Geometric Duality

Andrew Swann

IMADA / CP^3-Origins
University of Southern Denmark
swann@imada.sdu.dk

November 2009 / Odense

Outline

1 Geometry

- Metric geometry with torsion
- KT Geometry
- HKT Geometry

Outline

1. **Geometry**
   - Metric geometry with torsion
   - KT Geometry
   - HKT Geometry

2. **Twists**
   - T-duality as a Twist Construction

Outline

Outline

GEOMETRY FROM SUPERSYMMETRY

CLASSICAL GEOMETRIES

- Riemannian/Lorentzian metric $g = (g_{ij})$, has a unique covariant derivative ∇^{LC}, Levi-Civita connection, that is metric $\nabla^{LC} g = 0$ and torsion-free $\nabla^{LC}_X Y - \nabla^{LC}_Y X = [X, Y]$:

$$2g(\nabla^{LC}_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, Y]);$$

$$\nabla^{LC}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}, \qquad \Gamma^k_{ij} = \tfrac{1}{2} g^{k\ell}(g_{\ell i,j} + g_{j\ell,i} - g_{ij,\ell}) = \Gamma^k_{ji}.$$

Supersymmetry: usually parallel complex structures $J = (J_i^{\,j})$:

$$g(JX, JY) = g(X, Y), \qquad J^2 = -1, \qquad \nabla^{LC} J = 0;$$

$$J_i^{\,\ell} J_j^{\,k} g_{\ell k} = g_{ij}, \qquad J_i^{\,j} J_j^{\,k} = -\delta_i^k, \qquad J_i^{\,k}{}_{,j} = \Gamma^\ell_{ij} J_\ell^{\,k}.$$

One complex structure

- Kähler geometry: Riemann surfaces, $\mathbb{C}P(n)$, projective varieties $X = \bigcap_i (f_i = 0) \subset \mathbb{C}P(n)$, Hermitian symmetric spaces,…
- Calabi-Yau manifolds, Kähler with $c_1 = 0$: have $\mathrm{Ric} \equiv 0$ so Einstein; $X = (f = 0) \subset \mathbb{C}P(n)$, $\deg f = n + 1$; K3 surface $(x^4 + y^4 + z^4 + w^4 = 0) \subset \mathbb{C}P(3)$.

Multiple complex structures

- HyperKähler geometry I, J, K, $IJ = K = -JI$: are Calabi-Yau; K3 surfaces, $T^{4k} = \mathbb{R}^{4k}/\mathbb{Z}^{4k}$; Hilbert schemes; instanton moduli;…

Holonomy classification (Berger,…) essentially only get products of the above examples

Outline

Torsion Geometry

Metric geometry with torsion

- metric g, connection ∇, torsion $T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
- $\nabla g = 0$

Torsion Geometry

Metric geometry with torsion

- metric g, connection ∇, torsion $T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
- $\nabla g = 0$
- $c(X,Y,Z) = g(T^\nabla(X,Y),Z)$ a three-form

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \gamma_{ij}^k \frac{\partial}{\partial x^k},$$
$$T_{ij}^\ell = \gamma_{[ij]}^k,$$
$$c_{ijk} = c_{[ijk]} = g_{\ell k} T_{ij}^\ell,$$

Torsion Geometry

Metric geometry with torsion

- metric g, connection ∇, torsion $T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
- $\nabla g = 0$
- $c(X,Y,Z) = g(T^\nabla(X,Y),Z)$ a three-form

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \gamma_{ij}^k \frac{\partial}{\partial x^k},$$
$$T_{ij}^\ell = \gamma_{[ij]}^k,$$
$$c_{ijk} = c_{[ijk]} = g_{\ell k} T_{ij}^\ell,$$

$$\nabla = \nabla^{\mathrm{LC}} + \tfrac{1}{2} c$$
$$\gamma_{ij}^k = \Gamma_{ij}^k + \tfrac{1}{2} c_{ijk}$$

- Any $c \in \Omega^3(M)$ will do.
- ∇, ∇^{LC} have the same geodesics (dynamics).
- The geometry is *strong* if $dc = 0$.

Such geometries with extra structure from supersymmetry arise from:

- Wess-Zumino terms in the Lagrangian, superstrings with torsion, B-fields (Strominger, 1986)
- One-dimensional quantum mechanics with type B supersymmetry, blackhole dynamics and moduli (Michelson and Strominger, 2000; Coles and Papadopoulos, 1990; Hull, 1999; Gibbons et al., 1997)
- Constructions in supergravity (Grover et al., 2009)

Mathematically, one wishes to:

- clarify the basic definitions and relationships to known geometries,
- construct and classify examples in given categories.

