Twist geometry of elementary deformations

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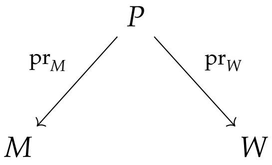
Happy Birthday, Helga Baum!

Joint work with Óscar Maciá (Valencia)

Outline

Twist construction

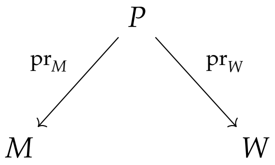


- $P \to M$ a principal S^1-bundle, symmetry Y, connection 1-form θ, curvature $\mathrm{pr}_M^* F = d\theta$
- $X \in \mathfrak{X}(M)$ generating S^1-action preserving F
- $X' = \tilde{X} + aY$ on P preserving θ and $Y{:}\tilde{X} \in \mathcal{H} = \ker\theta$, $(\mathrm{pr}_M)_* \tilde{X} = X$, and $da = -X \lrcorner F$
- $W = P/X'$, has action induced by Y

Twist data

Twist data

- M manifold
- $X \in \mathfrak{X}(M)$, circle action
- $F \in \Omega^2_{\mathbb{Z}}(M)^X$
- $a \in C^\infty(M)$ with $da = -X \lrcorner F$



horizontal distribution
$\mathcal{H} = \ker\theta \subset TP$

Definition

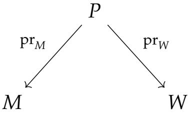
α tensor on M is $\mathcal{H}$-*related* to α_W on W if

$$\mathrm{pr}_M^*\alpha = \mathrm{pr}_W^*\alpha_W \quad \text{on } \mathcal{H}$$

Write $\alpha \sim_{\mathcal{H}} \alpha_W$

Twist computations

$\alpha \sim_{\mathcal{H}} \alpha_W$ if $\mathrm{pr}_M^* \alpha = \mathrm{pr}_W^* \alpha_W$ on $\mathcal{H} = \ker \theta$



- $\alpha \in \Omega^p(M)$:

$$d\alpha_W \sim_{\mathcal{H}} d_W \alpha := d\alpha - \tfrac{1}{a} F \wedge (X \lrcorner \alpha)$$

- I complex structure on M:
  I_W integrable if and only if $F \in \Lambda_I^{1,1}$

Example

$M = M(n) := \mathbb{C}P^n \times T^2$ Kähler, X on T^2, $F = \omega_{\mathrm{FS}}$:
$W = S^{2n+1} \times S^1$ non-Kähler.

How can we get Kähler, hyperKähler,…?

HyperKähler geometry

M^{4n} *hyperKähler*: g (pseudo-)Riemannian metric, $I, J, K\colon TM \to TM$ bundle endomorphisms with

- $I^2 = -1 = J^2 = K^2$, $IJ = K = -JI$
- $g(IX, IY) = g(X, Y)$ etc. and
- $\omega_I(X, Y) = g(IX, Y)$ etc. satisfy

$$d\omega_I = 0 = d\omega_J = d\omega_K$$

Then

- g is Ricci-flat,
- $\mathrm{Hol}_0(g) \leqslant Sp(n)$ and
- I, J, K are integrable.

HyperKähler quotients

Suppose X is a tri-holomorphic isometry of (M, g, I, J, K), $L_X g = 0$, $L_X I = 0 = L_X J = L_X K$, generating a circle action.

A *hyperKähler moment map* for X is $\mu = (\mu_I, \mu_J, \mu_K) \colon M \to \mathbb{R}^3$ such that

$$d\mu = (X \lrcorner \omega_I, X \lrcorner \omega_J, X \lrcorner \omega_K)$$

X is then *tri-Hamiltonian*

Theorem (Hitchin, Karlhede, Lindström and Roček 1987)

If X acts freely on $\mu^{-1}(0)$, then

$$M /\!\!/\!\!/ X = \mu^{-1}(0)/X$$

is a smooth hyperKähler manifold of dimension $\dim M - 4$.

HyperKähler modifications

Definition (Dancer-S)

The *hyperKähler modification* of M is

$$M_{\text{mod}} = (M \times \mathbb{H}) /\!\!/\!\!/ (X' = X - \tfrac{\partial}{\partial\theta})$$

where $\frac{\partial}{\partial\theta}$ generates $q \mapsto e^{i\theta} q$ on $\mathbb{H} = \mathbb{R}^4$.

- $\dim M_{\text{mod}} = \dim M$
- M complete, then M_{mod} complete
- $\pi_1(M) = 0$, then $b_2(M_{\text{mod}}) = b_2(M) + 1$

Example

$M = \mathbb{H}$, $X = \frac{\partial}{\partial\theta}$, $\mu = \mu_{\mathbb{H}} + c$, $c \neq 0$: $\quad M_{\text{mod}} = T^* \mathbb{C}P(1)$

Which hyperKähler metrics are modifications?

A double fibration

For $\Phi = \mu - \mu_{\mathbb{H}}$, $X' = X - \frac{\partial}{\partial\theta}$:

$$\begin{array}{ccc} & P = \Phi^{-1}(0) \overset{\iota}{\hookrightarrow} M \times \mathbb{H} & \\ {\scriptstyle \mathrm{pr}_1} \swarrow & & \searrow {\scriptstyle \mathrm{pr}(X')} \\ M & & M_{\mathrm{mod}} \end{array}$$

- $\mathrm{pr}(X')$ is a Riemannian submersion for $\iota^*(g + g_{\mathbb{H}})$
- pr_1 is *not* a Riemannian submersion, it induces the metric g^N on M:

$$g^N = g + \frac{1}{2\|\mu\|} g_\alpha, \qquad g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$$

$\alpha_0 = X^\flat = g(X, \cdot)$, $\alpha_I = I\alpha_0 = -\alpha_0(I\cdot)$ etc.

