Moment map geometry for three-forms

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Joint work with Thomas Bruun Madsen
`arXiv:1012.2048,1012.0402`

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1. Background
   - Symplectic Geometry
   - Strong Geometry
   - Covariant Moment Maps

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2. **Multi-moment maps**
   - Commuting vector fields
   - Lie kernels
   - Existence
   - $(2,3)$-trivial Lie algebras
3. **G_2 holonomy**
   - Reduction
   - Four-dimensional geometry

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The moment map $\mu\colon \mathcal{O} \to \mathfrak{g}^*$ is just inclusion.

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- Definition introduced and studied by Cariñena, Crampin, and Ibort (1991) and by Gotay, Isenberg, Marsden, and Montgomery (1998).
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- Problems include:
  1. Target space $\Omega^1(M, \mathfrak{g})$ depends both on M and $\mathfrak{g}$.
  2. Existence often requires some restrictive assumption such as $b_2(M) = 0$.

Outline

COMMUTING VECTOR FIELDS

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So the one-form $c(X,Y,\cdot)$ is $d\nu_{X,Y}$ for some local function $\nu_{X,Y}$.

Outline

1. Background
   - Symplectic Geometry
   - Strong Geometry
   - Covariant Moment Maps
2. Multi-moment maps
   - Commuting vector fields
   - Lie kernels
   - Existence
   - (2, 3)-trivial Lie algebras
3. G_2 holonomy
   - Reduction
   - Four-dimensional geometry

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Linearity in the basic calculation shows that

$$d(p \lrcorner c) = d(\sum_{i=1}^{r} c(X_i, Y_i, \cdot)) = -(\sum_{i=1}^{r} [X_i, Y_i]) \lrcorner c = 0.$$

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- For G Abelian, $\mathcal{P}_{\mathfrak{g}} = \Lambda^2 \mathfrak{g}$.
- For G semi-simple, $\Lambda^2 \mathfrak{g} \cong \mathfrak{g} \oplus \mathcal{P}_{\mathfrak{g}}$. In particular, for G compact and simple, $\mathcal{P}_{\mathfrak{g}}$ is the isotropy representation of the isotropy irreducible space $SO(\dim \mathfrak{g})/G$.

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is the description of $SU(3)$ as a hypercomplex (HKT) Swann bundle over the quaternionic Kähler $\mathbb{C}P(2)$.

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Outline

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- There exist unimodular $(2,3)$-trivial Lie groups admitting compact discrete quotients ($\dim G \geqslant 5$).

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with

$$\omega_0 = U_1 \lrcorner U_2 \lrcorner {*}\phi, \quad \omega_i = U_i \lrcorner \phi, \quad (g_{UU} g_{VV} - g_{UV}^2)\, h^2 = 1.$$

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$$h^2\,{\omega_0}^2 = g_{UU}^{-1}\,{\omega_1}^2 = g_{VV}^{-1}\,{\omega_2}^2 = 2\,\mathrm{vol}_M,$$
$$\omega_0 \wedge \omega_1 = 0 = \omega_0 \wedge \omega_2, \quad \omega_1 \wedge \omega_2 = 2g_{UV}\,\mathrm{vol}_M\,.$$

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Donaldson (2006) asks whether the underlying compact manifold is always hyperKähler.

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- $(2, 3)$-trivial Lie algebras may be classified in small dimensions and described and as certain one-dimensional solvable extensions of nilpotent algebras in general.
- G_2 holonomy manifolds with T^2-symmetry correspond via multi-moment map reduction to coherent symplectic triples on M^4.

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