HyperKähler manifolds with Abelian symmetry

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Outline

1. HyperKähler manifolds
2. Dimension four
3. Toric hyperKähler
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HyperKähler manifolds

$(M, \omega_I, \omega_J, \omega_K)$ is *hyperKähler* if:

1. each ω_A is a symplectic two-form: $d\omega_A = 0$ and ω_A is non-degenerate,
2. the tangent bundle endomorphisms $I = \omega_K^{-1}\omega_J$, $J = \omega_I^{-1}\omega_K$, $K = \omega_J^{-1}\omega_I$ satisfy
   - $I^2 = -1 = J^2 = K^2$, $IJ = K = -JI$, etc., and
   - $g = -\omega_A(A\,\cdot\,, \cdot\,)$ is independent of A and positive definite.

Consequences

- $\dim M = 4n$,
- $a\omega_I + b\omega_J + c\omega_K = g(\mathcal{I}_{a,b,c}\,\cdot\,, \cdot\,)$ is symplectic for each $(a, b, c) \in S^2$, $\mathcal{I}_{a,b,c} = aI + bJ + cK$,
- (Hitchin et al. 1987) $\mathcal{I}_{a,b,c}$ are integrable complex structures,
- g is Ricci-flat, with holonomy contained in $Sp(n) \leqslant SU(2n)$.

Symmetry considerations

Ricci-flatness implies:

- if M is compact, then any Killing vector field is parallel, so the holonomy of M reduces and M splits as a product,
- if M is homogeneous then g is flat, so M is a quotient of flat $\mathbb{R}^{4n}$ by a discrete group (Alekseevskiĭ and Kimel'fel'd 1975).

Concentrate on complete (non-compact) hyperKähler manifolds with an Abelian group G of symmetries preserving each symplectic structure: tri-holomorphic isometries. Assume the action is *tri-Hamiltonian*, so there is a *hyperKähler moment map*: a G-invariant map

$$\mu = (\mu_I, \mu_J, \mu_K)\colon M \to \mathbb{R}^3 \otimes \mathfrak{g}^*$$

with $d\langle \mu_A, X\rangle = X \lrcorner \omega_A$.

Gibbons-Hawking Ansatz

X a tri-Hamiltonian vector field on hyperKähler M^4
Away from M^X, locally

$$g = \frac{1}{V}(dt + \omega)^2 + V(dx^2 + dy^2 + dz^2)$$

where $V = 1/g(X, X)$, $dx = X \lrcorner \omega_I = d\mu_I$, etc., and

$$d\omega = -*_3 dV$$

on $\mathbb{R}^3$. In particular,

- $\mu = (\mu_I, \mu_J, \mu_K)$ is locally a conformal submersion to $(\mathbb{R}^3, dx^2 + dy^2 + dz^2)$,
- V is locally a harmonic function on $\mathbb{R}^3$.

Examples

$$V(p) = c + \frac{1}{2}\sum_{i\in Z}\frac{1}{\|p-p_i\|}, \qquad c \geqslant 0, \quad p_i \in \mathbb{R}^3 \text{ distinct}$$

- $c = 0$, $|Z| < \infty$: multi-Euguchi Hanson metrics

$\lvert Z\rvert$	1	2	...
space	flat $\mathbb{R}^4$	$T^*\mathbb{C}P(1)$	...

- $c > 0$, $|Z| < \infty$: multi-Taub-NUT metrics

$\lvert Z\rvert$	0	1	2	...
space	flat $S^1 \times \mathbb{R}^3$	Taub-NUT $\mathbb{R}^4$	$T^*\mathbb{C}P(1)$	...

- Z countably infinite: require $V(p)$ to converge at some $p \in \mathbb{R}^3$, get A_∞ metrics (Anderson et al. 1989; Goto 1994), e.g. $Z = \mathbb{N}_{>0}$, $p_n = (1/n^2, 0, 0)$, and their Taub-NUT deformations.

Classification

Theorem

The potentials

$$V(p) = c + \frac{1}{2}\sum_{i\in Z}\frac{1}{\|p - p_i\|}, \qquad c \geqslant 0, \quad p_i \in \mathbb{R}^3 \textit{ distinct},$$

with $0 < V(p) < \infty$ *for some* $p \in \mathbb{R}^3$*, classify all complete hyperKähler four-manifolds with tri-Hamiltonian circle action.*

When $|Z| < \infty$, this is due to Bielawski (1999), and the first parts of the proof are essentially the same.

PROOF STRUCTURE

Local considerations

- The only special orbits are fixed points
- $\mu\colon M/S^1 \to \mathbb{R}^3$ is a local homeomorphism, even at fixed points
- locally near a fixed point x,

$$V(\mu(y)) = \frac{1}{2}\frac{1}{\|\mu(y)-\mu(x)\|} + \phi(\mu(y))$$

with $\phi \geqslant 0$ harmonic (Bôcher's Theorem plus a Chern class argument).