In particular, we will be looking for *compact* simply-connected torsion geometries with compatible complex structures.

OUTLINE

KT Geometry

$g,\ \nabla = \nabla^{\mathrm{LC}} + \frac{1}{2}c, \quad c \in \Lambda^3 T^*M$

KT geometry

additionally

- I integrable complex structure
- $g(IX, IY) = g(X, Y)$
- $\nabla I = 0$

Two form $\omega_I(X, Y) = g(IX, Y)$

KT Geometry

$$g, \nabla = \nabla^{\mathrm{LC}} + \tfrac{1}{2}c, \quad c \in \Lambda^3 T^* M$$

KT geometry

additionally

- I integrable complex structure
- $g(IX, IY) = g(X, Y)$
- $\nabla I = 0$

Two form $\omega_I(X, Y) = g(IX, Y)$

∇ is unique

$$c = -I d\omega_I$$

the *Bismut connection*

KT Geometry

g, $\nabla = \nabla^{\mathrm{LC}} + \frac{1}{2}c$, $\quad c \in \Lambda^3 T^*M$

KT geometry

additionally

- I integrable complex structure
- $g(IX, IY) = g(X, Y)$
- $\nabla I = 0$

Two form $\omega_I(X, Y) = g(IX, Y)$

∇ is unique

$$c = -Id\omega_I$$

the *Bismut connection*

- KT geometry = Hermitian geometry + Bismut connection
- $c = 0$ is Kähler geometry
- strong KT is $\partial\bar{\partial}\omega_I = 0$

KT Geometry

$g, \nabla = \nabla^{\mathrm{LC}} + \frac{1}{2}c, \quad c \in \Lambda^3 T^* M$

KT geometry

additionally

- I integrable complex structure
- $g(IX, IY) = g(X, Y)$
- $\nabla I = 0$

Two form $\omega_I(X, Y) = g(IX, Y)$

∇ is unique

$$c = -I d\omega_I$$

the *Bismut connection*

- KT geometry = Hermitian geometry + Bismut connection
- $c = 0$ is Kähler geometry
- strong KT is $\partial\bar{\partial}\omega_I = 0$

Example

$M^6 = S^3 \times S^3 = SU(2) \times SU(2)$

Gauduchon (1991)

every compact Hermitian M^4 is conformal to strong KT

Outline

HKT Geometry

HKT structure

(g, ∇, I, J, K) with

- (g, ∇, A) KT, $\quad A = I, J, K$
- $IJ = K = -JI$

$c = -Ad\omega_A$ independent of A

HKT Geometry

HKT structure

(g, ∇, I, J, K) with

- (g, ∇, A) KT, $\quad A = I, J, K$
- $IJ = K = -JI$

$c = -Ad\omega_A$ independent of A

Martín Cabrera and Swann (2008)

$$Id\omega_I = Jd\omega_J = Kd\omega_K$$

implies I, J, K integrable, so HKT.

HKT Geometry

HKT structure

(g, ∇, I, J, K) with

- (g, ∇, A) KT, $\quad A = I, J, K$
- $IJ = K = -JI$

$c = -Ad\omega_A$ independent of A

Martín Cabrera and Swann (2008)

$$Id\omega_I = Jd\omega_J = Kd\omega_K$$

implies I, J, K integrable, so HKT.

Examples

Dim 4 T^4, K3, $S^3 \times S^1$ (Boyer, 1988)

Dim 8 Hilbert schemes, $SU(3)$, nilmanifolds, vector bundles over discrete groups (Verbitsky, 2003; Barberis and Fino, 2008)

HKT Geometry

HKT structure

(g, ∇, I, J, K) with

- (g, ∇, A) KT, $A = I, J, K$
- $IJ = K = -JI$

$c = -Ad\omega_A$ independent of A

Martín Cabrera and Swann (2008)

$$Id\omega_I = Jd\omega_J = Kd\omega_K$$

implies I, J, K integrable, so HKT.

Examples

Dim 4 T^4, K3, $S^3 \times S^1$ (Boyer, 1988)

Dim 8 Hilbert schemes, $SU(3)$, nilmanifolds, vector bundles over discrete groups (Verbitsky, 2003; Barberis and Fino, 2008)

Compact, simply-connected examples which are neither hyperKähler nor homogeneous?

