Toric Geometry of G_2-metrics

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Early history

Holonomy G_2 is one of two special Riemannian holonomies in Berger's classification 1955

These are Ricci-flat geometries on 7-dimensional manifolds

First non-trivial examples: Bryant (1987)

First complete examples: Bryant and Salamon (1989)

M^7	$\Lambda^2_-(S^4)$	$\Lambda^2_-(\mathbb{C}P^2)$	$S^3 \times \mathbb{R}^4$
Isom_0	$\mathrm{Sp}(2)$	$\mathrm{SU}(3)$	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$
$\mathrm{rank}(\mathrm{Isom})$	2	2	3

First compact examples: Joyce (1996)

Ricci-flat implies on compact manifolds that Killing vector fields are parallel, so will be interested in non-compact M^7

G_2 manifolds

M^7 with $\varphi \in \Omega^3(M)$ pointwise of the form

$$\varphi = e_{123} - e_{145} - e_{167} - e_{246} - e_{275} - e_{347} - e_{356},$$

$e_{ijk} = e_i \wedge e_j \wedge e_k$
Specifies metric $g = e_1^2 + \cdots + e_7^2$, orientation $\mathrm{vol} = e_{1234567}$ and four-form

$$*\varphi = e_{4567} - e_{2345} - e_{2367} - e_{3146} - e_{3175} - e_{1256} - e_{1247}$$

via

$$6g(X, Y)\,\mathrm{vol} = (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$$

There is also a cross-product

$$g(X \times Y, Z) = \varphi(X, Y, Z)$$

with $X \times Y \perp X, Y$

Holonomy of g is in G_2 when $d\varphi = 0 = d*\varphi$, a *parallel G_2-structure*

Multi-Hamiltonian actions

(M, α) manifold with closed $\alpha \in \Omega^p(M)$ preserved by $G = T^n$
This is *multi-Hamiltonian* if it there is a G-invariant $\nu\colon M \to \Lambda^{p-1}\mathfrak{g}^*$ with

$$d\langle \nu, X_1 \wedge \cdots \wedge X_{p-1}\rangle = \alpha(X_1, \ldots, X_{p-1}, \cdot\,)$$

for all $X_i \in \mathfrak{g}$

- take $n > p - 2$
- ν invariant $\iff$ α pulls-back to 0 on each T^n-orbit
- $b_1(M) = 0 \implies$ each T^n-action preserving α is multi-Hamiltonian

For (M, φ) a parallel G_2-structure, can take $\alpha = \varphi$ and/or $\alpha = *\varphi$

Multi-Hamiltonian parallel G_2-manifolds

Proposition

Suppose (M, φ) is a parallel G_2-manifold with T^n-symmetry multi-Hamiltonian for $\alpha = \varphi$ and/or $\alpha = *\varphi$. Then $2 \leqslant n \leqslant 4$.

q: dimension of orbit space M^7/T^n

k: dimension of target of multi-moment map $\Lambda^2\mathbb{R}^n$ and/or $\Lambda^3\mathbb{R}^n$

n	q	α	k	note
2	5	φ	1	Madsen and Swann (2012)
3	4	φ	3	
		$*\varphi$	1	
		φ & $*\varphi$	4	*toric*
4	3	φ	6	Baraglia (2010)
		$*\varphi$	4	

Toric G_2

Definition

A *toric G_2 manifold* is a parallel G_2-structure (M, φ) with an action of T^3 multi-Hamiltonian for both φ and $*\varphi$