Elementary deformations

g hyperKähler, X an isometry, $\alpha_0 = X^\flat$, $g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$

Definition

An *elementary deformation* g^N of g with respect to X is

$$g^N = fg + hg_\alpha$$

for some $f, h \in C^\infty(M)$

There are only two cases for X

1. *tri-holomorphic*: $L_X I = 0 = L_X J = L_X K$
2. *rotating*: $L_X I = 0$, $L_X J = K$

Tri-holomorphic actions

g hyperKähler, X tri-holomorphic, $\dim M > 4$
Locally X is tri-Hamiltonian with moment map $\mu = (\mu_I, \mu_J, \mu_K)$

Theorem (S)

An elementary deformation $g^N = fg + hg_\alpha$ twists via (X, F, a) to a hyperKähler metric g_W if and only if

- f *constant, so take* $f \equiv 1$,
- $h = h(\mu_I, \mu_J, \mu_K)$ *is harmonic*
- $F = d(h\alpha_0) + *_3 dh$
- $a = 1 + h\|X\|^2 \neq 0$

Proof method

1. $\omega_I^W \sim_{\mathcal{H}} \omega_I^N = f\omega_I + h\omega_I^\alpha$
2. impose $d\omega_I^W = 0$, i.e. $d_W \omega_I^N = 0$
3. impose $da = -X \lrcorner F$
4. impose $dF = 0$

g hyperKähler, X tri-Hamiltonian, $g^N = g + hg_\alpha$, h harmonic

Example

HyperKähler modification is $h = 1/(2\|\mu\|)$

Example

Taub-NUT deformation $W = (M \times (S^1 \times \mathbb{R}^3)) /\!\!/\!\!/ S^1$, diffeomorphic to M, is $h \equiv 1$

Example

$h > 0$: Z^4, $g_Z(h) = \frac{1}{h}(dt + \omega)^2 + h(dx^2 + dy^2 + dz^2)$, $d\omega = *_3 dh$ is a general hyperKähler 4-manifold with free tri-Hamiltonian action and $W = (M \times Z) /\!\!/\!\!/ S^1$

Inversion

Generally: Twist of M by data (X, F, a) to W is inverted by twist data on W $\mathcal{H}$-related to $(\frac{1}{a}X, -\frac{1}{a}F, \frac{1}{a})$.

Proposition (S)

HyperKähler twists above of the elementary deformation $g^N = g + hg_\alpha$ of g corresponding to h is inverted by the elementary deformation of g_W corresponding to $-h$.

- Modification inverted by $h = -1/(2\|\mu\|)$. To get positive definite, need $\|X\|^2 < 2\|\mu\|$. So flat $\mathbb{R}^4$ is *not* a modification.
- Taub-NUT deformation if and only if $\|X\|$ is bounded.

- $h > 0$: inversion corresponds to hyperKähler quotient of $(M \times Z^4, g \times -g_Z(h))$. quaternionic Lorentzian

Strong HKT

strong HKT: (g, I, J, K) with

- $Id\omega_I = Jd\omega_J = Kd\omega_K =: -c$ and
- $dc = 0$

Proposition

g hyperKähler, X tri-Hamiltonian, rank $d\alpha_0 \geqslant 16$. An elementary deformation $g^N = fg + hg_\alpha$ twists via (X, F, a) to a strong HKT metric g_W if and only if

- *f constant, so take $f \equiv 1$,*
- *$h = h(\mu_I, \mu_J, \mu_K)$ is harmonic*
- $F = d(h\alpha_0)$
- $a = 1 + h\|X\|^2 \neq 0$

Cf. hyperKähler twist $F = d(h\alpha_0) + *_3 dh$.

Rotating actions

g hyperKähler, $\dim M > 4$, $L_X I = 0$, $L_X J = K$
Locally there is a Kähler moment map $\mu \colon M \to \mathbb{R}$ for (ω_I, X).

Theorem (Maciá-S)

For X non-null, an elementary deformation $g^N = fg + hg_\alpha$ twists to quaternionic Kähler if and only if

- $f = 1/(\mu - c)$
- $h = -1/(\mu - c)^2$
- $F = d\alpha_0 + \omega_I$
- $a = \|X\|^2 - \mu + c$

This is the hK/qK correspondence Haydys 2008; Alexandrov, Persson and Pioline 2011; Hitchin 2013; Alekseevsky, Cortés, Dyckmanns and Mohaupt 2013. In the c-map context, g has signature $(4n, 4)$, but g^N is positive definite.

Show quaternionic Kähler by $d\Omega = 0$

$$\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$$

provided $\dim M \geqslant 12$.
For $\dim M = 8$, show

$$d \begin{pmatrix} \omega_I \\ \omega_J \\ \omega_K \end{pmatrix} = A \wedge \begin{pmatrix} \omega_I \\ \omega_J \\ \omega_K \end{pmatrix}$$

for some $A \in \Omega^1 \otimes \mathfrak{so}(3)$.

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