Moment Maps and Toric Special Holonomy

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Outline

The Delzant picture

Compact symplectic toric manifolds	$\longleftrightarrow$	Delzant polytopes

(M^{2n}, ω) symplectic with a *Hamiltonian* action of $G = T^n$:
have G-invariant $\mu\colon M \longrightarrow \mathfrak{g}^* \cong \mathbb{R}^n$ with

$$d\langle \mu, X\rangle = X \lrcorner \omega \qquad \forall X \in \mathfrak{g}$$

the *moment map*

- μ invariant $\iff$ ω pulls-back to 0 on each T^n-orbit
- $b_1(M) = 0 \implies$ each symplectic T^n-action is Hamiltonian
- $\dim(M/T^n)$ equals dimension of target space of μ
- Stabiliser of any point is a (connected) subtorus of dimension $n - \operatorname{rank} d\mu$

The Delzant polytope

$$\Delta = \{a \in \mathbb{R}^n \mid \langle a, u_k\rangle \leqslant \lambda_k,\ k = 1, \dots, m\} = \mu(M)$$

with faces $F_k = \{\langle \mu, u_k\rangle = \lambda_k\}$ satisfying

smoothness $F_{k_1} \cap \cdots \cap F_{k_r} \neq \varnothing \implies u_{k_1}, \dots, u_{k_r}$ are part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n \subset \mathbb{R}^n$

compactness the intersection of any n faces is a point

M is constructed as the symplectic quotient of $\mathbb{C}^m$ by the Abelian group N given by

$$0 \longrightarrow N \longrightarrow T^m \longrightarrow T^n \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{n} \longrightarrow \mathbb{R}^m \xrightarrow{\beta} \mathbb{R}^n \longrightarrow 0$$

$$\beta(e_k) = u_k$$

at level $\lambda = (\lambda_k)$

HyperKähler manifolds

$(M, g, \omega_I, \omega_J, \omega_K)$ is *hyperKähler* if each $(g, \omega_A = g(A\,\cdot\,,\,\cdot\,))$ is Kähler and $IJ = K = -JI$

Then $\dim M = 4n$ and g is Ricci-flat, holonomy in $Sp(n) \leqslant SU(2n)$

Ricci-flatness implies:

> *if M is compact, then any Killing vector field is parallel so the holonomy of M reduces*

So take (M, g) non-compact and complete instead

Swann (2016) and Dancer and Swann (2016), following Bielawski (1999), Bielawski and Dancer (2000) and Goto (1994)

Hypertoric is complete hyperKähler M^{4n} with tri-Hamiltonian $G = T^n$ action: have G-invariant map (*hyperKähler moment map*)

$$\mu = (\mu_I, \mu_J, \mu_K)\colon M \to \mathbb{R}^3 \otimes \mathfrak{g}^* \qquad d\langle \mu_A, X\rangle = X \lrcorner \omega_A$$

- $\dim(M/T^n)$ equals dimension $3n$ of target space of μ
- Stabiliser of any point is a (connected) subtorus of dimension $n - \frac{1}{3}\operatorname{rank} d\mu$

Locally (Lindström and Roček 1983)

$$g = (V^{-1})_{ij}\theta_i\theta_j + V_{ij}(d\mu_I^i d\mu_I^j + d\mu_J^i d\mu_J^j + d\mu_K^i d\mu_K^j),$$

with V_{ij} positive-definite and harmonic on every affine three-plane $X_{a,v} = a + \mathbb{R}^3 \otimes v$

Hypertoric configuration data

For M^{4n} hypertoric, μ is *surjective*:

$$\mu(M^{4n}) = \mathbb{R}^{3n} = \mathbb{R}^3 \otimes \mathbb{R}^n = \operatorname{Im}\mathbb{H} \otimes \mathbb{R}^n$$

All stabilisers are (connected) subtori and μ induces a homeomorphism $M/T^n \longrightarrow \mathbb{R}^{3n}$