Outline

Four-dimensional inspiration

HyperKähler M^4

$$ds^2 = V^{-1}(d\tau + \omega)^2 + V\gamma_{ij}dx^i dx^j$$
$$dV = *_3 d\omega$$

T duality

$\longleftrightarrow$

on
$X = \frac{\partial}{\partial\tau}$

Strong HKT W^4

$$ds^2 = V(d^2\tau + \gamma_{ij}dx^i dx^j)$$
$$c = -d\tau \wedge d\omega$$

FOUR-DIMENSIONAL INSPIRATION

HYPERKÄHLER M^4

$$ds^2 = V^{-1}(d\tau + \omega)^2 + V\gamma_{ij}dx^i dx^j$$

$$dV = *_3 d\omega$$

T duality $\longleftrightarrow$ on $X = \frac{\partial}{\partial\tau}$

STRONG HKT W^4

$$ds^2 = V(d^2\tau + \gamma_{ij}dx^i dx^j)$$

$$c = -d\tau \wedge d\omega$$

- Gibbons, Papadopoulos, and Stelle, 1997
- Callan, Harvey, and Strominger, 1991
- Bergshoeff, Hull, and Ortín, 1995

Four-dimensional inspiration

HyperKähler M^4

$$ds^2 = V^{-1}(d\tau + \omega)^2 + V\gamma_{ij}dx^i dx^j$$

$$dV = *_3 d\omega$$

T duality

$\longleftrightarrow$

on

$X = \frac{\partial}{\partial\tau}$

Strong HKT W^4

$$ds^2 = V(d^2\tau + \gamma_{ij}dx^i dx^j)$$

$$c = -d\tau \wedge d\omega$$

- Gibbons, Papadopoulos, and Stelle, 1997
- Callan, Harvey, and Strominger, 1991
- Bergshoeff, Hull, and Ortín, 1995

For circle actions have:

$$R \leftrightarrow 1/R \qquad \text{and here} \quad W = (M/S^1) \times S^1$$

T-duality as a Twist

- X_p generating a n-torus action on M
- $(P, \theta, Y_q) \xrightarrow{\pi} M$ an invariant principal T^n-bundle

T-duality as a Twist

- X_p generating a n-torus action on M
- $(P, \theta, Y_q) \xrightarrow{\pi} M$ an invariant principal T^n-bundle
- $X_p' = \tilde{X}_p + a_{pq} Y_q$ a lift of X_p generating a free torus action, $da_{pq} = -X_p \lrcorner F_q^\theta$

T-duality as a Twist

- X_p generating a n-torus action on M
- $(P, \theta, Y_q) \xrightarrow{\pi} M$ an invariant principal T^n-bundle
- $X'_p = \tilde{X}_p + a_{pq} Y_q$ a lift of X_p generating a free torus action, $da_{pq} = -X_p \lrcorner F^\theta_q$

Definition

A *twist* W of M with respect to X_p is

$$W := P / \langle X'_p \rangle$$

T-duality as a Twist

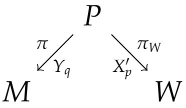
- X_p generating a n-torus action on M
- $(P, \theta, Y_q) \xrightarrow{\pi} M$ an invariant principal T^n-bundle
- $X_p' = \tilde{X}_p + a_{pq} Y_q$ a lift of X_p generating a free torus action, $da_{pq} = -X_p \lrcorner F_q^\theta$

Definition

A *twist* W of M with respect to X_p is

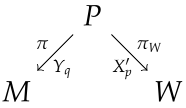
$$W := P / \langle X_p' \rangle$$

- Transverse locally free lifts always exist for $X_p \lrcorner F_q^\theta$ exact.
- W is at worst an orbifold.



T-duality as a Twist

- X_p generating a n-torus action on M
- $(P, \theta, Y_q) \xrightarrow{\pi} M$ an invariant principal T^n-bundle
- $X_p' = \tilde{X}_p + a_{pq} Y_q$ a lift of X_p generating a free torus action, $da_{pq} = -X_p \lrcorner F_q^\theta$
- Transverse locally free lifts always exist for $X_p \lrcorner F_q^\theta$ exact.
- W is at worst an orbifold.



