Twists and special holonomy

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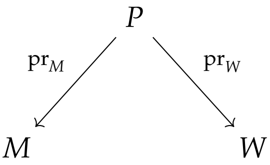
Including joint work with Óscar Maciá (Valencia) and Marco Freibert (Aarhus)

Outline

Twist construction

Twist data

- M manifold
- $X \in \mathfrak{X}(M)$, circle action
- $F \in \Omega^2_{\mathbb{Z}}(M)^X$
- $a \in C^\infty(M)$ with $da = -X \lrcorner F$



horizontal distribution $\mathcal{H} = \ker\theta \subset TP$

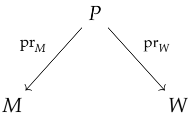
α tensor on M is *$\mathcal{H}$-related* to α_W on W if

$$\mathrm{pr}_M^*\alpha = \mathrm{pr}_W^*\alpha_W \quad \text{on } \mathcal{H}$$

Write $\alpha \sim_{\mathcal{H}} \alpha_W$

Twist computations

$\alpha \sim_{\mathcal{H}} \alpha_W$ if $\mathrm{pr}_M^* \alpha = \mathrm{pr}_W^* \alpha_W$ on $\mathcal{H} = \ker \theta$



- $\alpha \in \Omega^p(M)$: $d\alpha_W \sim_{\mathcal{H}} d\alpha - \frac{1}{a} F \wedge (X \lrcorner \alpha)$
- I complex structure on M:
  I_W integrable if and only if $F \in \Lambda_I^{1,1}$

Example

$M = M(n) := \mathbb{C}P^n \times T^2$ Kähler, X on T^2, $F = \omega_{\mathrm{FS}}$: $W = S^{2n+1} \times S^1$ Hermitian non-Kähler.

Example

$M = T^n$, F left-invariant: W is a nilmanifold corresponding to $\mathfrak{g}^* = (0^{n-1}, F)$.

How can we get Kähler, hyperKähler,. . . ?

Model: HyperKähler modification

X is a tri-Hamiltonian isometry of hyperKähler (M, g, I, J, K)

Definition (Dancer-Swann)

The *hyperKähler modification* of M is

$$M_{\text{mod}} = (M \times \mathbb{H}) /\!/\!/ (X' = X - \tfrac{\partial}{\partial\theta})$$

where $\frac{\partial}{\partial\theta}$ generates $q \mapsto e^{i\theta} q$ on $\mathbb{H} = \mathbb{R}^4$.

- $\dim M_{\text{mod}} = \dim M$
- M complete, then M_{mod} complete
- $\pi_1(M) = 0$, then $b_2(M_{\text{mod}}) = b_2(M) + 1$

Example

$M = \mathbb{H}$, $X = \frac{\partial}{\partial\theta}$,
$\mu = \mu_{\mathbb{H}} + c$, $c \neq 0$:
$M_{\text{mod}} = T^* \mathbb{C}P(1)$

A double fibration

For $\Phi = \mu - \mu_{\mathbb{H}}$, $X' = X - \frac{\partial}{\partial\theta}$:

$$\begin{array}{ccccc} & & P = \Phi^{-1}(0) & \xhookrightarrow{\iota} & M \times \mathbb{H} \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}(X')} & \\ M & & & & M_{\mathrm{mod}} \end{array}$$

- $\mathrm{pr}(X')$ is a Riemannian submersion for $\iota^*(g + g_{\mathbb{H}})$
- pr_1 is *not* a Riemannian submersion, it induces the metric g^N on M:

$$g^N = g + \frac{1}{2\|\mu\|} g_\alpha, \qquad g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$$

$\alpha_0 = X^\flat = g(X, \cdot)$, $\alpha_I = I\alpha_0 = -\alpha_0(I\cdot)$ etc.

Elementary deformations

g hyperKähler, X an isometry, $\alpha_0 = X^\flat$, $g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$

Definition

An *elementary deformation* g^N of g with respect to X is

$$g^N = fg + hg_\alpha$$

for some $f, h \in C^\infty(M)$

There are only two cases for X

1. *tri-holomorphic*: $L_X I = 0 = L_X J = L_X K$
2. *rotating*: $L_X I = 0$, $L_X J = K$

Tri-holomorphic actions

(M, g) hyperKähler, $\dim M > 4$, X tri-Hamiltonian with moment map $\mu = (\mu_I, \mu_J, \mu_K)$

Theorem (Swann)

An elementary deformation $g^N = fg + hg_\alpha$ twists via (X, F, a) to a hyperKähler metric g_W if and only if

- f *constant, so take* $f \equiv 1$,
- $h = h(\mu_I, \mu_J, \mu_K)$ *is harmonic in* $U \subset \mathbb{R}^3$,
- $F = d(h\alpha_0) + *_3 dh$,
- $a = 1 + h\|X\|^2 \neq 0$.

Proof method

1. $\omega_I^W \sim_{\mathcal{H}} \omega_I^N = f\omega_I + h\omega_I^\alpha$
2. impose $d\omega_I^W = 0$, i.e. $d\omega_I^N - \frac{1}{a} F \wedge (X \lrcorner \omega_I^N) = 0$
3. impose $da = -X \lrcorner F$
4. impose $dF = 0$

g hyperKähler, X tri-Hamiltonian, $g^N = g + hg_\alpha$, h harmonic on $U \subset \mathbb{R}^3$

Example

HyperKähler modification is $h = 1/(2\|\mu\|)$

Example

Taub-NUT deformation $W = (M \times (S^1 \times \mathbb{R}^3)) /\!\!/\!\!/ S^1$, diffeomorphic to M, is $h \equiv 1$

Example

$h > 0$: Z^4, $g_Z(h) = \frac{1}{h}(dt + \omega)^2 + h(dx^2 + dy^2 + dz^2)$, $d\omega = *_3 dh$ is a general hyperKähler 4-manifold with free tri-Hamiltonian action and $W = (M \times Z) /\!\!/\!\!/ S^1$
If M is complete and g_Z extends to a complete hyperKähler metric, then get a unique hyperKähler completion of the twist.
E.g. g_Z multi-Eguchi-Hanson, multi-Taub-NUT og A_∞.

