Geometric T-duality and the c-map

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Joint work with Óscar Maciá

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Swann, A. F. (2016), 'Twists versus modifications', *Adv. Math.* 303: 611–37
Swann, A. F. (2010), 'Twisting Hermitian and hypercomplex geometries', *Duke Math. J.* 155 (2): 403–31

Background

c-map introduced by Cecotti et al. (1989), further explicit local expressions in Ferrara and Sabharwal (1990).

Two forms

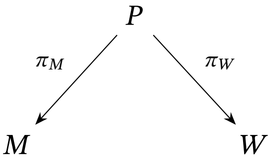
1. *rigid*: special Kähler manifold $\dim = 2n$ gives hyperKähler manifold $\dim 4n$
2. *local*: projective special Kähler manifold $\dim 2n$ gives quaternionic Kähler $\dim 4n+4$

de Wit and Van Proeyen (1992) used this to show the classification of Alekseevskiĭ (1975) completely solvable quaternionic Kähler Lie groups was not complete; fixed subsequently by Cortés (1996).

Freed (1999): mathematical descriptions of special Kähler manifolds, projective special Kähler manifolds and the rigid c-map, explaining ideas of Donagi and Witten (1995) for algebraic integrable systems.

Today: explain different ingredients of the c-maps, via moment maps and the twist

The twist construction



- $P \to M$ a principal S^1-bundle,
  symmetry Y, connection 1-form θ, curvature $\pi_M^* F = d\theta$
- S^1-action on M preserving F,
  symmetry X, horizontal lift $\tilde{X} \in \mathcal{H} = \ker \theta$
- $X' = \tilde{X} + aY$ on P preserving θ and Y
  $da = -X \lrcorner F$
- $W = P/X'$, has action induced by Y

GEOMETRIC DUALITY

$$\begin{array}{ccc} & P, \mathcal{H} & \\ \swarrow & & \searrow \\ M & & W \end{array}$$

$$X \lrcorner F = -da$$

α an invariant tensor on M
is *$\mathcal{H}$-related* to α_W on W
if $\pi_M^* \alpha$ and $\pi_W^* \alpha_W$ agree on $\mathcal{H}$

Exterior derivatives are related by

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - \tfrac{1}{a} F \wedge (X \lrcorner \alpha)$$

Swann (2010)

Reproduces formulas derived from T-duality in Gibbons et al. (1997)

HyperKähler quotients

$(M, g, \omega_I, \omega_J, \omega_K)$ is *hyperKähler* if ω_A are symplectic forms and g is a pseudo-Riemannian metric such that $I = g^{-1}\omega_I$, etc., satisfy $I^2 = -\mathrm{Id} = J^2 = K^2$ and $IJ = K = -JI$.

g is then Ricci-flat and I, J, K integrable.

Given a symmetry X preserving each of $g, \omega_I, \omega_J, \omega_K$, a *hyperKähler moment map* is an invariant function

$$\mu = (\mu_I, \mu_J, \mu_K)\colon M \to \mathbb{R}^3$$

such that $d\mu = (X \lrcorner\, \omega_I, X \lrcorner\, \omega_J, X \lrcorner\, \omega_K)$.

Hitchin et al. (1987): the corresponding quotient $M/\!/\!/X = \mu^{-1}(c)/X$ is hyperKähler if c is a regular value.

Example $M = \mathbb{R}^4 = \mathbb{H}$ flat, X generating $q \mapsto e^{\mathbf{i}t}q$,
$\mu(q) = \frac{1}{2}\overline{q}^T\mathbf{i}q = \big(\frac{1}{2}(|z|^2 - |w|^2), \mathrm{Re}(zw), \mathrm{Im}(zw)\big)$.

HyperKähler modifications

M hyperKähler with tri-Hamiltonian circle action. A *basic modification* (Dancer and Swann 2006) of M is the hyperKähler manifold

$$M_{\mathrm{mod}} = (M \times \mathbb{H})///\text{diagonal action.}$$

Example Each Gibbons-Hawking metric in dimension 4, is obtained from flat $\mathbb{R}^4$ or flat $S^1 \times \mathbb{R}^3$ by successive hyperKähler modifications.

Theorem (Swann 2016)

g_{mod} is the twist of

$$\widetilde{g} = g + \frac{1}{2\|\mu\|} g_{\mathbb{H}X}$$

where $g_{\mathbb{H}X} = (X^\flat)^2 + (IX^\flat)^2 + (JX^\flat)^2 + (KX^\flat)^2$.

General modifications are obtained by replacing $\mathbb{H}$ by an arbitrary hyperKähler four-manifold with tri-Hamiltonian symmetry.

Elementary deformations

An *elementary deformation* of g is

$$\widetilde{g} = f\,g + h\,g_{\mathbb{H}X}.$$

Theorem (Swann 2016)

If X is a tri-Hamiltonian symmetry, then (up to scale) the elementary deformation twists to a hyperKähler metric if and only if the twist is a general hyperKähler modification.

A *rotating* action is an isometry with

$$L_X\omega_I = 0, \quad L_X\omega_J = \omega_K, \quad L_X\omega_K = -\omega_J$$

Theorem (Maciá and Swann 2015)

If X is a rotating symmetry, then (up to scale) the twist of an elementary deformation is quaternionic Kähler if and only if $f = (\mu_I - c)^{-1}$ and $h = -(\mu_I - c)^{-2}$.