Definition

A *twist* W of M with respect to X_p is

$$W := P / \langle X_p' \rangle$$

Dually

M is a twist of W with respect to $X_q^W = (\pi_W)_* Y_q$, $\theta_p^W = (a^{-1})^{pq} \theta_q$

Definition

Tensors α on α_W on M and W are *$\mathcal{H}$-related*, $\alpha_W \sim_{\mathcal{H}} \alpha$ if their pull-backs agree on $\mathcal{H} = \ker \theta$

Move invariant geometry from M to W by using the corresponding $\mathcal{H}$-related tensors

$$g_W \sim_{\mathcal{H}} g, \qquad \omega_I^W \sim_{\mathcal{H}} \omega_I, \quad \text{etc.}$$

Definition

Tensors α on α_W on M and W are *$\mathcal{H}$-related*, $\alpha_W \sim_{\mathcal{H}} \alpha$ if their pull-backs agree on $\mathcal{H} = \ker\theta$

Move invariant geometry from M to W by using the corresponding $\mathcal{H}$-related tensors

$$g_W \sim_{\mathcal{H}} g, \qquad \omega_I^W \sim_{\mathcal{H}} \omega_I, \quad \text{etc.}$$

For invariant forms

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - F_q^\theta \wedge (a^{-1})^{pq} X_q \lrcorner \alpha$$

For the KT torsion form $c = -Id\omega_I$:

$$c_W \sim_{\mathcal{H}} c - (a^{-1})^{pq} I F_q^\theta \wedge X^p.$$

Outline

Superconformal Quantum Mechanics

N particles in 1 dimension

$$H = \frac{1}{2} P_a^* g^{ab} P_b + V(x)$$

Standard quantisation

$$P_a \sim -i \frac{\partial}{\partial x^a}, \quad a = 1, \ldots, N$$

Superconformal Quantum Mechanics

N particles in 1 dimension

$$H = \frac{1}{2} P_a^* g^{ab} P_b + V(x)$$

Standard quantisation

$$P_a \sim -i \frac{\partial}{\partial x^a}, \quad a = 1, \ldots, N$$

Michelson and Strominger (2000); Papadopoulos (2000)

- operator D with $[D, H] = 2iH \iff$ vector field X with $L_X g = 2g$ & $L_X V = -2V$
- K so $\operatorname{span}\{iH, iD, iK\} \cong \mathfrak{sl}(2, \mathbb{R}) \iff X^\flat = g(X, \cdot)$ is closed
- then $K = \frac{1}{2} g(X, X)$.

Choose a superalgebra containing $\mathfrak{sl}(2, \mathbb{R})$ in its even part.

Outline

The Superalgebras $D(2,1;\alpha)$

The classification of simple Lie superalgebras contains *one* continuous family

$D(2,1;\alpha)$

THE SUPERALGEBRAS $D(2,1;\alpha)$

The classification of simple Lie superalgebras contains *one* continuous family

$D(2,1;\alpha)$

- $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$

The Superalgebras $D(2,1;\alpha)$

The classification of simple Lie superalgebras contains *one* continuous family

$D(2,1;\alpha)$

- $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$
- $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{C})_+ + \mathfrak{sl}(2,\mathbb{C})_-$
- $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2_+ \otimes \mathbb{C}^2_- = \mathbb{C}^4_Q + \mathbb{C}^4_S$

The Superalgebras $D(2,1;\alpha)$

The classification of simple Lie superalgebras contains *one* continuous family

$D(2,1;\alpha)$

- $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$
- $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{C})_+ + \mathfrak{sl}(2,\mathbb{C})_-$
- $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2_+ \otimes \mathbb{C}^2_- = \mathbb{C}^4_Q + \mathbb{C}^4_S$
- $[S^a, Q^a] = D,$
- $[S^1, Q^2] = -\frac{4\alpha}{1+\alpha} R^3_+ - \frac{4}{1+\alpha} R^3_-$

The Superalgebras $D(2,1;\alpha)$

The classification of simple Lie superalgebras contains *one* continuous family

$D(2,1;\alpha)$

- $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$
- $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{C})_+ + \mathfrak{sl}(2,\mathbb{C})_-$
- $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2_+ \otimes \mathbb{C}^2_- = \mathbb{C}^4_Q + \mathbb{C}^4_S$
- $[S^a, Q^a] = D,$
- $[S^1, Q^2] = -\frac{4\alpha}{1+\alpha} R^3_+ - \frac{4}{1+\alpha} R^3_-$

Simple for $\alpha \neq -1, 0, \infty$.

The Superalgebras $D(2,1;\alpha)$

The classification of simple Lie superalgebras contains *one* continuous family

$D(2,1;\alpha)$

- $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$
- $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{C})_+ + \mathfrak{sl}(2,\mathbb{C})_-$
- $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2_+ \otimes \mathbb{C}^2_- = \mathbb{C}^4_Q + \mathbb{C}^4_S$
- $[S^a, Q^a] = D$,
- $[S^1, Q^2] = -\frac{4\alpha}{1+\alpha}R^3_+ - \frac{4}{1+\alpha}R^3_-$

Simple for $\alpha \neq -1, 0, \infty$.