Polytope faces are replaced by affine flats of codimension 3:

$$H_k = H(u_k, \lambda_k) = \{a \in \operatorname{Im}\mathbb{H} \otimes \mathbb{R}^n \mid \langle a, u_k \rangle = \lambda_k\},$$

$u_k \in \mathbb{Z}^n$, $\lambda_k \in \operatorname{Im}\mathbb{H}$

smoothness $H(u_{k_1}, \lambda_{k_1}) \cap \cdots \cap H(u_{k_r}, \lambda_{k_r}) \neq \emptyset \implies u_{k_1}, \dots, u_{k_r}$ is part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$

Get only finitely many distinct vectors u_k, but possibly infinitely many λ_k's

Construction and classification

Choose $\mathbb{L}$ finite or countably infinite and $\Lambda = (\Lambda_k)_{k\in\mathbb{L}} \in \mathbb{H}^{\mathbb{L}}$, so that with $\lambda_k = -\frac{1}{2}\overline{\Lambda_k}\mathbf{i}\Lambda_k$ have $\sum_{k\in\mathbb{L}}(1+|\lambda_k|)^{-1} < \infty$

Let $\mathbb{L}^2(\mathbb{H}) = \{v \in \mathbb{H}^{\mathbb{L}} \mid \sum_{k\in\mathbb{L}}|v_k|^2 < \infty\}$

Hilbert manifold $M_\Lambda = \Lambda + \mathbb{L}^2(\mathbb{H})$ is hyperKähler with action of Hilbert group $T_\lambda = \{g \in (S^1)^{\mathbb{L}} \mid \sum_{k\in\mathbb{L}}(1+|\lambda_k|)|1-g_k|^2 < \infty\}$

Suppose $u_k \in \mathbb{Z}^n$ are such that $\{H(u_k,\lambda_k) \mid k \in \mathbb{L}\}$ satisfy the smoothness condition and define the Hilbert group N_β by

$$\begin{array}{c} 0 \longrightarrow N_\beta \longrightarrow T_\lambda \longrightarrow T^n \longrightarrow 0 \\ 0 \longrightarrow \mathfrak{n}_\beta \longrightarrow \mathfrak{t}_\lambda \xrightarrow{\beta} \mathbb{R}^n \longrightarrow 0 \end{array} \qquad \beta(e_k) = u_k$$

Then the hyperKähler quotient of M_Λ by N_β is hypertoric and every simply-connected hypertoric manifold arises in this way up to adding a (positive semi-definite) constant matrix to (V_{ij})

G_2 manifolds

M^7 with $\varphi \in \Omega^3(M)$ pointwise of the form

$$\varphi = e_{123} - e_{145} - e_{167} - e_{246} - e_{257} - e_{347} - e_{356},$$

$e_{ijk} = e_i \wedge e_j \wedge e_k$
Specifies metric $g = e_1^2 + \cdots + e_7^2$, orientation $\mathrm{vol} = e_{1234567}$ and four-form

$$*\varphi = e_{4567} - e_{2345} - e_{2367} - e_{3146} - e_{3175} - e_{1256} - e_{1247}$$

via

$$6g(X,Y)\,\mathrm{vol} = (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$$

Holonomy of g is in G_2 when $d\varphi = 0 = d{*}\varphi$, a *parallel G_2-structure*
Then g is Ricci-flat

Multi-Hamiltonian actions

Joint work with Thomas Bruun Madsen

(M, α) manifold with closed $\alpha \in \Omega^p(M)$ preserved by $G = T^n$
This is *multi-Hamiltonian* if it there is a G-invariant $\nu\colon M \longrightarrow \Lambda^{p-1}\mathfrak{g}^*$ with

$$d\langle \nu, X_1 \wedge \cdots \wedge X_{p-1}\rangle = \alpha(X_1, \dots, X_{p-1}, \cdot\,)$$

for all $X_i \in \mathfrak{g}$

- take $n > p - 2$
- ν invariant $\iff$ α pulls-back to 0 on each T^n-orbit
- $b_1(M) = 0 \implies$ each T^n-action preserving α is multi-Hamiltonian

For (M, φ) a parallel G_2-structure, can take $\alpha = \varphi$ and/or $\alpha = *\varphi$

Multi-Hamiltonian parallel G_2-manifolds

Proposition

Suppose (M, φ) is a parallel G_2-manifold with T^n-symmetry multi-Hamiltonian for $\alpha = \varphi$ and/or $\alpha = *\varphi$. Then $2 \leqslant n \leqslant 4$.

