Toric methods for Ricci-flat metrics

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OUTLINE

Ricci-flat special holonomy

$$\mathrm{Ric} = 0$$

Ricci-flat geometries in the Berger holonomy classification 1955,…

Name	Holonomy	Dimension	Form degrees
Calabi-Yau	$SU(n)$	$2n$	$2, n, n$
HyperKähler	$Sp(n)$	$4n$	$2, 2, 2$
G_2 holonomy	G_2	7	$3, 4$
Spin(7) holonomy	$\mathrm{Spin}(7)$	8	4

In these cases

- the forms specify the geometry
- holonomy reduction $\iff$ the forms are closed

Symplectic constructions

ω symplectic form: degree 2, closed, non-degenerate

X symmetry: $\mathscr{L}_X \omega = 0$

moment map: $\mu_X : M \to \mathbb{R}, \quad d\mu_X = \omega(X, \cdot)$

Flat space

$M = \mathbb{R}^{2n}$ coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$

$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$

$$X = c_1\left(y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1}\right) + \cdots + c_n\left(y_n \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial y_n}\right)$$

$$\mu_X = \tfrac{1}{2}\big(c_1(x_1^2 + y_1^2) + \cdots + c_n(x_n^2 + y_n^2)\big) + c$$

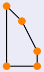
DELZANT CONSTRUCTION

(M^{2n}, ω) compact symplectic with Hamiltonian action of torus T^n: invariant function $\mu\colon M \to \mathbb{R}^n = \mathrm{Lie}(T^n)^*$ with

$$d\langle \mu, X\rangle = \omega(X, \cdot\,) \qquad \forall X \in \mathbb{R}^n = \mathrm{Lie}(T^n)$$

DELZANT (1988)

Compact symplectic toric manifolds correspond to Delzant polytopes



Polytopes in $\mathbb{R}^n$ with normal vectors in $\mathbb{Z}^n$ and the normals at intersections of faces being part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

- $b_1(M) = 0 \implies$ every symplectic torus T^k action is Hamiltonian
- then ω pulls back to 0 on each torus orbit, so $k \leqslant n$

The Delzant dimensions are such that

$$\dim(M^{2n}/T^n) = n = \dim(\text{codomain}\,\mu)$$

Can be used for non-compact M

Toric Calabi-Yau

Include symplectic quotients of $\mathbb{R}^{2N} = \mathbb{C}^N$ by subtori of T^N whose weights sum to zero.

$$\begin{aligned}\mathbb{C}^4 /\!/ \operatorname{diag}(e^{\mathbf{i}t}, e^{\mathbf{i}t}, e^{-\mathbf{i}t}, e^{-\mathbf{i}t}) &= \mu^{-1}(0)/T^1 \\ &= \big(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P(1)\big)\end{aligned}$$

Tian and Yau (1991),…, Goto (2012)
Futaki, Ono, and Wang (2009)

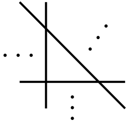
HyperKähler

$(M, \omega_I, \omega_J, \omega_K)$ is *hyperKähler* if each ω_A is symplectic and $I := \omega_K^{-1} \circ \omega_J$, etc., satisfy $IJ = K = -JI$ and $g = \omega_I(\,\cdot\,, I\,\cdot\,) > 0$
T^k action is *tri-Hamiltonian* if Hamiltonian for ω_I, ω_J and ω_K

$$\mu = (\mu_I, \mu_J, \mu_K) \colon M^{4n} \to \mathbb{R}^{3k}$$

- $b_1(M) = 0$ is sufficient
- $k \leqslant n$
- for $k = n$, $\dim M^{4n}/T^n = 3n = \dim(\text{codomain}\,\mu)$

Combinatorial data: arrangement of codimension 3 affine subspaces in $\mathbb{R}^{3n} = \mathbb{R}^3 \otimes \mathbb{R}^n$ with $\mathbb{Z}$-basis property for normal vectors in $\mathbb{R}^n$



Maximum of $n(n+1)/2$ normal directions

CLASSIFICATION

THEOREM (DANCER AND SWANN 2017)

Every complete hypertoric manifold (M^{4n}, T^n) *is the hyperKähler quotient of an affine Hilbert manifold by a subgroup of a standard torus*
Furthermore $\mu\colon M^{4n}/T^n \to \mathbb{R}^{3n}$ *is a homeomorphism*

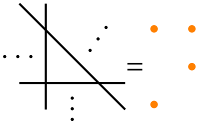
Calabi metric
$T^*\mathbb{C}P(n) = \mathbb{H}^{n+1} /\!/\!/ \operatorname{diag}(e^{\mathbf{i}t}, \dots, e^{\mathbf{i}t}) = \mathbb{C}^{2n+2} /\!/\!/ \operatorname{diag}(e^{\mathbf{i}t}, e^{-\mathbf{i}t}, \dots)$