- Over $\mathbb{C}$, isomorphisms between the cases $\alpha^{\pm 1}$, $-(1+\alpha)^{\pm 1}$, $-(\alpha/(1+\alpha))^{\pm 1}$.

The Superalgebras $D(2,1;\alpha)$

The classification of simple Lie superalgebras contains *one* continuous family

$D(2,1;\alpha)$

- $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$
- $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{C})_+ + \mathfrak{sl}(2,\mathbb{C})_-$
- $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2_+ \otimes \mathbb{C}^2_- = \mathbb{C}^4_Q + \mathbb{C}^4_S$
- $[S^a, Q^a] = D,$
- $[S^1, Q^2] = -\frac{4\alpha}{1+\alpha} R^3_+ - \frac{4}{1+\alpha} R^3_-$

Simple for $\alpha \neq -1, 0, \infty$.

- Over $\mathbb{C}$, isomorphisms between the cases $\alpha^{\pm 1}$, $-(1+\alpha)^{\pm 1}$, $-(\alpha/(1+\alpha))^{\pm 1}$.
- Real form $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) + \mathfrak{su}(2)_+ + \mathfrak{su}(2)_-$.
- Over $\mathbb{R}$, isomorphisms for $\alpha^{\pm 1}$

Outline

Superconformal Geometry

$\mathcal{N} = 4B$ quantum mechanics

with $D(2,1;\alpha)$ superconformal symmetry

$\leftrightarrow$

HKT manifold M

with X a *special homothety of type* (a,b)

- $L_X g = ag$,
- $L_{IX} J = bK$,
- $L_X I = 0$, $L_{IX} I = 0, \ldots$

Superconformal Geometry

$\mathcal{N} = 4B$ quantum mechanics

with $D(2,1;\alpha)$ superconformal symmetry

$\leftrightarrow$

HKT manifold M

with X a *special homothety of type* (a,b)

- $L_X g = ag$,
- $L_{IX} J = bK$,
- $L_X I = 0$, $L_{IX} I = 0, \ldots$

- $\alpha = \frac{a}{b} - 1$
- Action of $\mathbb{R} \times SU(2)$ rotating I, J, K

Superconformal Geometry

$\mathcal{N} = 4B$ quantum mechanics

with $D(2,1;\alpha)$ superconformal symmetry

$\leftrightarrow$

HKT manifold M

with X a *special homothety of type* (a,b)

- $L_X g = ag,$
- $L_{IX} J = bK,$
- $L_X I = 0,\ L_{IX} I = 0, \ldots$

- $\alpha = \frac{a}{b} - 1$
- Action of $\mathbb{R} \times SU(2)$ rotating I, J, K

For $a \neq 0$

- M is non-compact
- $\mu = \frac{2}{a(a-b)} \|X\|^2$ is an *HKT potential*

$$\omega_I = \tfrac{1}{2}(dd_I + d_J d_K)\mu = \tfrac{1}{2}(1 - J)dId\mu.$$

Superconformal Geometry II

Example

$M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n)$
$a = 2$, $b = -2$, $\alpha = -2$.

Superconformal Geometry II

Example

$M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n)$
$a = 2$, $b = -2$, $\alpha = -2$.

Poon and Swann (2003)

$a \neq 0$ corresponds to $Q = M/(\mathbb{R} \times SU(2)) = \mu^{-1}(1)/SU(2)$ a QKT orbifold (of special type).

Superconformal Geometry II

Example

$M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n)$
$a = 2$, $b = -2$, $\alpha = -2$.

Poon and Swann (2003)

$a \neq 0$ corresponds to $Q = M/(\mathbb{R} \times SU(2)) = \mu^{-1}(1)/SU(2)$ a QKT orbifold (of special type).

E.g. $Q = k\,\mathbb{C}\mathrm{P}(2)$.

Superconformal Geometry II

Example

$M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n)$
$a = 2$, $b = -2$, $\alpha = -2$.

Poon and Swann (2003)

$a \neq 0$ corresponds to $Q = M/(\mathbb{R} \times SU(2)) = \mu^{-1}(1)/SU(2)$ a QKT orbifold (of special type).

E.g. $Q = k\mathbb{CP}(2)$.

