A shear construction, solvable Lie algebras and SKT geometry

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12th Minimeeting on Differential Geometry
CIMAT, December 2020

Freibert, M. and Swann, A. F. (2016), 'Solvable groups and a shear construction', *J. Geom. Phys.* 106: 268–74
Freibert, M. and Swann, A. F. (2019), 'The shear construction', *Geom. Dedicata*, 198 (1): 71–101.
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Section 1

SKT GEOMETRY

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$(g, J, \sigma = g(J\,\cdot\,,\,\cdot\,))$ a Hermitian structure (J integrable)
Strong Kähler with torsion (SKT):

$$dJd\sigma = 0$$

Equivalently $\partial\bar{\partial}\sigma = 0$.

Originates from supersymmetric σ-models (Gates et al. 1984) / superstrings with torsion (Strominger 1986).

TORSION: the Bismut connection $\nabla^B = \nabla^{LC} + \frac{1}{2}T^B$ has torsion T^B given by

$$g(T^B(X,Y),Z) = d\sigma(JX,JY,JZ) =: c^B(X,Y,Z)$$

a three-form. Gauduchon (1997): ∇^B is the unique Hermitian connection with torsion a three-form.
STRONG: the torsion three-form c^B is closed.

FIRST EXAMPLES

SKT: (g, J, σ) Hermitian with $dJd\sigma = 0$

KÄHLER MANIFOLDS: are all SKT.

REAL DIMENSION 4: SKT same as the Lee form $Jd^*\sigma$ is co-closed. For compact M, Gauduchon (1984) gives a unique SKT metric in each conformal class. E.g. $M = S^1 \times S^3$

COMPACT LIE GROUPS OF EVEN DIMENSION: J any left-invariant complex structure, g any compatible bi-invariant metric, $\nabla^{LC}_X Y = \frac{1}{2}[X, Y]$, $c^B = -g([X, Y], Z)$ is SKT (Spindel et al. 1988). E.g. $M = \mathrm{SU}(3)$ or $\mathrm{SU}(2) \times \mathrm{SU}(2)$

SOME PREVIOUS HOMOGENEOUS CLASSIFICATIONS

Notation: $(0, 12)$ indicates the affine algebra given dually by $de^1 = 0$, $de^2 = e^1 \wedge e^2 =: e^{12}$, etc.

Non-Kähler examples

NILPOTENT LIE GROUPS

Dimension 4: $(0, 0, 0, 12)$
Dimension 6: Fino et al. (2004) (out of 34 algebras)

$$(0, 0, 0, 0, 0, 12), \quad (0, 0, 0, 0, 12, 34),$$
$$(0, 0, 0, 0, 12, 13 + 42), \quad (0, 0, 0, 0, 12 + 34, 13 + 42).$$

Dimension 8: Enrietti et al. (2012)

SOLVABLE LIE GROUPS

Dimension 4: Madsen and Swann (2011)
Almost Abelian algebras: characterised by Arroyo and Lafuente (2019)

Section 2

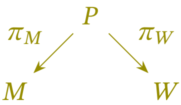
Twists and shears

The twist construction

$$\begin{array}{ccc} & P & \\ {}^{\pi_M}\swarrow & & \searrow^{\pi_W} \\ M & & W \end{array}$$

M a manifold with an action of a connected Abelian group A_M, e.g. $A_M = T^k$, infinitesimal action $\xi\colon \mathfrak{a}_M \to \mathfrak{X}(M)$.
P a principal $A_P = T^k$-bundle over M, infinitesimal action $\rho\colon \mathfrak{a}_P \to \mathfrak{X}(P)$. Connection one-form $\theta \in \Omega^1(P, \mathfrak{a}_P)$, curvature $\pi_M^*\omega = d\theta$, and horizontal distribution $\mathcal{H} = \ker\theta$. Horizontal lift $\tilde{\xi}$ of ξ.
The action $\mathring{\xi} = \tilde{\xi} + (\pi^* a)\rho$, $a \in \Omega^0(M, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$, commutes with A_P if and only if $\xi \lrcorner \omega = -da$ and $\xi^*\omega = 0$.
We then put $W = P/\langle\mathring{\xi}\rangle$ to be the *twist* of M.
A_P then induces an action on W, and M is also a twist of W.

$\mathcal{H}$-RELATED FORMS



Invariant p-forms α_M on M and α_W on W are *$\mathcal{H}$-related*, written $\alpha_M \sim_{\mathcal{H}} \alpha_W$, if

$$\pi_M^* \alpha_M \big|_{\Lambda^p \mathcal{H}} = \pi_W^* \alpha_W \big|_{\Lambda^p \mathcal{H}} .$$

EXTERIOR DERIVATIVES are then related by

$$d\alpha_W \sim_{\mathcal{H}} d\alpha_M - a^{-1} \omega \wedge (\xi \lrcorner \alpha_M)$$

ω curvature of $P \longrightarrow M$, ξ the infinitesimal action on M, $a \in \Omega^0(M, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$, $da = -\xi \lrcorner \omega$.

