Torus actions and Ricci-flat metrics

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To Eldar Straume on his 70th birthday

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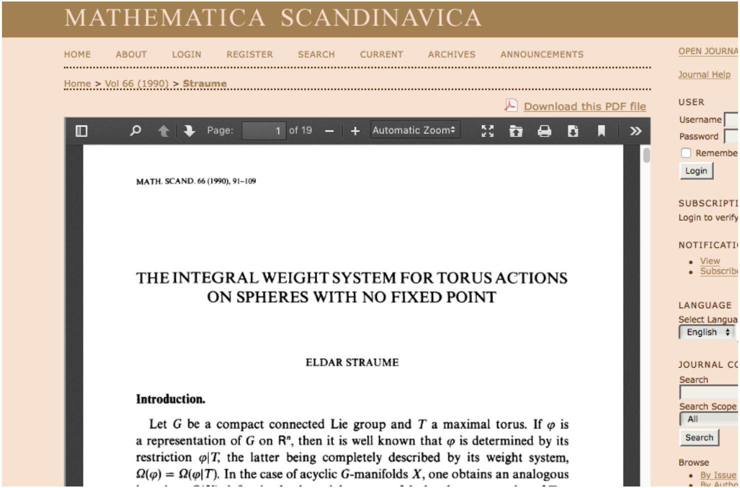

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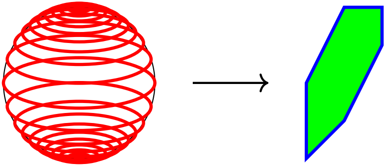
Outline

1. The Delzant picture
2. HyperKähler manifolds
   - Dimension four
   - Toric hyperKähler
3. G_2 manifolds

The Delzant picture

(M^{2n}, ω) symplectic with a *Hamiltonian* action of T^n
$\Longrightarrow$
The moment map $\mu\colon M \to \mathbb{R}^n \cong \mathfrak{t}$ identifies the orbit space M/T^n with a convex polytope in $\mathbb{R}^n$.



Each such (M^{2n}, ω) may be constructed as a symplectic quotient of $\mathbb{R}^{2m}$ by an Abelian subgroup of T^m.

SYMPLECTIC MOMENT MAPS

Let (M, ω) be symplectic: $d\omega = 0$.
If $X \in \mathfrak{X}(M)$ preserves ω, then Cartan's formula gives

$$0 = L_X\omega = d(X \lrcorner \omega),$$

so (locally) $X \lrcorner \omega = d\mu^X$ for some $\mu^X \colon M \to \mathbb{R}$.
For G Abelian acting on M preserving ω, the action is *Hamiltonian* if there is a G-invariant map

$$\mu \colon M \to \mathfrak{g}^*$$

such that $d\langle \mu, X\rangle = X \lrcorner \omega$ for each $X \in \mathfrak{g}$.
M simply-connected and G connected Abelian $\implies$ Hamiltonian $\iff$ all orbits are *isotropic*

$$\omega(X, Y) = 0 \qquad \text{for all } X, Y \in \mathfrak{g}.$$

The Delzant polytope

Faces of $\Delta = \mu(M)$ are of the form $F_k = \{\langle \mu, u_k \rangle = \lambda_k\}$ and the Delzant polytope is

$$\Delta = \{a \in \mathbb{R}^n \mid \langle a, u_k \rangle \leqslant \lambda_k \ \forall k\}.$$

$u_k \in \mathbb{R}^n$ with points over $(F_k)^\circ$ having stabiliser the subtorus with Lie algebra $\{v \in \mathbb{R}^n = \mathfrak{t} \mid \langle v, u_k \rangle = 0\}$, so $u_k \in \mathbb{Q}^n$.

Smoothness of M is equivalent to:
$F_{k_1} \cap \cdots \cap F_{k_r} \neq \varnothing \implies$ the corresponding $u_{k_1}, \ldots, u_{k_r}$ are part of a $\mathbb{Z}$-basis.

This restricts the possible u_i's *locally*.

HyperKähler manifolds

$(M, \omega_I, \omega_J, \omega_K)$ is *hyperKähler* if:

1. each ω_A is a symplectic two-form,
2. the tangent bundle endomorphisms $I = \omega_K^{-1}\omega_J$, etc., satisfy
   - $I^2 = -1 = J^2 = K^2$, $IJ = K = -JI$, etc., and
   - $g = -\omega_A(A\,\cdot\,, \cdot\,)$ is independent of A and positive definite.