For S 3-Sasaki, $M = S \times \mathbb{R}$ warped product, is hyperKähler with special homothety $\alpha = -2$

Superconformal Geometry II

Example

$M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n)$
$a = 2$, $b = -2$, $\alpha = -2$.

Poon and Swann (2003)

$a \neq 0$ corresponds to $Q = M/(\mathbb{R} \times SU(2)) = \mu^{-1}(1)/SU(2)$ a QKT orbifold (of special type).

E.g. $Q = k\,\mathbb{CP}(2)$.

For S 3-Sasaki, $M = S \times \mathbb{R}$ warped product, is hyperKähler with special homothety $\alpha = -2$

Get to $a = 0$, special isometry, by potential change

$$g_1 = \frac{1}{\mu} g - \frac{1}{2\mu^2} (d^{\mathbb{H}} \mu)^2$$

Superconformal Geometry II

Example

$M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n)$
$a = 2$, $b = -2$, $\alpha = -2$.

Poon and Swann (2003)

$a \neq 0$ corresponds to $Q = M/(\mathbb{R} \times SU(2)) = \mu^{-1}(1)/SU(2)$ a QKT orbifold (of special type).

E.g. $Q = k\mathbb{CP}(2)$.

For S 3-Sasaki, $M = S \times \mathbb{R}$ warped product, is hyperKähler with special homothety $\alpha = -2$

Get to $a = 0$, special isometry, by potential change

$$g_1 = \frac{1}{\mu} g - \frac{1}{2\mu^2} (d^{\mathbb{H}} \mu)^2$$

Discrete quotient

$$M = (\mu^{-1}(1) \times \mathbb{R}) / \mathbb{Z}(\varphi, 2)$$

with g_1 is HKT with special isometry X

Superconformal Geometry II

Example

$M = \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}(n)$
$a = 2$, $b = -2$, $\alpha = -2$.

Poon and Swann (2003)

$a \neq 0$ corresponds to $Q = M/(\mathbb{R} \times SU(2)) = \mu^{-1}(1)/SU(2)$ a QKT orbifold (of special type).

E.g. $Q = k\,\mathbb{C}P(2)$.

For S 3-Sasaki, $M = S \times \mathbb{R}$ warped product, is hyperKähler with special homothety $\alpha = -2$

Get to $a = 0$, special isometry, by potential change

$$g_1 = \frac{1}{\mu} g - \frac{1}{2\mu^2}(d^{\mathbb{H}}\mu)^2$$

Discrete quotient

$$M = (\mu^{-1}(1) \times \mathbb{R})/\mathbb{Z}(\varphi, 2)$$

with g_1 is HKT with special isometry X

In this case

- $dX^\flat = 0$
- $b_1(M) \geqslant 1$

Outline

Twisting HKT

Twist by

$$g_W \sim_{\mathcal{H}} g, \quad \omega_I^W \sim_{\mathcal{H}} \omega_I, \text{etc.}$$

Twisting HKT

Twist by

$$g_W \sim_{\mathcal{H}} g, \quad \omega_I^W \sim_{\mathcal{H}} \omega_I, \text{etc.}$$

Then

$$I d\omega_I^W \sim_{\mathcal{H}} I d\omega_I + \tfrac{1}{a} X^\flat \wedge I\omega_\theta$$

For HKT need

$$c = -I d\omega_I = -J d\omega_J = -K d\omega_K$$

Twisting HKT

Twist by

$$g_W \sim_{\mathcal{H}} g, \quad \omega_I^W \sim_{\mathcal{H}} \omega_I, \text{etc.}$$

Then

$$I d\omega_I^W \sim_{\mathcal{H}} I d\omega_I + \tfrac{1}{a} X^\flat \wedge I\omega_\theta$$

For HKT need

$$c = -I d\omega_I = -J d\omega_J = -K d\omega_K$$

Proposition

HKT twists to HKT via a circle if and only if $F_\theta \in S^2 E = \bigcap_I \Lambda_I^{1,1}$, i.e., an instanton

Twisting HKT

Twist by

$$g_W \sim_{\mathcal{H}} g, \quad \omega_I^W \sim_{\mathcal{H}} \omega_I, \text{etc.}$$

Then

$$I d\omega_I^W \sim_{\mathcal{H}} I d\omega_I + \tfrac{1}{a} X^\flat \wedge I\omega_\theta$$

For HKT need

$$c = -I d\omega_I = -J d\omega_J = -K d\omega_K$$

Proposition

HKT twists to HKT via a circle if and only if $F_\theta \in S^2E = \bigcap_I \Lambda_I^{1,1}$, i.e., an instanton