Let U_1, U_2, U_3 generate the T^3-action, then $\varphi(U_1, U_2, U_3) = 0$, with multi-moment maps $(\nu, \mu) = (\nu_1, \nu_2, \nu_3, \mu)\colon M \longrightarrow \mathbb{R}^4$

$$d\nu_i = U_j \wedge U_k \lrcorner \varphi = (U_j \times U_k)^\flat \quad (i\,j\,k) = (1\,2\,3)$$

$$d\mu = U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi$$

Recall $\varphi = e_{123} - e_{145} - e_{167} - e_{246} - e_{275} - e_{347} - e_{356}$
If U_i are linearly independent at p, then there is a G_2-basis so that $\mathrm{Span}\{U_1, U_2, U_3\} = \mathrm{Span}\{E_5, E_6, E_7\}$. The repeated cross-products of the U_i then generate TM and $(d\nu, d\mu)$ is of full rank 4, so (ν, μ) induces a local diffeomorphism

$$M_0/T^3 \longrightarrow \mathbb{R}^4$$

The flat model

$M = S^1 \times \mathbb{C}^3$
Standard flat $\varphi = \frac{i}{2} dx(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}}) + \mathrm{Re}(dz_{123})$
Preserved by $T^3 = S^1 \times T^2 \leqslant S^1 \times \mathrm{SU}(3)$
Stabilisers T^2 at $S^1 \times \{0\}$ and T^1 at $S^1 \times (z_i = 0 = z_j, i \neq j)$
Multi-moment maps

$$4(\nu_1 - i\mu) = z_1 z_2 z_3, \quad 4\nu_2 = |z_2|^2 - |z_3|^2, \quad 4\nu_3 = |z_3|^2 - |z_1|^2$$

Topologically $M/T^3 = \mathbb{C}^3/T^2 = C(S^5)/T^2 = C(S^5/T^2) = C(S^3) = \mathbb{R}^4$
The ring $P(\mathbb{R}^6)^{T^2}$ of invariant polynomials has basis μ, ν_1, ν_2, ν_3 and $t = |z_3|^2$. By Schwarz (1975) any smooth invariant function on $\mathbb{C}^3/T^2$ is a smooth function of these five invariant polynomials. However, they satisfy

$$t(t + 2\nu_2)(t - 2\nu_3) = \nu_1^2 + \mu^2, \quad t \geqslant \max\{0, -2\nu_2, 2\nu_3\} \tag{S}$$

The linear projection $(t, \nu, \mu) \mapsto (\nu, \mu)$ is a homeomorphism of this set on to $\mathbb{R}^4$

General picture

Proposition

All isotropy groups of the T^3 action are connected and act on the tangent space as maximal tori in (a) $1 \times \mathrm{SU}(3)$, (b) $1_3 \times \mathrm{SU}(2)$ *or* (c) 1_7

Local tangent space models are flat model around (a) $S^1 \times (0, 0, 0)$ or (b) $S^1 \times (1, 0, 0)$. (b) is the Hopf fibration, topologically rigid.
At (a), ν_2 and ν_3 agree with the flat model to order 3, ν_1 and μ to order 4. Analysis of the singularity (S) and degree arguments give

Theorem

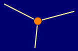
Let M be a toric G_2-manifold, then M/T^3 is homeomorphic to a smooth four-manifold. Moreover, the multi-moment map (ν, μ) induces a local homeomorphism $M/T^3 \longrightarrow \mathbb{R}^4$.

Configuration data

T^2 stabiliser $\mapsto$ a point in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$
S^1 stabiliser $\mapsto$ lines in (μ = constant) of rational slope
Any intersection is triple, with with the primitive slope vectors summing to zero. Thus we get a collection of trivalent graphs.