Converse construction

Theorem (Haydys 2008)

If X generates a circle action on quaternionic Kähler M, then for the tri-Hamiltonian lift X' to the hyperKähler cone $\mathcal{U}(M)$, we have $\mathcal{U}(M)///X'$ is hyperKähler with a rotating symmetry.

Special Kähler manifolds

Question Given (M, I), simplest way for T^*M to be hyperKähler?

M an n-manifold. Bundle $\pi\colon \mathrm{GL}(M) \to M$ of frames $u\colon \mathbb{R}^n \xrightarrow{\cong} T_aM$.
Canonical one-form $\theta_u(X) = u^{-1}(\pi_* X)$.

$$T^*M = \mathrm{GL}(M) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathbb{R}^n)^* \qquad v \circ u^{-1} \leftarrow\!\shortmid (u, v)$$

T^*M is symplectic, canonical $\omega_J = d(x\theta)$, where $x = \mathrm{Id}_{(\mathbb{R}^n)^*}$.

∇ any connection on M, connection one-form ω_∇.
Define $\alpha = dx - x\omega_\nabla$ on $\mathrm{GL}(M) \times (\mathbb{R}^n)^*$. Then

$$T(T^*M) = \mathcal{V} \oplus \mathcal{H} \qquad \text{with } \mathcal{H}_p = \ker\alpha \cong T_xM, \mathcal{V}_p \cong T_x^*M.$$

Lemma

$\omega_J = \alpha \wedge \theta$ *if and only if* ∇ *is torsion-free.*

Complex structures

(M, I) an almost complex, tangent spaces modelled on $(\mathbb{R}^n, \mathbf{i})$. Canonical complex-valued symplectic form

$$\omega_J + i\omega_K = d(x\theta) - id(x\mathbf{i}\theta).$$

Proposition

For ∇ torsion free, $\omega_K = -\alpha\mathbf{i} \wedge \theta$ if and only if I is integrable and ∇ is special:

$$(\nabla_X I)Y = (\nabla_Y I)X.$$

The rigid c-map

ω a Hermitian two-form on (M, I), giving metric of possibly indefinite signature, then

$$\pi^*\omega = -\theta^T \mathbf{s} \wedge \theta, \qquad \mathbf{s} = \operatorname{diag}(\pm\mathbf{i}_2, \dots, \pm\mathbf{i}_2).$$

Proposition

$$\omega_I = \frac{1}{2}\big(\alpha\mathbf{s} \wedge \alpha^T - \theta^T\mathbf{s} \wedge \theta\big), \quad \omega_J = \alpha \wedge \theta, \quad \omega_K = -\alpha \wedge \mathbf{i}\theta$$

is a hyperKähler triple if and only if (M, g, I, ∇) *is* special Kähler: (M, g, I) *Kähler with* ∇ *flat, symplectic, torsion-free and special.*

The *rigid c-map* is the construction of (T^*M, I, J, K) from special Kähler (M, g, I, ∇).

Extra symmetries

Special Kähler: (M, g, I, ∇) Kähler with ∇ flat, symplectic, torsion-free and special.
Flatness implies locally there is a horizontal section s of $\mathrm{Sp}(M)$.
So $s^*\omega_\nabla = 0$ and $s^*\theta$ satisfies $ds^*\theta = s^*(-\omega_\nabla \wedge \theta) = 0$. Thus there are local symplectic coordinates $y\colon M \to \mathbb{R}^n$, $dy = s^*\theta$.
Now

$$\omega_I = \frac{1}{2}(dx\mathbf{s} \wedge dx^T - dy^T\mathbf{s} \wedge dy), \quad \omega_J = dx \wedge dy, \quad \omega_K = -dx \wedge \mathbf{i}dy.$$

In particular, translations along the fibre, i.e. in x, are hyperKähler isometries.
There are $2n$ such translations giving a holomorphic completely integrable system.

Conic special Kähler manifolds

(M, g, I, ∇) special Kähler is *conic* if there is a non-null vector field Y with

$$\nabla Y = -I = \nabla^{\mathrm{LC}} Y.$$

Then Y is a holomorphic isometry, and IY is a homothety preserving I and ∇.

$\mu = g(Y, Y)/2$ is a Kähler moment map for Y.

$\mu^{-1}(c)/Y$ is a *projective special Kähler* manifold S.
(Intrinsically: Mantegazza 2021, group case Maciá and Swann 2019.)

The horizontal lift $\widetilde{Y}$ of Y to T^*M is a rotating symmetry of the hyperKähler structure. Using the twist we obtain a quaternionic Kähler manifold Q.

The *local c-map* is the passage from projective special Kähler S to quaternionic Kähler Q.

SIGNATURES AND BASIC EXAMPLES

S projective special Kähler, signature $(2n, 0)$.

$C \to S$ conic special Kähler, signature $(2n, 2)$.

$H = T^*C$ hyperKähler, signature $(4n, 4)$.

Q quaternionic Kähler twist of H, signature $(4n + 4, 0)$ or $(4n, 4)$.

Cortés et al. (2012): S complete implies Q complete.

EXAMPLE(S) $S = \mathbb{R}H(2)$ hyperbolic plane.
Homogeneous: affine group with Lie algebra $[A, B] = \lambda B$, orthonormal.
S is projective special Kähler only for $\lambda^2 = 4$ or $4/3$.
$\lambda^2 = 4$, the conic special Kähler manifold is flat, $Q = \mathrm{Gr}_2^+(\mathbb{C}^{2,2})$.
$\lambda^2 = 4/3$, get $Q = \mathrm{G}_2^*/\mathrm{SO}(4)$.

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