Toric ideas for special geometries

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Strings, geometry, and quantum fields zoominar
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Madsen, T. B. and Swann, A. F. (2019*a*), 'Toric geometry of G_2-manifolds', *Geom. Topol.* 23 (7): 3459–500
Madsen, T. B. and Swann, A. F. (2019*b*), 'Toric geometry of $\mathrm{Spin}(7)$-manifolds', to be published in *Int. Math. Res. Not. IMRN*, arXiv: 1810.12962 [math.DG]
Russo, G. and Swann, A. F. (2019), 'Nearly Kähler six-manifolds with two-torus symmetry', *J. Geom. Phys.* 138: 144–53

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Outline

1. Symplectic background
2. Multi-Hamiltonian actions
3. Spin(7)
4. Nearly Kähler

Symplectic constructions

ω symplectic form: degree 2, closed, non-degenerate

X symmetry: $\mathscr{L}_X\omega = 0$

moment map: $\mu_X : M \to \mathbb{R}, \quad d\mu_X = \omega(X, \cdot\,)$

Flat space

$M = \mathbb{R}^{2n}$ coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$

$\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$

$X = c_1\Bigl(y_1\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial y_1}\Bigr) + \dots + c_n\Bigl(y_n\frac{\partial}{\partial x_n} - x_n\frac{\partial}{\partial y_n}\Bigr)$

$\mu_X = \frac{1}{2}\bigl(c_1(x_1^2 + y_1^2) + \dots + c_n(x_n^2 + y_n^2)\bigr) + c$

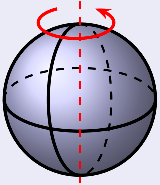
Delzant construction

(M^{2n}, ω) compact symplectic with Hamiltonian action of torus T^n: invariant function $\mu\colon M \to \mathbb{R}^n = \mathrm{Lie}(T^n)^*$ with

$$d\langle \mu, X\rangle = \omega(X, \cdot\,) \qquad \forall X \in \mathbb{R}^n = \mathrm{Lie}(T^n)$$

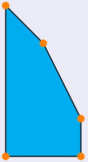
Delzant (1988)

Compact symplectic toric manifolds



correspond to

Delzant polytopes



Polytopes in $\mathbb{R}^n$ with normal vectors in $\mathbb{Z}^n$ and the normals at intersections of faces being part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

- $b_1(M^{2n}) = 0 \implies$ every symplectic torus T^k action is Hamiltonian
- then ω pulls back to 0 on each torus orbit, so $k \leqslant n$

The Delzant dimensions are such that

$$\dim(M^{2n}/T^n) = n = \dim(\text{codomain}\, \mu)$$

Can be used for non-compact M, e.g. Karshon and Lerman (2015)

Toric Calabi-Yau

Include symplectic quotients of $\mathbb{R}^{2N} = \mathbb{C}^N$ by subtori of T^N whose weights sum to zero.

$$\begin{aligned}\mathbb{C}^4 /\!/ \operatorname{diag}(e^{\mathbf{i}t}, e^{\mathbf{i}t}, e^{-\mathbf{i}t}, e^{-\mathbf{i}t}) &= \mu^{-1}(0)/T^1 \\ &= \big(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P(1)\big)\end{aligned}$$

Tian and Yau (1991), ..., Goto (2012)
Futaki, Ono, and Wang (2009)

Ricci-flat special holonomy

Ricci-flat geometries in the Berger holonomy classification 1955, …

Name	Holonomy	Dimension	Form degrees
Calabi-Yau	$\mathrm{SU}(n)$	$2n$	$2, n, n$
HyperKähler	$\mathrm{Sp}(n)$	$4n$	$2, 2, 2$
G_2 holonomy	G_2	7	$3, 4$
$\mathrm{Spin}(7)$ holonomy	$\mathrm{Spin}(7)$	8	4

In these cases

- the forms specify the geometry
- holonomy reduction $\iff$ the forms are closed

MULTI-HAMILTONIAN TORUS ACTIONS

(M, α) manifold with closed $\alpha \in \Omega^p(M)$ preserved by T^k is multi-Hamiltonian if there is an invariant

$$\nu\colon M \to \Lambda^{p-1}\big(\mathrm{Lie}(T^k)^*\big) \cong \mathbb{R}^N,$$
$$d\langle \nu, X_1 \wedge \cdots \wedge X_{p-1}\rangle = \alpha(X_1, \dots, X_{p-1}, \cdot\,)$$

for all $X_i \in \mathrm{Lie}(T^k)$.