Inversion

Generally: Twist of M by data (X, F, a) to W is inverted by twist data on W $\mathcal{H}$-related to $(\frac{1}{a}X, -\frac{1}{a}F, \frac{1}{a})$.

Proposition (Swann)

The hyperKähler twist above of the elementary deformation $g^N = g + hg_\alpha$ of g corresponding to h is inverted by the elementary deformation of g_W corresponding to $-h$.

- Modification inverted by $h = -1/(2\|\mu\|)$. To get positive definite, need $\|X\|^2 < 2\|\mu\|$. So flat $\mathbb{R}^4$ is *not* a modification.
- Taub-NUT deformation if and only if $\|X\|$ is bounded.

- $h > 0$: inversion corresponds to hyperKähler quotient of $(M \times Z^4, g \times -g_Z(h))$. quaternionic Lorentzian

HyperKähler to quaterninoic Kähler

g hyperKähler, $\dim M > 4$, X rotating: $L_X I = 0$, $L_X J = K$ and Hamiltonian for ω_I with Kähler moment map $\mu \colon M \to \mathbb{R}$.

Theorem (Maciá-Swann)

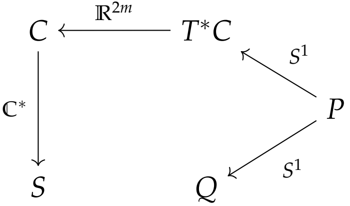
For X non-null, an elementary deformation $g^N = fg + hg_\alpha$ twists to quaternionic Kähler if and only if

- $f = 1/(\mu - c)$
- $h = -1/(\mu - c)^2$
- $F = d\alpha_0 + \omega_I$
- $a = \|X\|^2 - \mu + c$

This is a uniqueness result for the hK/qK correspondence of Haydys 2008; Alexandrov, Persson and Pioline 2011; Hitchin 2013; Alekseevsky, Cortés, Dyckmanns and Mohaupt 2013.

C-MAP

In the c-map context, g has signature $(4n, 4)$, but g^N is positive definite.



T^*C	hyperKähler
Q	quaternionic Kähler
P	twist bundle
C	conic special Kähler
S	projective special Kähler

Example

$S = \mathbb{R}H(1) = \mathrm{Aff}(\mathbb{R})$ has two left-invariant projective special Kähler structures giving Q as $\mathrm{Gr}_2(\mathbb{C}^{2,2})$ or $G_2^* / SO(4)$, as left-invariant quaternionic Kähler structures on solvable groups.

Detect that twist is quaternionic Kähler by $d\Omega = 0$

$$\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$$

provided $\dim M \geqslant 12$.

For $\dim M = 8$, show

$$d \begin{pmatrix} \omega_I \\ \omega_J \\ \omega_K \end{pmatrix} = A \wedge \begin{pmatrix} \omega_I \\ \omega_J \\ \omega_K \end{pmatrix}$$

for some $A \in \Omega^1 \otimes \mathfrak{so}(3)$.

Strong HKT

strong HKT: (g, I, J, K) with $Id\omega_I = Jd\omega_J = Kd\omega_K =: -c$ and $dc = 0$

Proposition (Swann)

g hyperKähler, X tri-Hamiltonian, rank $d\alpha_0 \geqslant 16$. An elementary deformation $g^N = fg + hg_\alpha$ twists via (X, F, a) to a strong HKT metric g_W if and only if

- *f constant, so take $f \equiv 1$,*
- *$h = h(\mu_I, \mu_J, \mu_K)$ is harmonic*
- *$F = d\alpha_0$*
- *$a = \|X\|^2 \neq 0$*

Cf. hyperKähler twist $F = d(h\alpha_0) + *_3 dh$.
Non-trivial examples from $M = T^*G^{\mathbb{C}}$ for each G compact Lie.

G_2 TO G_2

(M, g) holonomy G_2 with three-form φ.
For a symmetry X, we decompose

$$g = g_\perp + \frac{1}{\|X\|^2}\alpha_0^2$$

$$\varphi = \omega \wedge \frac{1}{\|X\|^2}\alpha_0 + \rho$$

with $\omega = X \lrcorner \varphi$, $X \lrcorner \rho = 0$. This has $d\omega = 0$.
Elementary deformation

$$g^N = f^2 g + \frac{(h^2 - f^2)}{\|X\|^2}\alpha_0^2$$

$$\varphi^N = f^2 h\omega \wedge \frac{1}{\|X\|^2}\alpha_0 + f^3\rho$$

$$g^N = f^2 g + \frac{(h^2 - f^2)}{\|X\|^2}\alpha_0^2, \; \varphi^N = f^2 h\omega \wedge \frac{1}{\|X\|^2}\alpha_0 + f^3\rho$$

Proposition (Freibert-Swann)

Let (M, g, φ) be a parallel G_2-structure with symmetry X. Then the elementary deformation (g^N, φ^N) twists to a parallel G_2-structure if and only if

- f *constant, so take* $f \equiv 1$,
- $h > 1$ *is arbitrary*
- $F = d((h-1)\alpha_0)$
- $a = h$

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