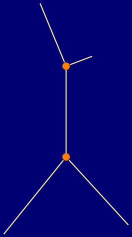
Example

Flat model $S^1 \times \mathbb{C}^3$:



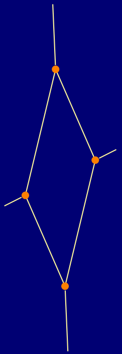
Example

Bryant-Salamon metrics on $S^3 \times \mathbb{R}^4$:



Example

Foscolo et al. (2018) examples on $M_{m,n}$ have $M_{m,n}$ a circle bundle over the canonical bundle of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with first Chern class $(m, -n)$ over the zero section, symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times S^1$:



Primitive directions

$(m-n, 0, n)$

$(0, n-m, m)$

$(n-m, m-n, -m-n)$

Hypertoric manifolds

Swann (2016) and Dancer and Swann (2017), following Bielawski (1999), Bielawski and Dancer (2000), Goto (1994) and Anderson et al. (1989)

$(M, g, \omega_I, \omega_J, \omega_K)$ is *hyperKähler* if each $(g, \omega_A = g(A\cdot\,,\,\cdot))$ is Kähler and $IJ = K = -JI$
Then $\dim M = 4n$ and g is Ricci-flat, holonomy in $\mathrm{Sp}(n) \leqslant \mathrm{SU}(2n)$, so also Calabi-Yau

Hypertoric is complete hyperKähler M^{4n} with tri-Hamiltonian $G = T^n$ action: have G-invariant map (*hyperKähler moment map*)

$$\mu = (\mu_I, \mu_J, \mu_K)\colon M \to \mathbb{R}^3 \otimes \mathfrak{g}^* \qquad d\langle \mu_A, X\rangle = X \lrcorner \omega_A$$

Hypertoric: M^{4n} with $G = T^n$ action $\mu = (\mu_I, \mu_J, \mu_K)\colon M \to \mathbb{R}^3 \otimes \mathfrak{g}^*$

- $\dim(M/T^n)$ is $3n$, the dimension of target space of μ
- stabiliser of any point is a (connected) subtorus of dimension $n - \frac{1}{3}\operatorname{rank} d\mu$
- $\mu(M) = \mathbb{R}^{3n}$ with configuration of flats (possibly infinitely many) $H(u_k, \lambda_k) = \{a \in \operatorname{Im}\mathbb{H} \otimes \mathbb{R}^n \mid \langle a, u_k \rangle = \lambda_k\}$

n	1	2	3
$\{u_k\}$	$\{(1)\}$	$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$

Locally (Gibbons and Hawking 1978; Lindström and Roček 1983)

$$g = (V^{-1})_{ij}\theta_i\theta_j + V_{ij}(d\mu_I^i d\mu_I^j + d\mu_J^i d\mu_J^j + d\mu_K^i d\mu_K^j),$$

with (V_{ij}) positive-definite, and harmonic on each $a + \mathbb{R}^3 \otimes v$

For $n = 1$:

$$g = V^{-1}\theta^2 + V(d\mu_I^2 + d\mu_J^2 + d\mu_K^2)$$
$$d\theta = *_3 dV, \qquad \Delta_{\mathbb{R}^3} V = 0$$
$$V(p) = c + \sum_{q \in Q \subset \mathbb{R}^3} \frac{1}{2\|p - q\|}$$

with $c \geqslant 0$, $Q \subset \mathbb{R}^3$ discrete, $V(p) < +\infty$ at some p

These are complete and contain examples with b_2 not finite.

Smooth equations for toric G_2

T^3 symmetry free on $M_0 \subset M$

$M_0 \longrightarrow M_0/T^3$ is a principal torus bundle with connection one-forms $\theta_i \in \Omega^1(M_0)$: $\theta_i(U_j) = \delta_{ij}$, $\theta_i(X) = 0\ \forall\, X \perp U_1, U_2, U_3$. On M_0, put

$$B = \bigl(g(U_i, U_j)\bigr) \quad \text{and} \quad V = B^{-1} = \frac{1}{\det B}\,\mathrm{adj}\,B$$