X a special isometry, $X \lrcorner F_\theta = 0$ twists to X_W a special isometry

Twisting HKT

Twist by

$$g_W \sim_{\mathcal{H}} g, \quad \omega_I^W \sim_{\mathcal{H}} \omega_I, \text{etc.}$$

Then

$$I d\omega_I^W \sim_{\mathcal{H}} I d\omega_I + \tfrac{1}{a} X^\flat \wedge I\omega_\theta$$

For HKT need

$$c = -I d\omega_I = -J d\omega_J = -K d\omega_K$$

Proposition

HKT twists to HKT via a circle if and only if $F_\theta \in S^2E = \bigcap_I \Lambda_I^{1,1}$, i.e., an instanton

X a special isometry, $X \lrcorner F_\theta = 0$ twists to X_W a special isometry

Theorem

M HKT with special isometry ($\alpha = -1$). Can

- *untwist locally to $dX^\flat = 0$ on $S \times S^1$*
- *change potential on $S \times \mathbb{R}$ to $a \neq 0$, ($\alpha = -2$)*

Twisting HKT

Twist by

$$g_W \sim_{\mathcal{H}} g, \quad \omega_I^W \sim_{\mathcal{H}} \omega_I, \text{etc.}$$

Then

$$I d\omega_I^W \sim_{\mathcal{H}} I d\omega_I + \tfrac{1}{a} X^\flat \wedge I\omega_\theta$$

For HKT need

$$c = -I d\omega_I = -J d\omega_J = -K d\omega_K$$

Proposition

HKT twists to HKT via a circle if and only if $F_\theta \in S^2E = \bigcap_I \Lambda_I^{1,1}$, i.e., an instanton

X a special isometry, $X \lrcorner F_\theta = 0$ twists to X_W a special isometry

Theorem

M HKT with special isometry ($\alpha = -1$). Can

- *untwist locally to $dX^\flat = 0$ on $S \times S^1$*
- *change potential on $S \times \mathbb{R}$ to $a \neq 0$, ($\alpha = -2$)*

$F_\theta = dX^\flat$ *is* an instanton

Twisting HKT

Twist by

$$g_W \sim_{\mathcal{H}} g, \quad \omega_I^W \sim_{\mathcal{H}} \omega_I, \text{etc.}$$

Then

$$I d\omega_I^W \sim_{\mathcal{H}} I d\omega_I + \tfrac{1}{a} X^\flat \wedge I\omega_\theta$$

For HKT need

$$c = -I d\omega_I = -J d\omega_J = -K d\omega_K$$

Proposition

HKT twists to HKT via a circle if and only if $F_\theta \in S^2E = \bigcap_I \Lambda_I^{1,1}$, i.e., an instanton

X a special isometry, $X \lrcorner F_\theta = 0$ twists to X_W a special isometry

Theorem

M HKT with special isometry ($\alpha = -1$). Can

- *untwist locally to $dX^\flat = 0$ on $S \times S^1$*
- *change potential on $S \times \mathbb{R}$ to $a \neq 0$, ($\alpha = -2$)*

$F_\theta = dX^\flat$ *is* an instanton

Many simply-connected examples when $b_2(S) \geqslant 1$
E.g., $Q = k\,\mathbb{C}P(2)$

Outline

Summary

- $D(2,1;\alpha)$ superconformal symmetry realised by HKT with $\mathbb{R} \times SU(2)$ action

Summary

- $D(2,1;\alpha)$ superconformal symmetry realised by HKT with $\mathbb{R} \times SU(2)$ action
- $\alpha \neq -1$ comes from $\mathbb{R} \times SO(3)$ bundles over certain QKT orbifolds

Summary

- $D(2,1;\alpha)$ superconformal symmetry realised by HKT with $\mathbb{R} \times SU(2)$ action
- $\alpha \neq -1$ comes from $\mathbb{R} \times SO(3)$ bundles over certain QKT orbifolds
- $\alpha = -1$ comes from previous examples via change of potential and twist

Summary

- $D(2,1;\alpha)$ superconformal symmetry realised by HKT with $\mathbb{R} \times SU(2)$ action
- $\alpha \neq -1$ comes from $\mathbb{R} \times SO(3)$ bundles over certain QKT orbifolds
- $\alpha = -1$ comes from previous examples via change of potential and twist
- construct non-homogeneous compact simply-connected examples with $\alpha = -1$

