T is for Twist

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Outline

1. **Motivation**
   - HKT and String Duals
   - Geometry with Torsion

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   - Joyce's Twist
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   - Joyce's Twist
   - Grantcharov-Poon
3. **General Twists**
   - Lifting Actions
   - Transformation Rules
4. **Examples**
   - HKT
   - Strong KT

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   - HKT
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An Example of T-duality

HyperKähler M^4

$$ds^2 = V^{-1}(d\tau + \omega)^2 + V\gamma_{ij}dx^i dx^j$$
$$dV = *_3 d\omega$$

T duality

$\longleftrightarrow$

on $X = \frac{\partial}{\partial\tau}$

Strong HKT W^4

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For circle actions have:

$$R \leftrightarrow 1/R \qquad \text{and here} \quad W = (M/S^1) \times S^1$$

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Metric geometry with torsion

- metric g
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Have

$$\nabla = \nabla^{\mathrm{LC}} + \tfrac{1}{2} c$$

- Any $c \in \Omega^3(M)$ will do
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Definition

The geometry is *strong* if $dc = 0$

KT Geometry

Metric geometry

g, $\nabla = \nabla^{\mathrm{LC}} + \frac{1}{2}c$, $c \in \Lambda^3 T^* M$

KT geometry

additionally

- I integrable complex structure
- $g(IX, IY) = g(X, Y)$
- $\nabla I = 0$

Here $I\colon TM \to TM$ with

$$I^2 = -1 \qquad N_I = 0$$

where $N_I(X, Y) =$
$[IX, IY] - I[IX, Y] - I[X, IY] - [X, Y]$

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Given (g, I) the connection ∇ is *unique*: $c = -IdF_I$, where $F_I(X,Y) = g(IX, Y)$

- KT geometry is just Hermitian geometry together with the Bismut connection ∇
- $c = 0$ is Kähler geometry
- strong KT geometry is $\partial\bar{\partial}F_I = 0$
- Gauduchon 1991: every compact Hermitian M^4 is conformal to strong KT

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- most commonly encountered hypercomplex structures (M, I, J, K) admit an HKT metric — but not all.
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Example

$G = SU(3) = M^8$, bi-invariant g is strong HKT

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Proof.

P pulls-back to a holomorphic bundle on the twistor space Z on M whose quotient is the twistor space of W. □

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Proof.

Let $\mathscr{H} = \ker\theta$ be the horizontal distribution in P.
Lift g, I, J and K to $\mathscr{H}$ and then push these forward to W.
Check the HKT conditions. □

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- $P \xrightarrow{\pi} M$ a principal S^1-bundle, generator Y
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Let $\tilde{X}$ be the horizontal lift of X. Then

$$X' = \tilde{X} + aY$$

with $da = -X^\theta$. □

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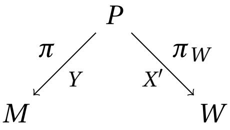
$$W := P/\langle X' \rangle$$



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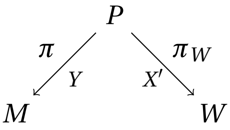

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- circle action generated by $X_W = (\pi_W)_* Y$
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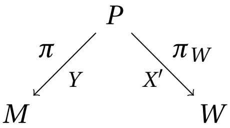

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Dually

M is a twist of W with respect to X_W

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Transformation Rules

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Tensors α on α_W on M and W are said to be $\mathscr{H}$*-related* if their pull-backs agree on $\mathscr{H} = \ker\theta$

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- For metrics

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Lemma

$d\alpha_W$ is $\mathscr{H}$-related to a form on M if and only if $L_X \alpha = 0$. Then $d\alpha_W \sim_{\mathscr{H}} d\alpha - F_\theta \wedge \frac{1}{a} X \lrcorner \alpha$.

Almost Hermitian Twist

Definition

Let (M, g, F_I) be an almost Hermitian structure invariant under X. This has *twist* (W, g_W, F_I^W) where

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Proposition

- *If I is integrable then I_W is integrable if and only if $F_\theta \in \Lambda^{1,1}$*
- *the forms $c = -IdF_I$ are related by*

$$c_W \sim_{\mathcal{H}} c - \frac{1}{a} X^\flat \wedge IF_\theta$$

Transformation Rules II

Corollary

If (M, g, I, J, K) *is hyperHermitian (resp. HKT) then* (W, g_W, I_W, J_W, K_W) *is hyperHermitian (resp. HKT) if and only if*

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Corollary

For M KT (resp. HKT) and F_θ an instanton, W is strong if and only if

$$dc = \frac{1}{a}(dX^\flat + X \lrcorner c - \frac{1}{a}\|X\|^2 F_\theta) \wedge F_\theta$$

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From a HyperKähler Metric

- $g = \frac{1}{V}\varphi^2 + Vh$
  hyperKähler, $c = 0$
- hyperKähler isometry X
  - $\varphi(X) = 1, \quad L_X\varphi = 0$
  - $X^\flat = V^{-1}\varphi, \quad V = \|X\|^{-2}$
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This is a twist via a trivial bundle with non-flat connection.

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$$g = \frac{1}{\rho} g_{\mathscr{U}} - \frac{1}{2\rho^2}(d^{\mathbb{H}}\rho)^2$$

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Twist of $\mathcal{U}(\mathbb{C}P(2))/\mathbb{Z}$ is strong HKT structure on $SU(3)$.

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Nilmanifold Examples

Theorem (Fino, Parton, and Salamon, 2004)

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Mejldal, 2004

The 8-dimensional nilmanifolds with $\mathfrak{g} = (0^6, 13-24+56, 12-2.23+3.34)$ are irreducible and lie in a 15-dimensional family of invariant strong KT structures.

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Remark

All known strong KT structures on nilmanifolds may be obtained via iterations of the above twist constructions starting from a flat torus.

Non-toral Base

- Twisting $M^6 = N^4 \times T^2$
- integrability condition $(F_1 + iF_2)^{0,2} = 0$
- if not instantons then $(F_1 + iF_2)^{0,2}$ is a global holomorphic form on N^4

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Let $\omega_I, \omega_J, \omega_K$ be the Kähler forms on N^4. The integrability condition gives

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with $\alpha \in \Lambda_I^{1,1}$ orthogonal to ω_I. The strong condition is

$$\alpha \wedge \bar{\alpha} = 4(|\lambda_1|^2 - 2|\lambda_2|^2)\,\mathrm{vol}_g$$

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Theorem

For linearly independent primitive F_i satisfying the conditions to the left, twist W^6 of $M^6 = N^4 \times T^2$ is a compact simply-connected strong KT manifold.

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- twisting by instantons preserves KT and HKT geometries
- strong structures may be obtained
- non-instanton twists are also necessary

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Exterior derivative of the torsion form

$$
\begin{aligned}
dc_W \sim_{\mathscr{H}} dc &- \tfrac{1}{a} dX^\flat \wedge IF_\theta + \tfrac{1}{a} X^\flat \wedge d(IF_\theta) \\
&- F_\theta \wedge \tfrac{1}{a} X \lrcorner\, c + F_\theta \wedge \tfrac{1}{a^2} \|X\|^2 IF_\theta - F_\theta \wedge \tfrac{1}{a} X^\flat \wedge X \lrcorner\, IF_\theta
\end{aligned}
$$