Twists, Torsion and T-duality

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Outline

1. **Motivation**
   - HKT and String Duals
   - Geometry with Torsion

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An Example of T-duality

HyperKähler M^4

$$ds^2 = V^{-1}(d\tau + \omega)^2 + V\gamma_{ij}dx^i dx^j$$
$$dV = *_3 d\omega$$

T duality $\longleftrightarrow$ on $X = \frac{\partial}{\partial\tau}$

Strong HKT W^4

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For circle actions have:

$$R \leftrightarrow 1/R \qquad \text{and here} \quad W = (M/S^1) \times S^1$$

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Metric geometry with torsion

- metric g
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Definition

The geometry is *strong* if $dc = 0$

KT Geometry

Metric geometry

g, $\nabla = \nabla^{\mathrm{LC}} + \frac{1}{2}c$, $c \in \Lambda^3 T^* M$

KT geometry

additionally

- I integrable complex structure
- $g(IX, IY) = g(X, Y)$
- $\nabla I = 0$

Here $I\colon TM \to TM$ with

$$I^2 = -1 \qquad N_I = 0$$

where $N_I(X, Y) =$
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Given (g, I) the connection ∇ is *unique*: $c = -IdF_I$, where $F_I(X, Y) = g(IX, Y)$

- KT geometry is just Hermitian geometry together with the Bismut connection ∇
- $c = 0$ is Kähler geometry
- strong KT geometry is $\partial\bar{\partial}F_I = 0$
- Gauduchon, 1991: every compact Hermitian M^4 is conformal to strong KT

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Example

$G = SU(3) = M^8$, bi-invariant g is strong HKT

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$$W := P/\langle X' \rangle$$

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Dually

M is a twist of W with respect to X_W

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Lifting Group Actions

- X a vector field on M
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Lemma

There is an X' on P preserving θ and projecting to X if and only if X^θ is exact.
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Proof.

Let $\tilde{X}$ be the horizontal lift of X. Then

$$X' = \tilde{X} + aY$$

with $da = -X^\theta$. $\square$

Lifting Circle Actions

Call an S^1-action generated by X *F-Hamiltonian* if X preserves $F \in \Omega^2(M)$ and $X \lrcorner F$ is exact.

Proposition (cf. Lashof, May, and Segal, 1983)

Suppose $F \in \Omega^2_{\mathbb{Z}}(M)$ is a closed 2-form with integral periods. For each F-Hamiltonian S^1-action and each principal circle bundle $P \to M$ with $c_1(P) \otimes \mathbb{R} = [F]$ there is a locally free circle action on P covering the action on M and an invariant principal connection θ such that $F_\theta = F$.

General circle actions on $\mathbb{C}\mathrm{P}(n)$ can not be lifted to *free* circle actions on P.

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Lemma

$d\alpha_W$ is $\mathcal{H}$-related to a form on M if and only if $L_X\alpha = 0$. Then $d\alpha_W \sim_{\mathcal{H}} d\alpha - F_\theta\wedge\frac{1}{a}X\lrcorner\alpha$.

Almost Hermitian Twist

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Let (M, g, F_I) be an almost Hermitian structure invariant under X. This has *twist* (W, g_W, F_I^W) where

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Proposition

- *If I is integrable then I_W is integrable if and only if $F_\theta \in \Lambda^{1,1}$*
- *the forms $c = -IdF_I$ are related by*

$$c_W \sim_{\mathcal{H}} c - \frac{1}{a} X^\flat \wedge IF_\theta$$

Transformation Rules II

Corollary

If (M, g, I, J, K) is hyperHermitian (resp. HKT) then (W, g_W, I_W, J_W, K_W) is hyperHermitian (resp. HKT) if and only if

$$F_\theta \in \bigcap_{A=I,J,K} \Lambda^{1,1}_A$$

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For M KT (resp. HKT) and F_θ an instanton, W is strong if and only if

$$dc = \frac{1}{a}(dX^\flat + X \lrcorner c - \frac{1}{a}\|X\|^2 F_\theta) \wedge F_\theta$$

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- $g = \frac{1}{V}\varphi^2 + Vh$
  hyperKähler, $c = 0$
- hyperKähler isometry X
  - $\varphi(X) = 1, \quad L_X\varphi = 0$
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This is a twist via a trivial bundle with non-flat connection.

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Twist of $\mathscr{U}(\mathbb{C}P(2))/\mathbb{Z}$: strong HKT structure on $SU(3)$.

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- Each stage is KT if each F_i is type $(1, 1)$
- Final twist is strong KT if $F_1^2 + F_2^2 + \cdots + F_r^2 = 0$

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The twist W of M is a compact nilmanifold $\Gamma \backslash G$ where $\mathfrak{g}$ has commutators given by

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Y central.

Can repeatedly twist using different central X_i and closed two-forms F_i.

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Nilmanifold Examples

Theorem (Fino, Parton, and Salamon, 2004)

The six-dimensional strong KT nilmanifolds have Lie algebras

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Mejldal, 2004

The 8-dimensional nilmanifolds with $\mathfrak{g} = (0^6, 13-24+56, 12-2.23+3.34)$ are irreducible and lie in a 15-dimensional family of invariant strong KT structures.

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Remark

All known strong KT structures on nilmanifolds may be obtained via iterations of the above twist constructions starting from a flat torus.

Non-toral Base

- Twisting $M^6 = N^4 \times T^2$
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Take N^4 a K3 surface, with Kähler forms $\omega_I, \omega_J, \omega_K$. Integrability,

$$F_1 + iF_2 = \alpha + \lambda_1\omega_I + \lambda_2(\omega_J + i\omega_K)$$

with $\alpha \in \Lambda_I^{1,1} \cap (\omega_I)^\perp$. Strong,

$$\alpha \wedge \bar{\alpha} = 4(|\lambda_1|^2 - 2|\lambda_2|^2)\,\mathrm{vol}_g$$

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Theorem

For linearly independent primitive F_i satisfying the conditions to the left, twist W^6 of $M^6 = N^4 \times T^2$ is a compact simply-connected strong KT manifold.

Outline

Other Generalisations

- Non-toral fibres: can twist $N \times M$ whenever M has a circle action using a two-form F on N. Get for example S^2-bundles over N.

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- n-torus twists: are governed by $da_{ij} = -X_i \lrcorner F_j$. Wider variety of phenomena.
- multiple twists: are not the same as n-torus twists.

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  - based on a double principal bundle $M \longleftarrow P \longrightarrow W$ with common Ehreshmann connection $\mathscr{H}$
- defining forms are $\mathscr{H}$-related
- twisting by instantons preserves KT and HKT geometries
- strong structures may be obtained
- non-instanton twists are also necessary

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Exterior derivative of the torsion form

$$
\begin{aligned}
dc_W \sim_{\mathscr{H}} dc &- \tfrac{1}{a} dX^\flat \wedge IF_\theta + \tfrac{1}{a} X^\flat \wedge d(IF_\theta) \\
&- F_\theta \wedge \tfrac{1}{a} X \lrcorner c + F_\theta \wedge \tfrac{1}{a^2} \|X\|^2 IF_\theta - F_\theta \wedge \tfrac{1}{a} X^\flat \wedge X \lrcorner IF_\theta
\end{aligned}
$$