General HKT with Torus Symmetry

- $M = N_1 \times N_2$
- N_2 with an HKT torus symmetry X_p
- $[F_q^\theta] \in H^2(N_1, \mathbb{Z})$, $F_q^\theta \in S^2E$ instanton

General HKT with Torus Symmetry

- $M = N_1 \times N_2$
- N_2 with an HKT torus symmetry X_p
- $[F_q^\theta] \in H^2(N_1, \mathbb{Z})$, $F_q^\theta \in S^2E$ instanton

Twists to $N_2 \to W \to N_1$ HKT with torus symmetry

General HKT with Torus Symmetry

- $M = N_1 \times N_2$
- N_2 with an HKT torus symmetry X_p
- $[F_q^\theta] \in H^2(N_1, \mathbb{Z})$, $F_q^\theta \in S^2E$ instanton

Twists to $N_2 \to W \to N_1$ HKT with torus symmetry

Example

N_1 a K3 surface
$N_2 = G$ compact Lie dim $= 4k$ or $N_2 = S \times S^1$, S 3-Sasaki
F_θ self-dual, primitive

Generate:

- large number of simply-connected examples, including new examples with reduced holonomy;
- all examples on compact nilmanifolds, N_i tori.

Hypercomplex vs. HKT

Theorem (Swann (2008))

There is a simply-connected T^4-bundle M over a K3 surface N that admits integrable I, J and K, but no compatible HKT metric.

This is constructed as a twist of $T^4 \times N$ using F_q^θ not of instanton type, but chosen so that integrability of the complex structures is preserved.

References I

M. L. Barberis and A. Fino. New strong HKT manifolds arising from quaternionic representations, May 2008. eprint `arXiv:0805.2335[math.DG]`.

E. Bergshoeff, C. Hull, and T. Ortín. Duality in the type-II superstring effective action. *Nuclear Phys. B*, 451(3):547–575, 1995. ISSN 0550-3213.

C. P. Boyer. A note on hyperHermitian four-manifolds. *Proc. Amer. Math. Soc.*, 102:157–164, 1988.

C. G. Callan, Jr., J. A. Harvey, and A. Strominger. Worldsheet approach to heterotic instantons and solitons. *Nuclear Phys. B*, 359(2-3):611–634, 1991. ISSN 0550-3213.

References II

R. A. Coles and G. Papadopoulos. The geometry of the one-dimensional supersymmetric nonlinear sigma models. *Classical Quantum Gravity*, 7(3):427–438, 1990. ISSN 0264-9381.

P. Gauduchon. Structures de Weyl et théorèmes d'annulation sur une varété conforme autoduale. *Ann. Sc. Norm. Sup. Pisa*, 18:563–629, 1991.

G. W. Gibbons, G. Papadopoulos, and K. S. Stelle. HKT and OKT geometries on soliton black hole moduli spaces. *Nuclear Phys. B*, 508(3):623–658, 1997. ISSN 0550-3213.

Jai Grover, Jan B. Gutowski, Carlos A. R. Herdeiro, and Wafic Sabra. HKT geometry and de Sitter supergravity. *Nuclear Phys. B*, 809(3):406–425, 2009. ISSN 0550-3213. doi: 10.1016/j.nuclphysb.2008.08.024. URL `http://dx.doi.org/10.1016/j.nuclphysb.2008.08.024`.

References III

C. M. Hull. The geometry of supersymmetric quantum mechanics, October 1999. eprint `arXiv:hep-th/9910028`.

F. Martín Cabrera and A. F. Swann. The intrinsic torsion of almost quaternion-hermitian manifolds. *Ann. Inst. Fourier*, 58(5):1455–1497, 2008. ISSN 0373-0956.

J. Michelson and A. Strominger. The geometry of (super) conformal quantum mechanics. *Comm. Math. Phys.*, 213(1): 1–17, 2000. ISSN 0010-3616.

G. Papadopoulos. Conformal and superconformal mechanics. *Classical Quantum Gravity*, 17(18):3715–3741, 2000. ISSN 0264-9381.

Y. S. Poon and A. F. Swann. Superconformal symmetry and hyperKähler manifolds with torsion. *Commun. Math. Phys.*, 241(1):177–189, 2003.

References IV

A. Strominger. Superstrings with torsion. *Nuclear Phys. B*, 274 (2):253–284, 1986. ISSN 0550-3213.

A. F. Swann. Twisting Hermitian and hypercomplex geometries. in preparation, 2008.

M. Verbitsky. Hyperkähler manifolds with torsion obtained from hyperholomorphic bundles, March 2003. eprint `arXiv:math.DG/0303129`.