Theorem

$$g = \frac{1}{\det V}\theta^t\,\mathrm{adj}(V)\,\theta + dv^t\,\mathrm{adj}(V)\,dv + \det(V)\,d\mu^2$$

$$\varphi = -\det(V)\,dv_{123} + d\mu dv^t\,\mathrm{adj}(V)\,\theta + \mathop{\mathfrak{S}}_{i,j,k} \theta_{ij}dv_k$$

$$*\varphi = \theta_{123}d\mu + \frac{1}{2\det(V)}\bigl(dv^t\,\mathrm{adj}(V)\,\theta\bigr)^2 + \det(V)\,d\mu \mathop{\mathfrak{S}}_{i,j,k} \theta_i dv_{jk}$$

Theorem (continued)

Such $(g, \varphi, *\varphi)$ defines a parallel G_2-structure if and only if $V \in C^\infty(M_0/T^3, S^2\mathbb{R}^3)$ is a positive-definite solution to

$$\sum_{i=1}^{3} \frac{\partial V_{ij}}{\partial v_i} = 0 \qquad j = 1, 2, 3 \qquad \text{(divergence-free)}$$

and

$$L(V) + Q(dV) = 0 \qquad \text{(elliptic)} \tag{1}$$

where $L = \frac{\partial^2}{\partial \mu^2} + \sum_{i,j} V_{ij} \frac{\partial^2}{\partial v_i \partial v_j}$ *and Q is a quadratic form with constant coefficients*

Cf. Chihara (2018)

L and Q are preserved up to scale by $\mathrm{GL}(3, \mathbb{R})$ change of basis; this specifies Q uniquely

Proposition

Solutions V to the divergence-free equation are given locally by $A \in C^\infty(M_0/T^3, S^2\mathbb{R}^3)$ via

$$V_{ii} = \frac{\partial^2 A_{jj}}{\partial {v_k}^2} + \frac{\partial^2 A_{kk}}{\partial {v_j}^2} - 2\frac{\partial^2 A_{jk}}{\partial v_j \partial v_k}$$

$$V_{ij} = \frac{\partial^2 A_{ik}}{\partial v_j \partial v_k} + \frac{\partial^2 A_{jk}}{\partial v_i \partial v_k} - \frac{\partial^2 A_{ij}}{\partial {v_k}^2} - \frac{\partial^2 A_{kk}}{\partial v_i \partial v_j}$$

$(i\,j\,k) = (1\,2\,3)$

Diagonal solutions

$V = \operatorname{diag}(V_1, V_2, V_3)$ (divergence-free) and off-diagonal terms in (elliptic)

$$\frac{\partial V_i}{\partial v_i} = 0 \qquad \frac{\partial V_i}{\partial v_j}\frac{\partial V_j}{\partial v_i} = 0 \quad (i \neq j)$$

Either $V = \operatorname{diag}(V_1(v_2, \mu), V_2(v_3, \mu), V_3(v_1, \mu))$ linear in each variable

E.g. $V = \mu 1_3$, $\mu > 0$, full holonomy G_2:

$$g = \frac{1}{\mu}(\theta_1^2 + \theta_2^2 + \theta_3^2) + \mu^2(dv_1^2 + dv_2^2 + dv_3^2) + \mu^3 d\mu^2$$

$$d\theta_i = dv_j \wedge dv_k \quad (ijk) = (123)$$

Or get elliptic hierarchy $V_3 = V_3(\mu)$, $V_2 = V_2(v_3, \mu)$, $V_1 = V_1(v_2, v_3, \mu)$

$$\frac{\partial^2 V_3}{\partial \mu^2} = 0 \quad \frac{\partial^2 V_2}{\partial \mu^2} + V_3\frac{\partial^2 V_2}{\partial {v_3}^2} = 0 \quad \frac{\partial^2 V_1}{\partial \mu^2} + V_2\frac{\partial^2 V_1}{\partial {v_2}^2} + V_3\frac{\partial^2 V_1}{\partial {v_3}^2} = 0$$

E.g. $V_3 = \mu, \quad V_2 = \mu^3 - 3v_3^2, \quad V_1 = 2\mu^5 - 15\mu^2 v_3^2 - 5v_2^2$

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