$n = 2, \alpha = \varphi$ $\nu\colon M \longrightarrow \Lambda^2\mathbb{R}^3 = \mathbb{R}^1$, $1 < 5 = \dim(M/T^n)$ (A)

$n = 3, \alpha = \varphi$ $\nu\colon M \longrightarrow \Lambda^2\mathbb{R}^3 = \mathbb{R}^3$, $3 < 4 = \dim(M/T^n)$

$n = 3, \alpha = *\varphi$ $\mu\colon M \longrightarrow \Lambda^3\mathbb{R}^3 = \mathbb{R}$, $1 < 4 = \dim(M/T^n)$

$n = 3, \alpha = \varphi$ and $\alpha = *\varphi$ $(\nu, \mu)\colon M \longrightarrow \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$,
$4 = 4 = \dim(M/T^n)$ (T)

$n = 4, \alpha = \varphi$ $\nu\colon M \longrightarrow \Lambda^2\mathbb{R}^4 = \mathbb{R}^6$, $6 > 3 = \dim(M/T^n)$ (B)

$n = 4, \alpha = *\varphi$ $\nu\colon M \longrightarrow \Lambda^3\mathbb{R}^4 = \mathbb{R}^4$, $4 > 3 = \dim(M/T^n)$

(A) Madsen and Swann (2012)
(B) Baraglia (2010)

Toric G_2

Definition

A *toric G_2 manifold* is a parallel G_2-structure (M, φ) with an action of T^3 multi-Hamiltonian for both φ and $*\varphi$

Let U_1, U_2, U_3 generate the T^3-action, then $\varphi(U_1, U_2, U_3) = 0$, multi-moment maps $(\nu, \mu) = (\nu_1, \nu_2, \nu_3, \mu)\colon M \longrightarrow \mathbb{R}^4$

$$d\nu_i = U_j \wedge U_k \lrcorner \varphi \quad (i\,j\,k) = (1\,2\,3) \qquad d\mu = U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi$$

Example

$M = S^1 \times \mathbb{C}^3$ has standard flat $\varphi = \frac{i}{2} dx(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}}) + \mathrm{Re}(dz_{123})$ preserved by $S^1 \times T^2 \leqslant S^1 \times SU(3)$

$$4(\nu_1 - i\mu) = z_1 z_2 z_3, \quad 4\nu_2 = |z_2|^2 - |z_3|^2, \quad 4\nu_3 = |z_3|^2 - |z_1|^2$$

Proposition

All isotropy groups of the T^3 action are connected and act on the tangent space as maximal tori in $1 \times SU(3)$, $1_3 \times SU(2)$ or 1_7

At a point p in a principal orbit U_1, U_2, U_3 are contained in a coassociative subspace of T_pM and $(d\nu, d\mu)$ has full rank 4
Let M_0 be the points with trivial isotropy
Then (ν, μ) induces a local diffeomorphism $M_0/T^3 \to \mathbb{R}^4$

Proposition

For the flat structure on $M = S^1 \times \mathbb{C}^3$ the quotient M/T^3 is homeomorphic to $\mathbb{R}^4$ and (ν, μ) induces a homeomorphism

Corollary

For any toric G_2-manifold M, the quotient M/T^3 is a topological manifold

Local form

(M, φ) toric G_2 with generating vector fields U_i
$M_0 \to M_0/T^3$ is a principal torus bundle with connection one-forms $\theta_i \in \Omega^1(M_0)$ satisfying $\theta_i(U_j) = \delta_{ij}, \quad \theta_i(X) = 0 \ \forall\, X \perp U_1, U_2, U_3$
On M_0, put

$$B = \big(g(U_i, U_j)\big) \quad \text{and} \quad V = B^{-1} = \frac{1}{\det B} \operatorname{adj} B$$