$\dim M = 4 \qquad g = V^{-1}\theta^2 + V(d\mu_I^2 + d\mu_J^2 + d\mu_K^2)$

$$V(p) = c + \sum_{q\in Q} \frac{1}{2\operatorname{dist}_{\mathbb{R}^3}(p,q)}$$

- $Q \subset \mathbb{R}^3$ discrete
- $c \geqslant 0$ the Taub-NUT parameter
- only constraint: $\exists\, p \in \mathbb{R}^3$ such that $V(p) < \infty$

Maximal arrangements

dim = 8: 

dim = 12: 

dim = 16:  and 

dim = 20 has 4 maximal configurations

Multi-Hamiltonian torus actions

(M, α) manifold with closed $\alpha \in \Omega^p(M)$ preserved by T^k is *multi-Hamiltonian* if there is an invariant

$$\nu\colon M \to \Lambda^{p-1}\bigl(\mathrm{Lie}(T^k)^*\bigr) \cong \mathbb{R}^N,$$
$$d\langle \nu, X_1 \wedge \cdots \wedge X_{p-1}\rangle = \alpha(X_1, \dots, X_{p-1}, \,\cdot\,)$$

for all $X_i \in \mathrm{Lie}(T^k)$.

- $b_1(M) = 0 \implies$ each T^k-action preserving α is multi-Hamiltonian
- then α pulls-back to 0 on each T^k-orbit

Geometry is *toric* if multi-Hamiltonian for T^k and

$$\dim(M/T^k) = \dim(\text{codomain}\,\nu)$$

A RICCI-FLAT HIERARCHY

Dimensions	4	6	7	8
Holonomies	Sp(1)	SU(3)	G_2	Spin(7)
Extension		$M^4 \times S^1 \times \mathbb{R}$	$M^6 \times S^1$	$M^7 \times S^1$
Closed forms	$(\omega_1, \omega_2, \omega_3)$	$(\omega, \Omega_+, \Omega_-)$	$(\varphi, *_7\varphi)$	Φ
Degrees	(2, 2, 2)	(2, 3, 3)	(3, 4)	4
Toric group	S^1	T^2	T^3	T^4

$$\omega_1 = e^{45} + e^{67}, \quad \omega_2 = e^{46} + e^{75}, \quad \omega_3 = e^{47} + e^{56}$$

$$\varphi = e^{123} - e^1 \wedge \omega_1 - e^2 \wedge \omega_2 - e^3 \wedge \omega_3$$

$$\Phi = e^0 \wedge \varphi + *_7\varphi$$

$$\varphi = e^1 \wedge \omega + \Omega_+ \quad *_7\varphi = \Omega_- \wedge e^1 + \tfrac{1}{2}\omega^2$$

TORIC Spin(7)

(M^8, Φ) with toric T^4 generated by U_i, $i = 0, 1, 2, 3$, multi-moment map $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$

$$\begin{aligned} d\nu_i &= (-1)^i \Phi(U_j \wedge U_k \wedge U_\ell, \,\cdot\,) \\ &= (-1)^i (U_j \times U_k \times U_\ell)^\flat \qquad (ijk\ell) = (0123) \end{aligned}$$

On the open dense set M_0 where T^4 acts freely, have rank $d\nu = 4$ so ν induces local diffeomorphism $M_0/T^4 \to \mathbb{R}^4$

THEOREM

ν induces a local homeomorphism

$$M/T^4 \to \mathbb{R}^4$$

Local homeomorphism of leaf space

Each stabiliser $\mathrm{Stab}_{T^4}(p)$ is a connected subtorus of T^4 of rank $\leqslant 2$

rank $\mathrm{Stab}_{T^4}(p) = 2$

Flat model $M = T^2 \times \mathbb{C}^3$, $T^4 \leqslant T^2 \times SU(3)$

$$\nu_0 = \mathrm{Im}(z_1 z_2 z_3), \qquad \nu_1 = \mathrm{Re}(z_1 z_2 z_3),$$
$$\nu_2 = \tfrac{1}{2}(|z_3|^2 - |z_2|^2), \qquad \nu_3 = \tfrac{1}{2}(|z_1|^2 - |z_3|^2)$$

Putting $t = |z_3|^2$ the singularity is

$$t(t + 2\nu_2)(t - 2\nu_3) = \nu_0^2 + \nu_1^2 \qquad t \geqslant \max\{0, -2\nu_2, 2\nu_3\} \qquad \text{(S)}$$

Projection (S) $\ni (t, \nu) \mapsto \nu \in \mathbb{R}^4$ is a homeomorphism
Applies to non-flat case via analysis of $(\nabla^{m_j} \nu_i)_p$, local estimates, rescaling and degree arguments

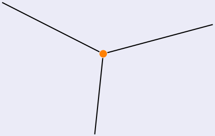
IMAGE OF SINGULAR SETS

$$d\nu_i = (-1)^i (U_j \times U_k \times U_\ell)^\flat \quad (ijk\ell) = (0123)$$

