I(X, Y) = g(IX, Y)$

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- KT geometry = Hermitian geometry + Bismut connection
- $c = 0$ is Kähler geometry
- strong KT is $\partial\bar{\partial}F_I = 0$

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Example

$M^6 = S^3 \times S^3 = SU(2) \times SU(2)$

Gauduchon (1991)

every compact Hermitian M^4 is conformal to strong KT

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HKT structure

(g, ∇, I, J, K) with

- (g, ∇, A) KT, $\quad A = I, J, K$
- $IJ = K = -JI$

$c = -AdF_A$ is independent of A

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Dim 4 T^4, K3, $S^3 \times S^1$ (Boyer, 1988)

Dim 8 Hilbert schemes, $SU(3)$, nilmanifolds, vector bundles over discrete groups (Verbitsky, 2003; Barberis and Fino, 2008)

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Compact, simply-connected examples which are neither hyperKähler nor homogeneous?

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Superconformal Quantum Mechanics

N particles in 1 dimension

$$H = \frac{1}{2} P_a^* g^{ab} P_b + V(x)$$

Standard quantisation

$$P_a \sim -i \frac{\partial}{\partial x^a}, \quad a = 1, \ldots, N$$

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Michelson and Strominger (2000); Papadopoulos (2000)

- operator D with $[D, H] = 2iH \iff$ vector field X with $L_X g = 2g$ & $L_X V = -2V$
- K so $\mathrm{span}\{iH, iD, iK\} \cong \mathfrak{sl}(2,\mathbb{R}) \iff X^\flat = g(X, \cdot)$ is closed
- then $K = \frac{1}{2} g(X, X)$.

Choose a superalgebra containing $\mathfrak{sl}(2,\mathbb{R})$ in its even part.

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The Superalgebras $D(2,1;\alpha)$

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- Real form $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) + \mathfrak{su}(2)_+ + \mathfrak{su}(2)_-$.
- Over $\mathbb{R}$, isomorphisms for $\alpha^{\pm 1}$

Outline

Superconformal Geometry

$\mathcal{N} = 4B$ quantum mechanics

with $D(2,1;\alpha)$ superconformal symmetry

$\leftrightarrow$

HKT manifold M

with X a *special homothety of type* (a,b)

- $L_X g = ag$,
- $L_{IX} J = bK$,
- $L_X I = 0$, $L_{IX} I = 0$,...

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For $a \neq 0$

- M is non-compact
- $\mu = \frac{2}{a(a-b)} \|X\|^2$ is an *HKT potential*

$$F_I = \tfrac{1}{2}(dd_I + d_J d_K)\mu = \tfrac{1}{2}(1 - J)dId\mu.$$

Superconformal Geometry II

Example

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$a = 2$, $b = -2$, $\alpha = -2$.

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Get to $a = 0$, special isometry, by potential change

$$g_1 = \frac{1}{\mu} g - \frac{1}{2\mu^2} (d^{\mathbb{H}} \mu)^2$$

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In this case

- $dX^\flat = 0$
- $b_1(M) \geqslant 1$

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T-duality as a Twist

- X generating a circle action on M
- $(P, \theta, Y) \xrightarrow{\pi} M$ an invariant principal S^1-bundle

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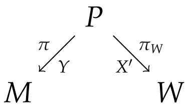

Definition

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Transverse locally free lifts always exist for $X \lrcorner F_\theta$ exact.

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$$\begin{array}{ccc} & P & \\ {}^{\pi}\swarrow{}_{Y} & & {}^{X'}\searrow{}^{\pi_W} \\ M & & W \end{array}$$

Dually

M is a twist of W with respect to $X_W = (\pi_W)_* Y$, $\theta_W = \frac{1}{a}\theta$

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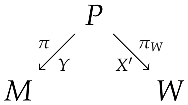

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Tensors α on α_W on M and W are *$\mathcal{H}$-related*, $\alpha_W \sim_{\mathcal{H}} \alpha$ if their pull-backs agree on $\mathcal{H} = \ker \theta$

$d\alpha_W \sim_{\mathcal{H}} d\alpha - F_\theta \wedge \frac{1}{a} X \lrcorner \alpha$
if invariant

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- *untwist locally to $dX^\flat = 0$ on $S \times S^1$*
- *change potential on $S \times \mathbb{R}$ to $a \neq 0$, ($\alpha = -2$)*

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Many simply-connected examples when $b_2(S) \geqslant 1$
E.g., $Q = k\,\mathbb{C}P(2)$

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General HKT with Circle Symmetry

- $M = N_1 \times N_2$
- N_2 with an HKT circle symmetry X
- $[F_\theta] \in H^2(N_1, \mathbb{Z})$, $F_\theta \in S^2E$

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Generate simply-connected examples

Example

N_1 a K3 surface
F_θ self-dual, primitive

Generalises to torus actions

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F_θ self-dual, primitive

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HKT nilmanifold $M = G/\Gamma$

General HKT with Circle Symmetry

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Proposition

Every HKT nilmanifold may be obtained by successive twists of a torus T^{4n}.

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$$\sum_{i,j} (a^{-1})_{ij} X_i \wedge I F_j \qquad \text{independent of } I$$

gives further non-compact HKT examples

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