Consequences

- $\dim M = 4n$,
- g is Ricci-flat, with holonomy contained in $Sp(n) \leqslant SU(2n)$.

Symmetry considerations

Ricci-flatness implies:

- if M is compact, then any Killing vector field is parallel, so the holonomy of M reduces and M splits as a product,
- if M is homogeneous then g is flat, so M is a quotient of flat $\mathbb{R}^{4n}$ by a discrete group (Alekseevskiĭ and Kimel'fel'd 1975).

Take (M, g) complete and G Abelian group of tri-holomorphic isometries.
Assume the action is *tri-Hamiltonian*, so there is a *hyperKähler moment map*: a G-invariant map

$$\mu = (\mu_I, \mu_J, \mu_K) \colon M \to \mathbb{R}^3 \otimes \mathfrak{g}^*$$

with $d\langle \mu_A, X \rangle = X \lrcorner \omega_A$.
This forces $4 \dim G \leqslant \dim M$.

Gibbons-Hawking Ansatz in 4d

X a tri-Hamiltonian vector field on hyperKähler M^4
Away from M^X, locally

$$g = \frac{1}{V}(dt + \omega)^2 + V(dx^2 + dy^2 + dz^2)$$

where $V = 1/g(X, X)$, $dx = X \lrcorner \omega_I = d\mu_I$, etc., and

$$d\omega = -*_3 dV$$

on $\mathbb{R}^3$. In particular,

- $\mu = (\mu_I, \mu_J, \mu_K)$ is locally a conformal submersion to $(\mathbb{R}^3, dx^2 + dy^2 + dz^2)$,
- V is locally a harmonic function on $\mathbb{R}^3$.

Examples

$$V(p) = c + \frac{1}{2}\sum_{i\in Z} \frac{1}{\|p-p_i\|}, \qquad c \geqslant 0, \quad p_i \in \mathbb{R}^3 \text{ distinct}$$

- $c = 0$, $|Z| < \infty$: multi-Euguchi Hanson metrics

$\lvert Z\rvert$	1	2	...
space	flat $\mathbb{R}^4$	$T^* \mathbb{C}P(1)$	...

- $c > 0$, $|Z| < \infty$: multi-Taub-NUT metrics

$\lvert Z\rvert$	0	1	2	...
space	flat $S^1 \times \mathbb{R}^3$	Taub-NUT $\mathbb{R}^4$	$T^* \mathbb{C}P(1)$	...

- Z countably infinite: require $V(p)$ to converge at some $p \in \mathbb{R}^3$, get A_∞ metrics (Anderson et al. 1989; Goto 1994), e.g. $Z = \mathbb{N}_{>0}$, $p_n = (n^2, 0, 0)$, and their Taub-NUT deformations.

Classification

Theorem (Swann 2016)

The above potentials

$$V(p) = c + \frac{1}{2} \sum_{i \in Z} \frac{1}{\|p - p_i\|}, \qquad c \geqslant 0, \quad p_i \in \mathbb{R}^3 \text{ distinct},$$

with $0 < V(p) < \infty$ *for some* $p \in \mathbb{R}^3$*, classify all complete hyperKähler four-manifolds with tri-Hamiltonian circle action.*

When $|Z| < \infty$, this is due to Bielawski (1999), and the first parts of the proof are essentially the same.

Proof structure

- The only special orbits are fixed points
- $\mu\colon M/S^1 \to \mathbb{R}^3$ is a local homeomorphism
- near a fixed point x, $V(\mu(y)) = \frac{1}{2}\|\mu(y) - \mu(x)\|^{-1} + \phi(\mu(y))$ with ϕ positive harmonic (Bôcher's Theorem; a Chern class)
- $\overline{\mu}\colon N^3 = (M \setminus M^X)/S^1 \to \mathbb{R}^3$ is conformal: conformal factor V, positive harmonic
- can adjust to $\overline{V}$ superharmonic, so that N becomes complete and use Schoen and Yau (1994) to show $\overline{\mu}\colon N \to \mathbb{R}^3$ is injective with image Ω having boundary that is polar
- V is then given by a Martin integral representation supported on $\partial\Omega$; completeness of M forces $\partial\Omega$ to be discrete.