Image under $\nu \colon M \to \mathbb{R}^4$

- $\dim(\mathrm{Stab}_{T^4}) = 1$: straight lines with rational slopes
- $\dim(\mathrm{Stab}_{T^4}) = 2$: points
  with three straight lines meeting at point, sum of primitive tangents is zero

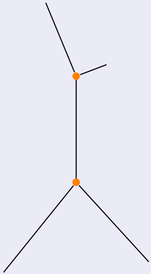
FLAT MODEL



planar graph

Example

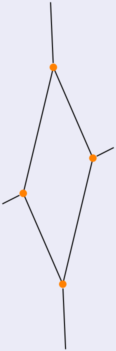
$S^1\times$ (Bryant-Salamon G_2 metric on $S^3 \times \mathbb{R}^4$):



non-planar, but lies in $\mathbb{R}^3 \subset \mathbb{R}^4$

Example

$S^1\times$ Foscolo, Haskins, and Nordström (2018) G_2-example on $M_{m,n}$: $M_{m,n}$ a circle bundle over the canonical bundle of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with first Chern class $(m, -n)$ over the zero section, symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times S^1$



Primitive directions
$(m - n, 0, n)$
$(0, n - m, m)$
$(n - m, m - n, -m - n)$
planar

Smooth behaviour

$$\Phi = \det(V)\Big(\mathfrak{S}_{ijk\ell}(-1)^i(\theta_i \wedge dv_{jk\ell} + \theta_{jk\ell} \wedge dv_i) + \tfrac{1}{2}(dv^t V^{-1}\theta)^2\Big)$$

$$g = \theta^t V^{-1}\theta + \det(V) dv^t V^{-1} dv$$

for $V = (g(U_i, U_j))^{-1}$, $\theta = (\theta_0, \theta_1, \theta_2, \theta_3)$ connection one-forms

Theorem

Holonomy contained in $\mathrm{Spin}(7)$ (*i.e.* $d\Phi = 0$) *if and only if* $\operatorname{div} V = 0$ *and* $L(V) + Q(dV) = 0$

$$\operatorname{div}(V)_a = \sum_{i=0}^{3} \frac{\partial V_{ia}}{\partial v_i}$$

$$L(V)_{ab} = \sum_{i,j=0}^{3} V_{ij}\frac{\partial^2 V_{ab}}{\partial v_i \partial v_j}, \qquad Q(dV)_{ab} = -\sum_{i,j=0}^{3} \frac{\partial V_{ia}}{\partial v_j}\frac{\partial V_{jb}}{\partial v_i}$$

$$\operatorname{div}(V)_a = \sum_{i=0}^{3} \frac{\partial V_{ia}}{\partial \nu_i} = 0, \quad (L(V) + Q(dV))_{ab} = \sum_{i,j=0}^{3} V_{ij} \frac{\partial^2 V_{ab}}{\partial \nu_i \partial \nu_j} - \frac{\partial V_{ia}}{\partial \nu_j} \frac{\partial V_{jb}}{\partial \nu_i} = 0$$

Holonomy in G_2, SU(3) and Sp(1) from $V = \begin{pmatrix} 1_r & 0 \\ 0 & * \end{pmatrix}$, $r = 1, 2, 3$

dim 4, Sp(1), $V = \begin{pmatrix} 1_3 & 0 \\ 0 & V_{33} \end{pmatrix}$

$\operatorname{div} V = 0 \iff V_{33} = V_{33}(\nu_0, \nu_1, \nu_3) \implies Q(dV) = 0$ and $L(V) = 0$ has only one scalar equation $\Delta_{\mathbb{R}^3} V_{33} = 0$, so V_{33} is harmonic on $\mathbb{R}^3$.

dim 6, SU(3), $V = \begin{pmatrix} 1_2 & 0 & 0 \\ 0 & V_{22} & V_{23} \\ 0 & V_{23} & V_{33} \end{pmatrix}$

Li (2019) produced Taub-NUT family of complete metrics on $\mathbb{C}^3$

Explicit full holonomy

Holonomy = Spin(7)

$V = \operatorname{diag}(\nu_1, \nu_2, \nu_3, \nu_0) > 0$

$$g = \sum_{i=0}^{3} \frac{1}{\nu_{i+1}} \theta_i^2 + \nu_{i+2}\nu_{i+3}\nu_i d\nu_i^2 \qquad d\theta_i = (-1)^{i+1}\nu_{i+2} d\nu_{i+2} \wedge d\nu_{i+3}$$

Holonomy = G_2

$V = \operatorname{diag}(1, \nu_0, \nu_0, \nu_0) > 0$

$$g = \frac{1}{\nu_0} \sum_{i=1}^{3} \theta_i^2 + \nu_0^3 d\nu_0^2 + \nu_0^2 \sum_{i=1}^{3} d\nu_i^2 \qquad d\theta_i = d\nu_j \wedge d\nu_k \ (ijk) = (123)$$

Cf. Hein, Sun, Viaclovsky, and Zhang (2018)

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