- $b_1(M) = 0 \implies$ each T^k-action preserving α is multi-Hamiltonian
- then α pulls-back to 0 on each T^k-orbit

Geometry is toric if multi-Hamiltonian for T^k and

$$\dim(M/T^k) = \dim(\operatorname{codomain} \nu)$$

A Ricci-flat hierarchy

dimension	4	6	7	8
holonomies	Sp(1)	SU(3)	G_2	Spin(7)
closed forms	$\omega_1, \omega_2, \omega_3$	$\omega, \Omega_+, \Omega_-$	$\varphi, *_7\varphi$	Φ
degrees	2, 2, 2	2, 3, 3	3, 4	4
Toric group	S^1	T^2	T^3	T^4
Extension		$M^4 \times S^1 \times \mathbb{R}$	$M^6 \times S^1$	$M^7 \times S^1$

$$\omega_1 = e^{45} + e^{67}, \quad \omega_2 = e^{46} + e^{75}, \quad \omega_3 = e^{47} + e^{56}$$

$$\omega = e^{23} - \omega_1, \quad \Omega_+ = -e^2 \wedge \omega_2 - e^3 \wedge \omega_3, \quad \Omega_- = -e^3\omega_2 + e^2 \wedge \omega_3$$

$$\varphi = e^1 \wedge \omega + \Omega_+ \qquad *_7\varphi = \Omega_- \wedge e^1 + \tfrac{1}{2}\omega^2$$

$$\Phi = e^0 \wedge \varphi + *_7\varphi$$

TORIC Spin(7)

(M^8, Φ) with toric T^4 generated by U_i, $i = 0, 1, 2, 3$, multi-moment map $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$

$$d\nu_i = (-1)^i \Phi(U_j \wedge U_k \wedge U_\ell, \cdot) \qquad (ijk\ell) = (0123)$$

On the open dense set M_0 where T^4 acts freely

$$\Phi = \det(V)\Big(\mathop{\mathfrak{S}}_{ijk\ell} (-1)^i (\theta_i \wedge d\nu_{jk\ell} + \theta_{jk\ell} \wedge d\nu_i) + \tfrac{1}{2}(d\nu^t V^{-1} \theta)^2\Big)$$

$$g = \theta^t V^{-1} \theta + \det(V) d\nu^t V^{-1} d\nu$$

for $V = (g(U_i, U_j))^{-1}$, $\quad \theta = (\theta_0, \theta_1, \theta_2, \theta_3)$ connection one-forms

Smooth behaviour

Theorem

Holonomy is contained in Spin(7) (*i.e.* $d\Phi = 0$) *if and only if*

$$\operatorname{div} V = 0 \quad \textit{and} \quad L(V) + Q(dV) = 0$$

$$\operatorname{div}(V)_a = \sum_{i=0}^{3} \frac{\partial V_{ia}}{\partial \nu_i}$$

$$L(V)_{ab} = \sum_{i,j=0}^{3} V_{ij} \frac{\partial^2 V_{ab}}{\partial \nu_i \partial \nu_j}, \qquad Q(dV)_{ab} = -\sum_{i,j=0}^{3} \frac{\partial V_{ia}}{\partial \nu_j} \frac{\partial V_{jb}}{\partial \nu_i}$$

Note

$$L(V) + Q(dV) = \sum_{i,j=0}^{3} \frac{\partial^2}{\partial \nu_i \partial \nu_j} (V_{ij} V_{ab} - V_{ib} V_{ja})$$

Specialisations

Holonomy in G_2, SU(3) and Sp(1) from $V = \begin{pmatrix} * & 0 \\ 0 & 1_r \end{pmatrix}$, $r = 1, 2, 3$

dim 4, Sp(1), $V = \begin{pmatrix} V_{00} & 0 \\ 0 & 1_3 \end{pmatrix}$

$\operatorname{div} V = 0 \iff V_{00} = V_{00}(v_1, v_2, v_3) \implies Q(dV) = 0$ and $L(V) = 0$ has only one scalar equation $\Delta_{\mathbb{R}^3} V_{33} = 0$, V_{33} is harmonic on domain(s) in $\mathbb{R}^3$.

dim 6, SU(3), $V = \begin{pmatrix} V_{00} & V_{01} & 0 \\ V_{01} & V_{11} & 0 \\ 0 & 0 & 1_2 \end{pmatrix}$

$$\frac{\partial V_{00}}{\partial v_0} + \frac{\partial V_{01}}{\partial v_1} = 0, \qquad \frac{\partial V_{01}}{\partial v_0} + \frac{\partial V_{11}}{\partial v_1} = 0$$

$$\frac{\partial^2}{\partial v_i \partial v_j} \det V + (-1)^{i+j} \left(\frac{\partial^2}{\partial {v_2}^2} + \frac{\partial^2}{\partial {v_3}^2} \right) V_{1-i,1-j} = 0 \quad i, j = 0, 1$$

Explicit full holonomy

Holonomy = Spin(7)

$$V = \operatorname{diag}(\nu_1, \nu_2, \nu_3, \nu_0) > 0$$

$$g = \sum_{i=0}^{3} \frac{1}{\nu_{i+1}} \theta_i^2 + \nu_{i+2}\nu_{i+3}\nu_i d\nu_i^2 \qquad d\theta_i = (-1)^{i+1} \nu_{i+2} d\nu_{i+2} \wedge d\nu_{i+3}$$

Holonomy = G_2

$$V = \operatorname{diag}(1, \nu_0, \nu_0, \nu_0) > 0$$

$$g = \frac{1}{\nu_0} \sum_{i=1}^{3} \theta_i^2 + \nu_0^3 d\nu_0^2 + \nu_0^2 \sum_{i=1}^{3} d\nu_i^2 \qquad d\theta_i = d\nu_j \wedge d\nu_k \ (ijk) = (123)$$