Proof structure 2

Injectivity of μ
Let $M' = M \setminus M^X$ be the set on which S^1 acts freely.
μ induces a conformal local diffeomorphism

$$\overline{\mu}\colon (N = M'/S^1, V(dx^2 + dy^2 + dz^2)) \to \mu(M') \subset (\mathbb{R}^3, g_{\mathbb{R}^3}).$$

Near each fixed point $x \in M^X$, $V = \phi + 1/2r$ and we may replace V by $\overline{V} > 0$, $\overline{V} \propto 1/2r^2$ and superharmonic, so that $(N, \overline{V}(dx^2 + dy^2 + dz^2))$ is complete with non-negative scalar curvature.

Schoen and Yau (1994) implies that $\overline{\mu}\colon N \to \mathbb{R}^3$ is injective and that the boundary $\partial\Omega$ of $\Omega = \overline{\mu}(N) = \mu(M')$ is polar, i.e. bounded harmonic functions have unique extension across $\partial\Omega$.

Proof structure 3

Martin boundary
$\Omega = \mu(M')$ having polar boundary implies $\Omega \subset \mathbb{R}^3$ is dense with Green's functions $G(p,q) = 1/\|p-q\|$.Assuming $p_0 = 0 \in \Omega$, the Martin kernel is

$$M(p,q) = \frac{G(p,q)}{G(p_0,q)} = \frac{\|q\|}{\|p-q\|}, \quad p,q \in \Omega.$$

The minimal Martin boundary

$$\Delta = \left\{ \lim_{q'} (p \mapsto M(p,q')) : q' \to q \notin \Omega \right\}$$

is

$$\Delta = \partial\Omega \cup \{\infty\}.$$

Proof structure 4

Potential theory

V is positive harmonic on $\Omega = \mu(M')$, so there is a positive measure $d\mu_V(q)$ such that

$$V(p) = \int_{\Delta=\partial\Omega\cup\{\infty\}} M(p,q)\,d\mu_V(q).$$

$F = \mu(M^X)$ is discrete, so Borel, and contained in $\partial\Omega$, so

$$W(p) = \int_F M(p,q)\,d\mu_V(q) = \frac{1}{2}\sum_{q\in F}\frac{1}{\|p-q\|}$$

is positive harmonic and finite on Ω, so F has no accumualtion points.

Completeness of M gives $\partial\Omega \setminus F$ is then empty. So $\Delta = F \cup \{\infty\}$ and $V = W + c$, $c \geqslant 0$ constant.

Toric hyperKähler

(with Andrew Dancer)

M^{4n} complete hyperKähler with tri-Hamiltonian action of T^n. Is given locally be the Pedersen-Poon Ansatz:

$$g = (V^{-1})_{ij}(dt + \omega_i)(dt + \omega_j) + V_{ij}(dx_i dx_j + dy_i dy_j + dz_i dz_j),$$

with $V_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$ with F a positive function on $\mathbb{R}^3 \otimes \mathbb{R}^n$ harmonic on every affine three-plane $X_{a,v} = a + \mathbb{R}^3 \otimes v$.

For generic $X_{a,v}$, then $Y = \mu^{-1}(X_{a,v})$ is smooth with free T^{n-1} action, Y/T^{n-1} is complete hyperKähler with S^1-action. Above analysis then fixes V on $X_{a,v}$, and F, providing a classification.

HyperKähler modification

(N^4, μ^N, g^N, X^N) a complete hyperKähler four-manifold with tri-Hamiltionian X^N.
X is a tri-Hamiltonian circle action on hyperKähler (M, g, I, J, K) of arbitrary dimension. The *hyperKähler modification* of M by N is

$$M_{\text{Nmod}} = (M \times N) /\!/\!/ (X' = X - X^N) = (\mu - \mu^N)^{-1}(0)/X'.$$

- $\dim M_{\text{Nmod}} = \dim M$
- M complete, then M_{Nmod} complete
- $\pi_1(M) = 0$, then $b_2(M_{\text{Nmod}}) = 1 + b_2(M) + b_2(N)$

Example

$M = \mathbb{H} = N$, $X = X^N$ generating $e^{it}q$, $\mu = \mu_{\mathbb{H}} + c$, $\mu_{\mathbb{H}} = \bar{q}iq$, $c \neq 0$: $M_{\text{Nmod}} = T^*\mathbb{C}P(1)$

A double fibration

For $\Phi = \mu - \mu^N$, $X' = X - X^N$:

$$\begin{array}{ccccc} & & P = \Phi^{-1}(0) & \xhookrightarrow{\iota} & M \times \mathbb{H} \\ & \overset{\mathrm{pr}_1}{\swarrow} & & \overset{\mathrm{pr}(X')}{\searrow} & \\ M & & & & M_{\mathrm{Nmod}} \end{array}$$