Similarly, for other tensors.

SKT TWISTS

If M is SKT, can now compute when the twist is SKT.

Typical examples obtained from $M = N \times T^{2k}$, N SKT or Kähler, $\omega = \sum_{i=1}^{2k} \omega_i \otimes e_i$, $\omega_i \in \Omega^{1,1}(N)$ (*instanton case*) integral with $\sum_{i,j=1}^{2k} \gamma_{ij} \omega_i \wedge \omega_j = 0$ for $(\gamma_{ij}) \in M_{2k}(\mathbb{R})$ invertible. There is also a non-instanton case, with more involved conditions on the $\omega_i \in \Omega^2(N)$.

Get non-trivial SKT structures on various torus bundles over Kähler manifolds, some examples where the manifold admits no Kähler structure: includes compact Lie groups as torus bundles over flag manifolds, Grantcharov et al. (2008) examples $(k-1)(S^2 \times S^4)\#k(S^3 \times S^3)$ for all $k \geqslant 1$ and each nilmanifold example in dimension 6.

TWO-STEP NILPOTENT TWISTS

$G = T^n$, $\mathfrak{g} = \mathbb{R}^n$

$\mathfrak{a}$ a k-dimensional subalgebra, $\xi\colon \mathfrak{a} \to \mathfrak{g}$ inclusion.

Take $\omega \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{a}$ closed with $\xi \lrcorner \omega = 0$.

Then for $a = \mathrm{id}_{\mathfrak{a}}$, the twist is two-step nilpotent with non-trivial derivatives given by $-\omega$.

EXAMPLE $\mathfrak{g} = \mathbb{R}^3$, $\mathfrak{a} = \mathrm{Span}\{E_3\}$, $\omega = -e^1 \wedge e^2 \otimes E_3$ has twist $(0, 0, e^{12})$, the Heisenberg group.

EXAMPLE $\mathfrak{g} = \mathbb{R}^6$, $\mathfrak{a} = \mathrm{Span}\{E_5, E_6\}$,
$\omega = -e^1 \wedge e^2 \otimes E_5 - (e^1 \wedge e^3 + e^4 \wedge e^2) \otimes E_6$ has twist $(0, 0, 0, 0, 12, 13 + 42)$, an SKT algebra.

SHEAR DATA

A foliated version of the twist construction.
$\pi_E \colon E \to M$ a bundle with flat connection ∇ and a bundle morphism $\xi \colon E \to TM$ that is *torsion free*

$$\xi(\nabla_{\xi e_1} e_2 - \nabla_{\xi e_2} e_1) = [\xi e_1, \xi e_2].$$

$\pi_F \colon F \to M$ a second flat bundle connection ∇, a bundle isomorphism $a \colon E \to F$ and a two-form $\omega \in \Omega^2(M, F)$ with

$$d^\nabla \omega = 0, \quad \xi \lrcorner \omega = -d^\nabla a \quad \text{and} \quad \xi^* \omega = 0.$$

A *shear total space* for ω is a foliated manifold P, with leaf space $\pi \colon P \to M$, such that $\mathcal{V} = \ker d\pi$ is isomorphic to $\pi^* F$, and there a "connection" $\theta \in \Omega^1(P, \pi^* F)$ realising $\mathcal{V} \cong \pi^* F$, and with $d^\nabla \theta = \pi^* \omega$. Again $\mathcal{H} = \ker \theta$ is a horizontal subbundle.

THE SHEAR CONSTRUCTION

Shear data ensures that the bundle morphism $\mathring{\xi}\colon \pi^*E \to TP$ given by

$$\mathring{\xi} = \tilde{\xi} + \rho \circ \pi^* a,$$

where $\tilde{\xi}\colon \pi^*E \to \mathcal{H} \subset TP$ is the horizontal lift, is such that $\mathring{\xi}(\pi^*E)$ is an integrable distribution on P.

The *shear* of (M, ξ, a, ω) is then

$$S = P/\mathring{\xi}(\pi^*E).$$

One can work with $\mathcal{H}$-related forms satisfying the invariance condition

$$\mathcal{L}^{\nabla}_{\xi}\alpha := \xi \lrcorner d\alpha + d^{\nabla}(\xi \lrcorner \alpha) = 0.$$

The previous formula for the exterior derivative then holds.

Section 3

Solvable algebras

SHEARS OF ABELIAN ALGEBRAS

$\mathfrak{g}$ any Lie algebra, $\mathfrak{a}_P$ Abelian.
Any extension

$$\mathfrak{a}_P \hookrightarrow \mathfrak{p} \to \mathfrak{g}$$

has

$$[X,Y]_{\mathfrak{p}} = [X,Y]_{\mathfrak{g}} - \omega(X,Y) \quad \text{and} \quad [X,Z]_{\mathfrak{p}} = \eta(X)Z$$

for $X, Y \in \mathfrak{g}$, $Z \in \mathfrak{a}_P$. Thus it is specified by

$$\omega \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{a}_P \quad \text{and} \quad \eta \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_P).$$

The Jacobi identity is

$$d\omega = -\eta \wedge \omega.$$

Regarding η as a connection one-form for ∇, this equation is $d^\nabla\omega = 0$.