Cf. Hein, Sun, Viaclovsky, and Zhang (2018)

Singular behaviour

Theorem

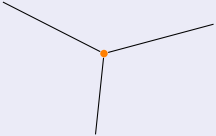
$\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$ *induces a local homeomorphism* $M/T^4 \to \mathbb{R}^4$

Each stabiliser $\mathrm{Stab}_{T^4}(p)$ is a connected subtorus of T^4 of rank $\leqslant 2$
Image under $\nu\colon M \to \mathbb{R}^4$

- $\dim(\mathrm{Stab}_{T^4}) = 1$: straight lines with rational slopes
- $\dim(\mathrm{Stab}_{T^4}) = 2$: points
  with three straight lines meeting at point, sum of primitive tangents is zero

Flat model

$M = T^2 \times \mathbb{C}^3, T^4 \leqslant T^2 \times SU(3)$

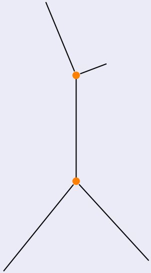


planar graph

Li (2019) produced Taub-NUT family of complete metrics on $\mathbb{C}^3$ with above orbit structure
Other Calabi-Yau examples Mooney (2020)

Example

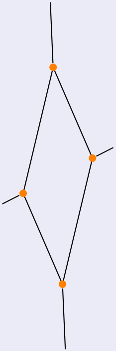
$S^1 \times$ (Bryant-Salamon G_2 metric on $S^3 \times \mathbb{R}^4$):



non-planar, but lies in $\mathbb{R}^3 \subset \mathbb{R}^4$

Example

$S^1\times$ Foscolo, Haskins, and Nordström (2018) G_2-example on $M_{m,n}$: $M_{m,n}$ a circle bundle over the canonical bundle of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with first Chern class $(m, -n)$ over the zero section, symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times S^1$



Primitive directions
$(m - n, 0, n)$
$(0, n - m, m)$
$(n - m, m - n, -m - n)$
planar

NEARLY KÄHLER IN DIMENSION 6

$(N^6, g, J, \sigma, \psi_+, \psi_-)$ is (strictly) nearly Kähler if $M^7 = C(N^6)$, $\phi = t^2 dt \wedge \sigma + t^3 \psi_+$ has holonomy in G_2. Equivalently

$$d\sigma = 3\psi_+ \quad \text{and} \quad d\psi_- = -2\sigma \wedge \sigma$$

GRAY (1976) (N^6, g) is positive Einstein

BUTRUILLE (2005) the homogeneous examples are

$$S^6 = G_2/\mathrm{SU}(3), \quad \mathbb{C}P(3) = \mathrm{Sp}(2)/\mathrm{U}(2),$$
$$F_{1,2}(\mathbb{C}^3) = \mathrm{SU}(3)/T^2 \quad \text{and} \quad S^3 \times S^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)$$

FOSCOLO AND HASKINS (2017) two new examples.

All known examples have symmetry group of rank at least 2, so T^2 symmetry

CONJECTURE (MOROIANU AND NAGY 2019): T^3 symmetry implies $N^6 = S^3 \times S^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)$ cf. Dixon (2020)

TWO-TORUS SYMMETRY

T^2 generated by vector fields U, V
Multi-moment map $\nu = \sigma(U, V)$ $\quad d\nu = 3\psi_+(U, V, \cdot)$

1. 0 is an interior point of the interval $\nu(N) \subset \mathbb{R}$
2. the T^2 action is free on $\nu^{-1}(s)$ whenever s is a regular value
3. $\nu^{-1}(s)/T^2$ is then a smooth three manifold, parallelised by a coframe $\beta_0, \beta_1, \beta_2$ satisfying

   $$f\, d\beta_0 = \beta_1 \wedge \beta_2, \quad d\beta_1 \wedge \beta_0 = 0 = d\beta_2 \wedge \beta_0$$

   $f = 4/(1 - s^2/\det G)$, G metric on T^2-fibre
4. can recover N via a geometric flow of the data $\beta_0, \beta_1, \beta_2, G$

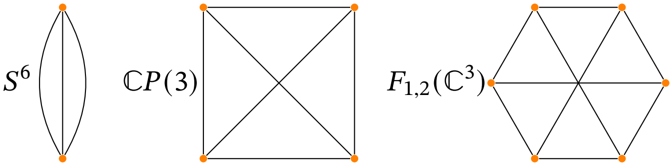
Singular behaviour

Stabilisers are either T^2, S^1 or $\mathbb{Z}_k$

T^2 and S^1 stabilisers occur only in $\nu^{-1}(0)$ and the quotient $\nu^{-1}(0)/T^2$ is homeomorphic to a smooth three-manifold

Images of points with S^1 stabiliser give curves in the three-manifold

Images of points with T^2 stabiliser give vertices at which three such curves meet.



Non-zero critical points lie on holomorphic tori.

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