Theorem

On M_0

$$g = \frac{1}{\det V} \theta^t \operatorname{adj}(V)\, \theta + dv^t \operatorname{adj}(V)\, dv + \det(V)\, d\mu^2$$

$$\varphi = -\det(V)\, dv_{123} + d\mu dv^t \operatorname{adj}(V)\, \theta + \mathfrak{S}_{i,j,k} \theta_{ij} dv_k$$

$$*\varphi = \theta_{123} d\mu + \frac{1}{2\det(V)} \big(dv^t \operatorname{adj}(V)\, \theta\big)^2 + \det(V)\, d\mu \mathfrak{S}_{i,j,k} \theta_i dv_{jk}$$

Theorem (continued)

Such a $(g, \varphi, *\varphi)$ defines a parallel G_2-structure if and only if $V \in C^\infty(M_0/T^3, S^2\mathbb{R}^3)$ is a positive-definite solution to

$$\sum_{i=1}^{3} \frac{\partial V_{ij}}{\partial v_i} = 0 \qquad j = 1, 2, 3 \qquad \text{(divergence-free)}$$

and

$$L(V) + Q(dV) = 0 \qquad \text{(elliptic)}$$

where

$$L = \frac{\partial^2}{\partial \mu^2} + \sum_{k,\ell} V_{ij} \frac{\partial^2}{\partial v_i \partial v_j}$$

and Q is a quadratic form with constant coefficients

Proposition

Solutions V to the divergence-free equation are given locally by $A \in C^\infty(M_0/T^3, S^2\mathbb{R}^3)$ via

$$V_{ii} = \frac{\partial^2 A_{jj}}{\partial {v_k}^2} + \frac{\partial^2 A_{kk}}{\partial {v_j}^2} - \frac{\partial^2 A_{jk}}{\partial v_j \partial v_k}$$

$$V_{ij} = \frac{\partial^2 A_{ik}}{\partial v_j \partial v_k} + \frac{\partial^2 A_{jk}}{\partial v_i \partial v_k} - \frac{\partial^2 A_{ij}}{\partial {v_k}^2} - \frac{\partial^2 A_{kk}}{\partial v_i \partial v_j}$$

$(i\,j\,k) = (1\,2\,3)$

Example solutions

Example

Bryant-Salamon metrics and their generalisations by Brandhuber et al. (2001) on $S^3 \times \mathbb{R}^4$: complete, cohomogeneity one with symmetry group $SU(2) \times SU(2) \times S^1 \times \mathbb{Z}/2$

Example

Diagonal $V = \mathrm{diag}(V_1, V_2, V_3)$. (divergence-free) $\partial V_i / \partial v_i = 0$.
Off-diagonal terms in (elliptic) give $\frac{\partial V_i}{\partial v_j} \frac{\partial V_j}{\partial v_i} = 0$.
Either $V = \mathrm{diag}(V_1(v_2, \mu), V_2(v_3, \mu), V_3(v_1, \mu))$ linear in each variable
Or have an elliptic hierachy $V_3 = V_3(\mu)$, $V_2 = V_2(v_3, \mu)$, $V_1 = V_1(v_2, v_3, \mu)$

$$\frac{\partial^2 V_3}{\partial \mu^2} = 0 \quad \frac{\partial^2 V_2}{\partial \mu^2} + V_3 \frac{\partial^2 V_2}{\partial {v_3}^2} = 0 \quad \frac{\partial^2 V_1}{\partial \mu^2} + V_2 \frac{\partial^2 V_1}{\partial {v_2}^2} + V_3 \frac{\partial^2 V_1}{\partial {v_3}^2} = 0$$

E.g. $V_3 = \mu, \quad V_2 = \mu^3 - 3v_3^2, \quad V_1 = 2\mu^5 - 15\mu^2 v_3^2 - 5v_2^2$

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