TWO-STEP SOLVABLE ALGEBRAS

$\mathfrak{g}$ Abelian. Can take $\xi = \mathrm{inc}\colon \mathfrak{a} = \mathfrak{a}_G \to \mathfrak{g}$ inclusion, $\mathfrak{a}_P = \mathfrak{a}$ and $a = \mathrm{id}_{\mathfrak{a}}\colon \mathfrak{a}_G \to \mathfrak{a}_P$.
Then ω determines the rest of the shear data. Writing $\mathfrak{g} = \mathfrak{a} \oplus U$,

$$\omega \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{a}$$
$$\omega_{-1} + \omega_0 + \omega_1 \in \left(\Lambda^2\mathfrak{a}^* \oplus (U^* \wedge \mathfrak{a}^*) \oplus \Lambda^2 U^*\right) \otimes \mathfrak{a}.$$

PROPOSITION

This is shear data if and only if $\omega_{-1} = 0$ and

$$\mathcal{A}(\omega(\omega(\,\cdot\,,\,\cdot\,),\,\cdot\,)) = 0$$

where $\mathcal{A}$ is skew-symmetrisation.

(Corresponds to $\eta = -\omega_0$ and connection form for E being 0).
The shear algebra $\mathfrak{h}$, which is $\mathfrak{p}$ quotiented by the diagonal copy of $\mathfrak{a}$, has Lie brackets given by $-\omega$. It is two-step solvable.

DATA FOR TWO-STEP SOLVABLE SKT ALGEBRAS

$\mathfrak{g}$ Abelian, even-dimensional, with flat Kähler structure (g, J, σ). If $(\mathfrak{a}, \omega)$ is two-step shear data on $\mathfrak{g}$, then the shear $\mathfrak{h}$ is SKT if and only if

$$\omega(J\,\cdot\,, J\,\cdot\,) = \omega(\,\cdot\,,\,\cdot\,) + J(\omega(J\,\cdot\,,\,\cdot\,) + \omega(\,\cdot\,, J\,\cdot\,)) \quad \text{and}$$
$$\mathcal{A}\big(g(\omega(J\,\cdot\,, J\,\cdot\,), \omega(\,\cdot\,,\,\cdot\,)) + 2g(\omega(J\omega(\,\cdot\,,\,\cdot\,), J\,\cdot\,),\,\cdot\,)\big) = 0$$

Put

$$\mathfrak{a}_J = \mathfrak{a} \cap J\mathfrak{a}, \quad \mathfrak{a}_r = \mathfrak{a}_J^{\perp} \cap \mathfrak{a},$$
$$U_r = J\mathfrak{a}_r, \quad U_J = (\mathfrak{a} \oplus J\mathfrak{a}_r)^{\perp}, \quad U = U_r \oplus U_J$$

and split $\omega = \omega_0 + \omega_1 \in (U^* \wedge \mathfrak{a}^* \oplus \Lambda^2 U^*) \otimes \mathfrak{a}$ accordingly.

SIMULTANEOUS DIAGONALISATION

For $X \in \mathfrak{a}_r$, put $A_X = -\omega_0(JX, \cdot\,) \in \operatorname{End}(\mathfrak{a})$ and K_X the part of A_X in $\operatorname{End}(\mathfrak{a}_J)$.

PROPOSITION

There is a unitary basis Y_i of $\mathfrak{a}_J$ and $\alpha_j \in \mathfrak{a}_r^* \otimes \mathbb{C}$ such that

$$K_X(Y_i) = \alpha_i(X)Y_i, \qquad \textit{for all } X \in \mathfrak{a}_r.$$

CLASSIFICATION RESULTS

Write $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ for the derived algebra.

THEOREM

There are explicit classifications for two-step solvable SKT Lie algebras $\mathfrak{g}$ *in the following cases:*

$\mathfrak{g}$ *almost Abelian, i.e.* $\mathfrak{g}$ *has a codimension one Abelian ideal,*

$\operatorname{codim} \mathfrak{g}' = 2$ *with* $J\mathfrak{g}' \neq \mathfrak{g}'$,

$\mathfrak{g}'$ *totally real, i.e.* $J\mathfrak{g}' \cap \mathfrak{g}' = \{0\}$, *with* $\operatorname{codim}_{\mathfrak{g}'} [\mathfrak{g}', J\mathfrak{g}'] \leqslant 2$,

and

$\dim \mathfrak{g}' \leqslant 2$.

Specialising and extending one also obtains a classification of all two-step solvable SKT Lie algebras in dimension 6, with the exception of the case when $\dim \mathfrak{g}' = 4$ *with* $J\mathfrak{g}' = \mathfrak{g}'$.

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