Toric hyperKähler

(Dancer and Swann 2016)

M^{4n} complete hyperKähler with tri-Hamiltonian action of T^n. Is given locally by the Pedersen-Poon Ansatz:

$$g = (V^{-1})_{ij}(dt + \omega_i)(dt + \omega_j) + V_{ij}(dx_i dx_j + dy_i dy_j + dz_i dz_j),$$

with $V_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$ with F a positive function on $\mathbb{R}^3 \otimes \mathbb{R}^n$ harmonic on every affine three-plane $X_{a,v} = a + \mathbb{R}^3 \otimes v$.

For generic $X_{a,v}$, then $Y = \mu^{-1}(X_{a,v})$ is smooth with free T^{n-1} action, Y/T^{n-1} is complete hyperKähler of dimension 4 with S^1-action. Above analysis then fixes V on $X_{a,v}$, and F, providing a classification.

All examples may be constructed as hyperKähler quotients of flat affine subspaces of Hilbert spaces, cf. Goto 1994; Hattori 2011.

Hypertoric configuration data

In the hypertoric situation μ is *surjective*:

$$\mu(M^{4n}) = \mathbb{R}^{3n} = \mathbb{R}^3 \otimes \mathbb{R}^n = \operatorname{Im}\mathbb{H} \otimes \mathbb{R}^n.$$

Polytope faces are replaced by affine flats of codimension 3:

$$H_k = H(u_k, \lambda_k) = \{a \in \operatorname{Im}\mathbb{H} \otimes \mathbb{R}^n \mid \langle a, u_k \rangle = \lambda_k\},$$

$u_k \in \mathbb{R}^n$, $\lambda_k \in \operatorname{Im}\mathbb{H}$.

Again stabilisers of points mapping to H_k are contained in the subtorus with Lie algebra orthogonal to u_k, forcing $u_k \in \mathbb{Q}^n$.

This time $H(u_{k_1}, \lambda_k) \cap \cdots \cap H(u_{k_n}, \lambda_{k_r}) \neq \varnothing$ whenever $u_{k_1}, \ldots, u_{k_r}$ are linearly independent.

Smoothness implies each such set $u_{k_1}, \ldots, u_{k_r}$ is part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$, giving a *global* restriction on the u_k's.

Get only finitely many distinct vectors u_k, but possibly infinitely many λ_k's.

G_2 manifolds

M^7 with $\varphi \in \Omega^3(M)$ pointwise of the form

$$\varphi = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{356} - e_{347},$$

$e_{ijk} = e_i \wedge e_j \wedge e_k$.

- φ specifies a metric g and an orientation
- The holonomy of g lies in G_2 when $d\varphi = 0 = d{*}\varphi$
- g is then Ricci-flat

Multi-moment maps: T^2 symmetry

(Madsen and Swann 2012)

Suppose T^2 acts preserving (M, φ), holonomy in G_2, with generating vector fields U, V.
The Cartan formula implies $U \lrcorner V \lrcorner \varphi$ is closed.
A function $\nu\colon M \to \mathbb{R}$ with $d\nu = U \lrcorner V \lrcorner \varphi$ is called a *multi-moment map*.

At regular values $X^4 = \nu^{-1}(x)/T^2$ is a four manifold carrying three symplectic forms of the same orientation induced by

$$U \lrcorner \varphi, \quad V \lrcorner \varphi, \quad U \lrcorner V \lrcorner *\varphi.$$

These do *not* form a hyperKähler structure in general.
(M, φ) may be recovered from the four-manifold X^4 by building a T^2-bundle Y^6, constructing an $SU(3)$ geometry (σ, ψ_+) on this bundle and then using an adaptation of the Hitchin flow for 'time' derivatives of these forms.

Multi-moment maps: T^3 symmetry

Suppose T^3 acts preserving (M, φ), holonomy in G_2, with generating vector fields U, V, W.
Multi-moment maps ν_U, ν_V, ν_W given by

$$d\nu_U = V \lrcorner W \lrcorner \varphi, \quad \text{etc.}$$

Hamiltonian condition is that this should be T^3-invariant, is equivalent to $\varphi(U, V, W) = 0$.
There is a fourth multi-moment μ associated to $*\varphi$ via

$$d\mu = U \lrcorner V \lrcorner W \lrcorner *\varphi.$$

$(\nu_U, \nu_V, \nu_W, \mu) \colon M^7 \to \mathbb{R}^4$ has generic fibre T^3.

What is the analogue of the Gibbons-Hawking Ansatz?

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