Nearly Kähler manifolds with symmetry

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Joint work with Giovanni Russo

Russo, G., and Swann, A. F. (2019), 'Nearly Kähler six-manifolds with two-torus symmetry', *J. Geom. Phys.* 138: 144–53
Russo, G. (2021), 'Multi-moment maps on nearly Kähler six-manifolds', *Geom. Dedicata*, 213: 57–81
Russo, G., and Swann, A. F. (2023b), *The nearly Kähler structure of $S^3 \times S^3$*, in progress
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Nearly Kähler geometry

Gray (1965): *nearly Kähler manifold*: almost Hermitian manifold (M, g, J) with

$$(\nabla_X J)X = 0 \quad \text{and} \quad \nabla J \neq 0.$$

Nagy (2002): complete, $\pi_1(M) = 0$, nearly Kähler are products of

- Kähler manifolds,
- three-symmetric spaces,
- *nearly Kähler six-manifolds*,

and/or

- twistor spaces of positive quaternionic Kähler manifolds.

Dimension 6: are positive Einstein (Gray 1976); complete gives compact, $|\pi_1(M)| < \infty$.

Butruille (2005) homogeneous are three-symmetric spaces: S^6, $\mathbb{C}P(3)$, $F_{1,2}$, $S^3 \times S^3$.

Foscolo and Haskins (2017) new compact examples: cohomogeneity one ($G = SU(2) \times SU(2)$) on $S^3 \times S^3$ and S^6.

Multi-moment maps for T^2-symmetry

Nearly Kähler equations: (M^6, g, J), 2-form $\sigma = g(J\,\cdot\,, \,\cdot\,)$, 3-forms $\psi = \psi_+ + i\psi_- \in \Omega^{3,0}(M)$,

$$d\sigma = 3\psi_+ \quad \text{and} \quad d\psi_- = -2\sigma^2.$$

T^2-symmetry generated by U, V, call

$$\nu = \sigma(U, V), \qquad \nu\colon M \to \mathbb{R} = \Lambda^2 \operatorname{Lie}(T^2)^*,$$

the *multi-moment map* (Madsen and Swann 2012).

Satisfies

$$d\nu = 3\psi_+(U, V, \,\cdot\,) \quad \text{and} \quad \Delta\nu = 24\nu.$$

$\nu(M) = [a, b]$ is a compact interval containing 0 in its interior.

Reduction at regular values

For $s \neq 0$ a regular value of $\nu = \sigma(U, V)$, $Q^3 = \nu^{-1}(s)/T^2$ is smooth. Define connection one-form $\vartheta = (\vartheta_1, \vartheta_2)$ and basic forms

$$\alpha_0 = \psi_-(U, V, \cdot), \quad \alpha_1 = s\vartheta_1 + V \lrcorner \sigma, \quad \alpha_2 = s\vartheta_2 - U \lrcorner \sigma.$$

Theorem

$$d\alpha_0 = f\,\alpha_1 \wedge \alpha_2, \quad d(f\alpha_1) \wedge \alpha_0 = 0 = d(f\alpha_2) \wedge \alpha_0,$$

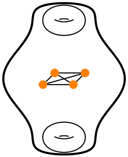
$f > 4$ smooth on Q^3, together with a choice of metric H on $\mathrm{Span}\{\alpha_1, \alpha_2\}$ *determine via a geometric flow a nearly Kähler manifold (often incomplete) with T^2-symmetry.*

Nearly Kähler metric

$$g = \frac{1}{9(h^2 - s^2)} ds^2 + \vartheta^T H \vartheta + \frac{1}{h^2 - s^2}(\alpha_0^2 + \alpha^T H \alpha), \quad \begin{aligned} h^2 &= \det H \\ &= s^2 f/(f - 4). \end{aligned}$$

Morse theory picture

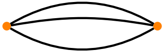
Nearly Kähler with T^2-symmetry



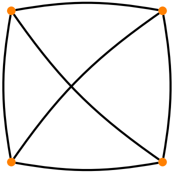
Critical set $\nu \neq 0$: U and V are linearly dependent over $\mathbb{C} = \mathbb{R} + J\mathbb{R}$. Discrete stabilisers.
Critical set $\nu = 0$: quotient space a trivalent graph. Vertices at stabiliser T^2, edges along stabiliser S^1.

Known examples have graphs

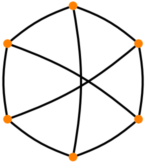
S^6



$\mathbb{C}P(3)$



$F_{1,2}(\mathbb{C}^3)$



$S^3 \times S^3$

$\varnothing$,

, or

Left-invariant flow

Q^3 is homogeneous with invariant data, then Q is any non-Abelian unimodular group.
Compact case: $Q = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, dual basis $de_a = e_b \wedge e_c$, cyclically.
Flow diagonalises: $\alpha_1 = v_1 e_1, \alpha_2 = v_2 e_2, \alpha_0 = u e_3$,

$$u' = -\frac{4s}{3(h^2 - s^2)}u, \quad v_i' = -\frac{s}{3}\left(\frac{8}{h^2 - s^2} - \frac{v_i}{u v_j h_j}\right)v_i, \quad \{i, j\} = \{1, 2\},$$
$$h_i' = \frac{h_i}{s}\left(-1 + \frac{(h^2 - s^2)v_i}{3u v_j h_j}\right), \quad f = \frac{u}{v_1 v_2} = \frac{4h^2}{h^2 - s^2}.$$

Generically 4-parameter family of solutions.

Theorem

Although the equations are singular at $s = 0$, the solutions that are smooth over $s = 0$ form a 4-parameter family, and near $s = 0$ this family is open in the set of generic solutions.

Standard three-symmetric solution

$$\alpha_0 = \frac{8}{27}\sin(2t)e_1, \qquad s = -\frac{4}{3\sqrt{3}}\cos(2t) \text{ with } t \in (0,\pi/2),$$

$$\alpha_1 = -\frac{4}{3\sqrt{3}}\frac{\sin(t)\sin(2t)}{2-\cos(2t)}e_2, \quad \alpha_2 = -\frac{4}{3\sqrt{3}}\frac{\cos(t)\sin(2t)}{2+\cos(2t)}e_3,$$

$$h_1 = \frac{4}{9}(2-\cos(2t)), \quad h_2 = \frac{4}{9}(2+\cos(2t)).$$

Gives cohomogeneity-one action of $T^2 \times \mathrm{SU}(2)$ on $S^3 \times S^3$.
Missing from topological classification of Podestà and Spiro (2010).

Moroianu and Nagy (2019): for any compact nearly Kähler 6-manifold, rank of symmetry group is at most 3. Only known example with such symmetry is the three-symmetric solution on $S^3 \times S^3$.

Previous slide, many local solutions with this symmetry rank.

Singular behaviour

For the left-invariant flow on $Q = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$, there are at most 2 critical values. Critical set is essentially the base of

$$S^1 \longrightarrow T^2 \times \mathrm{SU}(2) \longrightarrow S^1 \times S^3.$$

Have a topological parameter for this bundle.

Theorem

Around each critical value there is a two-parameter family of solutions for each topological parameter.

Question Can one match up these solutions at $s = 0$ to get new compact solutions?

Hoelscher (2010a,b) topologically result is $S^3 \times S^3$. But for non-standard topological parameters the $T^2 \times \mathrm{SU}(2)$-action is not that of the standard example.

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