- $\mathrm{pr}(X')$ is a Riemannian submersion for $\iota^*(g + g_{\mathbb{H}})$
- pr_1 is *not* a Riemannian submersion, it induces the metric $\tilde{g}$ on M:

$$\tilde{g} = g + V^N(\mu) g_\alpha, \qquad g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$$

$$\alpha_0 = X^\flat = g(X, \cdot), \ \alpha_I = I\alpha_0 = -\alpha_0(I\cdot) \text{ etc.}$$

Elementary deformations

$$\tilde{g} = g + V^N(\mu) g_\alpha,$$

g hyperKähler, X an isometry, $\alpha_0 = X^\flat$, $g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$

Definition

An *elementary deformation* $\tilde{g}$ of g with respect to X is

$$\tilde{g} = fg + hg_\alpha$$

for some $f, h \in C^\infty(M)$

Which elementary deformations define new hyperKähler metrics through such a double fibration picture?

Twist construction

Twist data

- M manifold
- $X \in \mathfrak{X}(M)$, circle action
- $F \in \Omega^2_{\mathbb{Z}}(M)^X$
- $a \in C^\infty(M)$ with $da = -X \lrcorner F$

$$M \xleftarrow{\mathrm{pr}_M} P \xrightarrow{\mathrm{pr}_W} W$$

horizontal distribution
$\mathcal{H} = \ker\theta \subset TP$

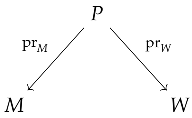
α tensor on M is $\mathcal{H}$*-related* to α_W on W if

$$\mathrm{pr}_M^*\alpha = \mathrm{pr}_W^*\alpha_W \quad \text{on } \mathcal{H}$$

Write $\alpha \sim_{\mathcal{H}} \alpha_W$

Twist computations

$\alpha \sim_{\mathcal{H}} \alpha_W$ if $\mathrm{pr}_M^* \alpha = \mathrm{pr}_W^* \alpha_W$ on $\mathcal{H} = \ker \theta$



- $\alpha \in \Omega^p(M)$: $d\alpha_W \sim_{\mathcal{H}} d\alpha - \frac{1}{a} F \wedge (X \lrcorner \alpha)$
- I complex structure on M:
  I_W integrable if and only if $F \in \Lambda_I^{1,1}$

Example

$M = M(n) := \mathbb{C}P^n \times T^2$
Kähler, X on T^2, $F = \omega_{\mathrm{FS}}$:
$W = S^{2n+1} \times S^1$ Hermitian non-Kähler.

Example

$M = T^n$, F left-invariant:
W is a nilmanifold corresponding to
$\mathfrak{g}^* = (0^{n-1}, F)$.

Tri-Hamiltonian actions

(M, g) hyperKähler, $\dim M > 4$, X tri-Hamiltonian with moment map $\mu = (\mu_I, \mu_J, \mu_K)$

Theorem

An elementary deformation $\tilde{g} = fg + hg_\alpha$ twists via (X, F, a) to a hyperKähler metric g_W if and only if

- *f constant, so take $f \equiv 1$,*
- *$h = h(\mu_I, \mu_J, \mu_K)$ is harmonic in $U \subset \mathbb{R}^3$,*
- *$F = d(h\alpha_0) + *_3 dh$,*
- *$a = 1 + h\|X\|^2 \neq 0$.*

HyperKähler modification is $h = V^N(\mu)$

Proof method

1. $\omega_I^W \sim_{\mathcal{H}} \omega_I^N = f\omega_I + h\omega_I^\alpha$
2. impose $d\omega_I^W = 0$, i.e. $d\omega_I^N - \frac{1}{a} F \wedge (X \lrcorner \omega_I^N) = 0$
3. impose $da = -X \lrcorner F$
4. impose $dF = 0$

Inversion

Generally: Twist of M by data (X, F, a) to W is inverted by twist data on W $\mathcal{H}$-related to $(\frac{1}{a}X, -\frac{1}{a}F, \frac{1}{a})$.

Proposition

The hyperKähler twist above of the elementary deformation $\tilde{g} = g + hg_\alpha$ of g corresponding to h is inverted by the elementary deformation of g_W corresponding to $-h$.

- Modification by $N = \mathbb{H}$, $V^N = 1/2\|\mu\|$, is inverted by $h = -1/(2\|\mu\|)$. To get positive definite, need $\|X\|^2 < 2\|\mu\|$. So flat $\mathbb{R}^4$ is *not* a modification.
- $h > 0$: inversion corresponds to hyperKähler quotient of $(M \times N^4, g \oplus -g^N(h))$, quaternionic Lorentzian

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